Lecture 8 : The Geometric Distribution

The geometric distribution is a special case of negative binomial, it is the case $r = 1$. It is so important we give it special treatment.

Motivating example

Suppose a couple decides to have children until they have a girl. Suppose the probability of having a girl is P. Let

$$X = \text{the number of boys that precede the first girl}$$

Find the probability distribution of X. First X could have any possible whole number value (although $X = 1,000,000$ is very unlikely)

$$P(X = k) = P(\underbrace{\underline{B}\ \underline{B}\ \underline{B}\ \_\ \underline{B}}_{k}\ \underset{\uparrow\, p}{\underline{G}})$$

$$= q^k p \quad (\text{where } q = 1 - p)$$

We have suppose birth are independent.
We have motivated.

Definition

Suppose a discrete random variable X has the following pmf

$$P(X = k) = q^k P, \quad 0 \leq k < \infty$$

The X is said to have geometric distribution with parameter P.

Remark

Usually this is developed by replacing "having a child" by a Bernoulli experiment and having a girl by a "success" (PC). I could have used coin flips.

Proposition

Suppose X has geometric distribution with parameter p.
Then

(i) $E(X) = \dfrac{q}{p}$

(ii) $V(X) = \dfrac{q}{p^2}$

Proof of (i) (you are not responsible for this).

$$
\begin{aligned}
E(X) &= (0)(p) + (1)(qp) + (2)(q^2 p) + \cdots + (k)(q^k p) + \cdots \\
&= p(q + 2q + \cdots + kq^k + \cdots
\end{aligned}
$$

Now

$$\frac{X}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots + kx^k + \cdots$$

↑ why?

So

$$EX() = p\left(\frac{q}{(1-q)^2}\right) = p\left(\frac{q}{p^2}\right) = \frac{q}{p}$$

□

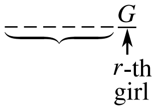
The Negative Binomial Distribution

Now suppose the couple decides they want more girls - say r girls, so they keep having children until the r-th girl appears. Let X = the number of boys that precede the r-th girl.
Find the probability distribution of X.

Remark

Sometimes (eg. pg. 13-14) it is better to write X_r instead of X.

Let's compute $P(X = k)$



What do we have preceding the r-th girl. Of course we must have $r - 1$ girls and since we are assuming $X = k$ we have k boys so $ktr - 1$ children.
All orderings of boys and girls have the some probability so

$$P(X = k) = (?)P(\underbrace{B \ldots B}_{k-1}\,\underbrace{G \ldots G}_{r-1}\,G)$$

or

$$P(X = k) = (?)q^k \cdot p^{r-1} \cdot q = (?)q^k p^r$$

(?) is the number of words of length $ktr - 1$ in B and G using k B's (where $r - 1$ G's).
Such a word is determined by choosing the slots occupied by the boys so there are $\binom{k+r-1}{k}$ words so

$$P(X = k) = \binom{k + r - 1}{k} p^r q^k$$

So we have motivated the following.

Definition

A discrete random variable X is said to have negative binomial distribution with parameters r and p if

$$P(X = k) = \binom{k+r-1}{k} p^r q^k, \ 0 \leq k < \infty$$

The text denotes this proof by $nb(x; r, p)$ so

$$nb(x; r, p) = \binom{x+r-1}{k} p^r q^x, \ 0 \leq x \leq \infty.$$

Proposition

Suppose X has negative binomial distribution with parameters r and p. Then

(i) $E(X) = r\dfrac{q}{p}$

(ii) $V(X) = \dfrac{rq}{p^2}$

Waiting Times

The binomial, geometric and negative binomial distributions are all tied to repeating a given Bernoulli experiment (flipping a coin, having a child) infinitely many times.
Think of discrete time $0, 1, 2, 3, \ldots$ and we repeat the experiment at each of these discrete times. - Eg., flip a coin every minute.

Now you can do the following things

1. Fix a time say n and let $X = \sharp$ of successes in that time period. Then $X \sim \mathrm{Bin}(n, p)$. We should write X_n and think of the family of random variable parametrized by the discrete time n as the "binomial process". (see page. 18 - the Poisson process).
2. ((discrete) waiting time for the first success)
   Let Y be the amount of time up to the time the first success occurs.

This is the geometric random variable. Why?
Suppose we have in out boy/girl example

$$\underbrace{\underset{0}{\underline{B}}\ \underset{1}{\underline{B}}\ \underset{2}{\underline{B}}\ \underset{3}{\underline{\ \ }}\ \underline{B}}_{k}\ \ \underset{k}{\underline{G}}$$

So in this case
$X = \sharp$ of boys = k
Y = waiting time = k
so $Y = X$.

Waiting time for r-th success

Now let $Y_n =$ the waiting time up to the r-th success then there is a difference between X_r and Y_r.
Suppose $X_r = k$ so there are k boys before the r-th girl arrives.

$$\underbrace{\begin{array}{ccccc} & B & & & \\ \overline{0} & \overline{1} & \overline{2} & \quad & \overline{k+r-2} \end{array}} \quad \begin{array}{c} G \\ \overline{k+r-1} \end{array}$$

k B's $r-1$ G's so $ktr-1$ slots.
But start at 0 so the last slot is $k+r-$ ② so

$$Y_r = X_r + r - 1$$

The Poisson Distribution

For a change we won't start with a motivating example but will start with the definition.

Definition

A discrete random variable X is said to have Poisson distribution with parameter λ.

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!},\ 0 \le k < \infty$$

We will abbreviate this to $X \sim P(\lambda)$.
I will now try to motivate the formula which looks complicated.

Why is the factor of $e^{-\lambda}$ there? It is there to make to total probability equal to 1.

Total Probability $= \sum_{k=0}^{\infty} P(X = k)$

$$= \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

But from calculus

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

Total probability $= e^{-\alpha} \cdot e^{\alpha} = 1$ as it has to be.

Proposition

Suppose $X \sim P(\lambda)$. *Then*

(i) $E(X) = \lambda$

(ii) $V(X) = \lambda$

Remark

It is remarkable that $E(X) = V(X)$.

Example (3.39)

Let X *denote the number of creatures of a particular type captured during a given time period. Suppose* $X \sim P(4.5)$. *Find* $P(X = 5)$ *and* $P(X \leq 5)$.

Solution

$$P(X = 5) = e^{-4.5} \frac{(4.5)^5}{5!}$$

(just plug into the formula using $\lambda = 4.5$)

$$\begin{aligned} P(X \leq 5) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &\quad + P(X = 3) + P(X - 4) + P(X = 5) \\ &= e^{-\lambda} + e^{-\lambda}\lambda + e^{-\lambda}\frac{\lambda^2}{2} \\ &\quad \underbrace{+ e^{-\lambda}\frac{\lambda^3}{3!} + e^{-\lambda}\frac{\lambda^4}{4!} + e^{-\lambda^2}\frac{\lambda^5}{5!}}_{\text{don't try to evaluate this}} \end{aligned}$$

The Poisson Process

A very important application of the Poisson distribution arises in counting the number of occurrences of a certain event in time t

1. Animals in a trap.
2. Calls coming into a telephone switch board.

Now we could let t vary so we get a one-parameter family of Poisson random variable X_t, $0 \le t < \infty$.
Now a Poisson process is completely determined once we know its mean λ.

So far each t, X_t is a Poisson random variable. So $X_t \sim P(\lambda(t))$.
So the Poisson parameter λ is a function of t.
In the *Poisson process* one assume that $\lambda(t)$ is the simplest possible function of t (aside from a constant function) namely a linear function

$$\lambda(t) = \alpha t.$$

Necessarily

$$\alpha = \lambda(1) = \text{the average number of observations in unit time.}$$

Remark

In the text, page 124, the author proposes 3 axioms on a one parameter family of random variables X_t. So that X_t is a Poisson process i.e.,

$$X_t \sim P(\alpha t)$$

Example

(from an earlier version of the text)
The number of tickets issued by a meter reader can be modelled by a Poisson process with a rate of 10 ticket every two pairs.

(a) What is the probability that exactly 10 tickets are given out during a particular 12 hour period.

Solution

We want $P(X_{12} = 10)$.

First find α = *average* $\sharp$ *of tickets by unit time.*

So $\alpha = \dfrac{10}{2} = 5$

So $X_t \sim P(5t)$

Solution (Cont.)

So $X_{12} \sim P((5)(12)) = P(60)$

$$P(X_{12} = 10) = e^{-\lambda} \frac{\lambda^{10}}{(10)!}$$
$$= e^{-60} \frac{(60)^{10}}{(10)!}$$

(b) What is the probability that at least 10 tickets are given out during a 12 hour time period.

We wait

$$P(X_{12} \geq 10) = 1 - P(X \leq 9)$$
$$= 1 - \sum_{k=0}^{9} e^{-\lambda} \frac{\lambda^k}{k!}$$
$$= 1 - \underbrace{\sum_{k=0}^{9} e^{-60} \frac{(60)^k}{k!}}_{\substack{\text{not something you} \\ \text{want to try to} \\ \text{evaluate by hand.}}}$$

Waiting Times

Again there are waiting time random variables associated to the Poisson process.
Let Y = waiting time until the first animal is caught in the trap.
and Y_r = waiting time until the r-th animal is caught in the trap.
Now Y and Y_r are *continuous* random variables which we are about to study. Y is *exponential* and Y_r has a special kind *gomma* distribution.