

# VOLUMES OF DEFINABLE SETS IN O-MINIMAL EXPANSIONS AND AFFINE GAGA THEOREMS

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ABSTRACT. In this expository note, I prove the following fact, which is originally due to Nguyen and Valette: any  $d$ -dimensional definable subset of  $S \subseteq \mathbb{R}^n$  in an o-minimal expansion of the ordered field of real numbers satisfies the volume estimate  $\mathcal{H}^d(\{x \in S : \|x\| < r\}) \leq Cr^d$ , where  $\mathcal{H}^d$  denotes the  $d$ -dimensional Hausdorff measure on  $\mathbb{R}^n$  and  $C$  is a constant depending on  $S$ . Coupled with the theorem of Bishop and Stoll, this result gives a quick proof of the GAGA theorem of Peterzil and Starchenko in the case of o-minimal expansions of the ordered field of real numbers.

## 1. INTRODUCTION

The GAGA theorem of Peterzil–Starchenko [PS08] says that a closed analytic subset of  $\mathbb{C}^n$ , which is definable in an o-minimal expansion of the ordered field  $\mathbb{R}$ , is algebraic. It is a crucial ingredient in (at least) two, closely related, recent advances in Hodge theory: the paper by Bakker, Klingler and Tsimerman [BKT18], which gives a new proof of the theorem of Cattani, Deligne and Kaplan [CDK95] on the algebraicity of the Hodge locus, and the paper by Bakker, Brunebarbe and Tsimerman proving a conjecture of Griffiths on the algebraicity of the image of the period map [BBT18]. Given these important results, it seems desirable to have an understanding of the Peterzil–Starchenko theorem from several points of view.

The point of this expository note is to show that the Peterzil–Starchenko theorem follows directly from a GAGA theorem originally due to Stoll [Sto64] and a volume estimate for definable sets. This volume estimate was known to experts in o-minimal structures for some years now: it is a special case of Proposition 3.1 of a 2018 paper by Nguyen and Valette [NV18]. I will state it precisely in Theorem 4 below and prove it from scratch in §5, but essentially it says the following: Suppose  $S$  is a  $d$ -dimensional subset of  $\mathbb{R}^n$ , which is definable with respect to an o-minimal expansion of  $\mathbb{R}_{\text{alg}}$  (for example,  $\mathbb{R}_{\text{an,exp}}$ ). Then the Hausdorff measure of the set  $S(r) := \{x \in S : \|x\| < r\}$  viewed as a function of  $r$  is in  $O(r^d)$ . In other words, the volume of the intersection of  $S$  with a ball of radius  $r$  grows at most as fast as a constant multiple of  $r^d$ . In §4.2, I use it to give a very quick proof of the Peterzil–Starchenko theorem.

Peterzil and Starchenko published two proofs of their theorem. The first, in [PS08], works for o-minimal expansions of arbitrary real closed fields and is based on results from model theory. The second proof, in [PS09], like the proof presented in this note, relies on results from complex analysis. More precisely, it relies on a paper of Shiffman that is directly related to Stoll’s theorem [Shi68] in that Shiffman’s results are ultimately about bounds on volumes of complex analytic sets. Aside from brevity, the main advantage of my approach is that it makes it clear that the proof uses volume estimates that hold for all definable sets, not just complex analytic sets. Still, while I think the viewpoint and the brevity of this paper are worthwhile, the techniques are similar to the techniques of [PS09].

The proof of Theorem 4 given in this paper is also very similar to the proof given by Nguyen and Valette in [NV18], which I learned about after the first version of this paper appeared on the ArXiv. I decided to keep my original proof of Theorem 4 in this note because it is self-contained and the exposition is aimed at readers who are not experts in o-minimal structures. However, the reader should be aware that neither Theorem 4 nor its proof are new.

To help make this paper approachable for algebraic geometers, I review the theory of o-minimal structures in section §2. In §3, I review the notions of Hausdorff dimension and Stoll’s theorem, which is also sometimes called the Bishop–Stoll theorem. (Bishop [Bis64] generalized and extended the result of Stoll used in this paper.) In §4, I state the main volume estimate, Theorem 4.

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## 2. O-MINIMAL STRUCTURES

In defining o-minimal structures, I follow the book by van den Dries [vdD98].

**Definition 1.** An o-minimal structure on  $\mathbb{R}$  is a sequence  $\mathcal{S} = (\mathcal{S}_n)_{n \in \mathbb{N}}$  of sets such that, for each  $n$ :

- (i)  $\mathcal{S}_n$  is a boolean algebra of subsets of  $\mathbb{R}^n$ ;
- (ii)  $A \in \mathcal{S}_n$  implies that  $A \times \mathbb{R}$  and  $\mathbb{R} \times A$  are in  $\mathcal{S}_{n+1}$ ;
- (iii) If  $1 \leq i < j \leq n$ , then  $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i = x_j\} \in \mathcal{S}_n$ .
- (iv) If  $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  denotes the projection onto a factor, then  $A \in \mathcal{S}_{n+1} \implies \pi(A) \in \mathcal{S}_n$ ;
- (v) For each  $r \in \mathbb{R}$ ,  $\{r\} \in \mathcal{S}_1$ . Moreover,  $\{(x, y) \in \mathbb{R}^2 : x < y\} \in \mathcal{S}_2$ ;
- (vi) The only subsets in  $\mathcal{S}_1$  are the finite unions of intervals and points.

Call a sequence  $\mathcal{S}$  a *structure* if it satisfies all of the hypotheses of Definition 1 except possibly the last two [vdD98, p.13]. If  $X \in \mathcal{S}_n$  for some  $n$ , then we say that  $X$  is a *definable* subset of  $\mathbb{R}^n$  with respect to the structure  $\mathcal{S}$ . Similarly, if  $f : X \rightarrow Y$  is a function with  $X \subset \mathbb{R}^n$  and  $Y \subset \mathbb{R}^m$  for  $n, m \in \mathbb{N}$ , then we say  $f$  is *definable* if its graph (viewed as a subset of  $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$ ) is definable.

It is clear that, if we let  $\mathcal{S}_n = \mathcal{P}(\mathbb{R}^n)$ , i.e., the power set of  $\mathbb{R}^n$ , then we get a structure. (But obviously not an o-minimal one.) It is also relatively easy to see that the intersection of structures is a structure. So, given an arbitrary collection  $\mathcal{T}_n \subset \mathcal{P}(\mathbb{R}^n)$  (for  $n \in \mathbb{N}$ ), there is a smallest structure  $(\mathcal{S}_n)_{n \in \mathbb{N}}$  containing  $\mathcal{T}_n$ . This is the structure *generated* by the  $\mathcal{T}_n$ .

If  $\{\mathcal{S}_n\}$  and  $\{\mathcal{S}'_n\}$  are both structures with  $\mathcal{S}'_n \subset \mathcal{S}_n$  for all  $n$ , then  $\{\mathcal{S}_n\}$  is called an *expansion* of  $\{\mathcal{S}'_n\}$ . If  $\{\mathcal{T}_n\}$  is any collection with  $\mathcal{T}_n \subset \mathcal{P}(\mathbb{R}^n)$ , the *structure generated by*  $\{\mathcal{S}_n \cup \mathcal{T}_n\}_{n \in \mathbb{N}}$  is called the *expansion of*  $\{\mathcal{S}_n\}$  *generated by*  $\{\mathcal{T}_n\}$ .

One classical example of an o-minimal structure on  $\mathbb{R}$  is the structure  $\mathbb{R}_{\text{alg}}$  consisting of all semi-algebraic sets. (The fact that  $\mathbb{R}_{\text{alg}}$  satisfies (iv) is the content of the Tarski-Seidenberg theorem.) The example that is most important for the recent applications to Hodge theory mentioned in the introduction is the one called  $\mathbb{R}_{\text{an,exp}}$ . This is the expansion of  $\mathbb{R}_{\text{alg}}$  generated by the graph of the real exponential function  $x \mapsto e^x$  and the collection of all graphs of analytic functions on  $[0, 1]$ . (See [vdDM96] for references. The o-minimality of the expansion  $\mathbb{R}_{\text{exp}}$  of  $\mathbb{R}_{\text{alg}}$  generated by the graph of the real exponential function is a celebrated theorem of Wilkie [Wil96].)

It is convenient to think about definable sets in a structure in terms of logic as subsets of  $\mathbb{R}^n$  defined by the formulas in a language  $\mathcal{L}$  interpreted in the field of real numbers. This point of view is explained (a little informally) in [vdD98, Chapter 1]. (For a more precise explanation of the model theory point of view, see, for example, Marker's book [Mar02]). Here subsets  $\psi$  of  $\mathbb{R}^n$  are thought of as properties  $\psi(x_1, \dots, x_n)$  of  $n$ -tuples  $(x_1, \dots, x_n)$  of real numbers with  $\psi(x_1, \dots, x_n)$  being the property that  $(x_1, \dots, x_n) \in \psi$ . Suppose  $S = \{\psi_i\}$  is a collection of such subsets, with  $\psi_i \subset \mathbb{R}^{n_i}$ . Then the expansion of  $\mathbb{R}_{\text{alg}}$  generated by  $\psi$  consists of the subsets of  $\mathbb{R}^n$  definable by formulas involving the field operations on  $\mathbb{R}$ , the real numbers (viewed as constants), the symbols  $<$  and  $=$ , variables  $(x_i)_{i=1}^\infty$ , and the  $\psi_i$  along with the  $\forall, \exists$  and the usual logical connectives.

## 3. VOLUMES AND THE BISHOP–STOLL THEOREM

My main reference for this section is G. Stolzenberg's book [Sto66].

Let  $X = (X, d_X)$  be a metric space. For  $\emptyset \neq S \subset X$ , the diameter of  $S$  is  $\text{diam } S := \sup\{d_X(x, y) : x, y \in S\}$ . By convention, write  $\text{diam } \emptyset = -\infty$ .

Suppose  $S \subset X$ , and  $\epsilon$  is a positive real number. An  $\epsilon$ -covering of  $S$  is a countable collection  $\{S_i\}_{i=1}^\infty$  of subsets of  $S$  of diameter less than  $\epsilon$  such that  $S \subset \cup_{i=1}^\infty S_i$ . Fix a non-negative real number  $d$  and set

$$I(d, \epsilon, S) := \inf \left\{ \sum_{i=1}^\infty (\text{diam } S_i)^d : \{S_i\}_{i=1}^\infty \text{ is an } \epsilon\text{-covering of } S \right\}.$$

The  $d$ -Hausdorff measure of  $S$  is

$$\mathcal{H}_d(S) := \frac{1}{2^d} \lim_{\epsilon \rightarrow 0^+} I(d, \epsilon, S).$$

If  $d$  is a non-negative integer and  $S$  is a closed  $d$ -dimensional sub-manifold of  $\mathbb{R}^n$ , then the volume  $\text{vol}_d S$  (defined in the usual way with respect to the standard metric on  $\mathbb{R}^n$ ) is given by

$$(2) \quad \text{vol}_d(S) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)} \mathcal{H}_d(S).$$

As it turns out, we want to refer to volume instead of  $\mathcal{H}_d(S)$  in general. So we use the equation (2) as a definition to define  $\text{vol}_d(S)$  for an arbitrary non-negative real number  $d$ . (Note that Federer's normalization for Hausdorff measure in his book [Fed69, §2.10.2] differs from that of Stolzenberg. For Federer, the  $d$ -dimensional Hausdorff measure  $\mathcal{H}_d(S)$  is just what we call  $\text{vol}_d(S)$ .)

Let's also make the convention that we always regard a subset  $S \subset \mathbb{R}^n$  as a metric subspace of  $\mathbb{R}^n$  with its standard metric. For a positive real number  $r$ , set  $B(r) = B_n(r) := \{x \in \mathbb{R}^n : |x| < r\}$ . Then, if  $S \subset \mathbb{R}^n$ , set

$$S(r) := S \cap B(r).$$

We use the Big-O notation: if  $f, g$  are two real valued functions defined on an interval of the form  $(a, \infty)$ , then we write  $f = O(g)$  if there exists a constant  $C$  and a real number  $b > a$  such that

$$x > b \Rightarrow |f(x)| \leq Cg(x).$$

**Theorem 3** (Stoll). *Suppose that  $Z$  is a closed analytic subset of  $\mathbb{C}^n$  of pure dimension  $d$ . If  $\text{vol}_{2d} Z(r) = O(r^{2d})$ , then  $Z$  is algebraic.*

See [Sto66, p. 2, Theorem D] for the statement. A proof is given in [Sto66, Chapter IV]. According to Cornalba and Griffiths [CG75, E. 4.2], the converse also holds. (This also follows directly from the main result of this note, Theorem 4 below.)

#### 4. VOLUMES OF DEFINABLE SETS

My main goal in this letter is to prove the Peterzil–Starchenko GAGA theorem (Theorem 5 below) using Theorem 3 and a general fact about definable sets and Hausdorff measures. To explain this general fact, let me first explain cells. To do this, fix an o-minimal expansion  $\mathbb{R}_{\text{alg},*}$  of  $\mathbb{R}_{\text{alg}}$ .

**4.1. Cells.** These are certain special definable subsets (with respect to  $\mathbb{R}_{\text{alg},*}$ ) of  $\mathbb{R}^n$  defined inductively. See page 50 of [vdD98] for a complete definition, but, roughly speaking, cells in  $\mathbb{R}^n$  are defined inductively with respect to  $n$  as either

- (a) graphs of continuous definable functions  $f$  on cells in  $\mathbb{R}^{n-1}$  or,
- (b) nonempty open regions in between graphs of continuous definable functions in  $\mathbb{R}^{n-1}$ .

There is a dimension function  $d$  defined inductively on the set of all cells by setting  $d(S) = d(T)$  if  $S \subset \mathbb{R}^n$  is constructed inductively from  $T \subset \mathbb{R}^{n-1}$  via procedure (a) and setting  $d(S) = d(T) + 1$  if it is constructed via procedure (b). Moreover, as a consequence of the cell decomposition theorem [vdD98, 2.11 on p. 52], every definable subset of  $\mathbb{R}^n$  can be written as a finite disjoint union of cells. (This is one of the most crucial properties of definable sets in o-minimal structures.) If  $X$  is then any definable subset of  $\mathbb{R}^n$ , van den Dries defines  $\dim X$  to be the maximum of the dimensions  $d(S)$  as  $S$  ranges over all cells contained in  $X$  [vdD98, p. 63]. Since, by [vdD98, p. 64],  $\dim(X \cup Y) = \max(\dim X, \dim Y)$ ,  $\dim X$  is also the maximum of the dimensions  $d(S)$  of the cells  $S$  appearing in a decomposition of  $X$  into disjoint cells.

Now, I am ready to state the main theorem of this note.

**Theorem 4.** *Suppose  $S \subset \mathbb{R}^n$  is a set which is definable in an o-minimal expansion of  $\mathbb{R}_{\text{alg}}$ . Set  $d = \dim S$ . Then*

$$\text{vol}_d S(r) = O(r^d).$$

**4.2. Peterzil–Starchenko.** Before proving Theorem 4, I want to use it to prove the Peterzil–Starchenko GAGA theorem.

**Theorem 5** (Peterzil–Starchenko). *Let  $A$  be a closed complex analytic subset of  $\mathbb{C}^n$ , which is definable with respect to an o-minimal expansion of  $\mathbb{R}_{\text{alg}}$ . Then  $A$  is an algebraic subset of  $\mathbb{C}^n$ .*

*Proof of Theorem 5 using Theorem 4.* Take a closed complex analytic subset  $Z \subset \mathbb{C}^n$  of complex dimension  $d$ , and assume  $Z$  is definable. Then  $Z$  has a definable dense open subset  $U$  which is submanifold of  $\mathbb{C}^n$ . It follows that the dimension of  $Z$  as a definable subset of  $\mathbb{C}^n$  (in the sense of [vdD98, Definition 4.1.1]) is  $2d$ . Then, by Theorem 4, we have  $\text{vol}_{2d} Z(r) = O(r^{2d})$ . So, by Theorem 3,  $Z$  is algebraic.  $\square$

In the next section, I prove Theorem 4. The elementary proof mainly relies on the Gauss map and change of variables.

## 5. PROOF OF THE VOLUME ESTIMATE

In what follows it will be convenient to note that, since linear transformations between finite dimensional real vector spaces are definable in any expansion of  $\mathbb{R}_{\text{alg}}$ , any finite dimensional real vector space  $V$  comes equipped with a canonical definable structure (via any linear isomorphism to  $\mathbb{R}^{\dim V}$ ).

If  $X$  is a definable subset of  $\mathbb{R}^n$  of dimension  $d$ , we say that *Theorem 4 holds for  $X$*  if  $\text{vol}_d X(r) = O(r^d)$ . Note that, if Theorem 4 holds for  $X$ , then, for  $d' > d$ , we have  $\text{vol}_{d'} X(r) = 0$  for all  $r$ . (See [Fed69, §2.10.2].) For each nonnegative integer  $d$ , write  $P(d)$  for the assertion that Theorem 4 holds for all definable sets  $X$  of dimension  $\leq d$ . The goal of the section is to prove  $P(d)$  for each nonnegative integer  $d$  by induction on  $d$ . Since zero dimensional definable sets are finite,  $P(0)$  obviously holds.

Write  $\text{Gr}(d, n)$  for the Grassmannian of real  $d$ -dimensional planes through the origin in  $\mathbb{R}^n$ . The set  $\text{Gr}(d, n)$  has a natural structure of a definable  $C^\infty$ -manifold. In the language of [vdD98, Chapter 10],  $\text{Gr}(d, n)$  is a definable space, which is also (compatibly) a compact  $C^\infty$ -manifold. (See also [Fis08] for a precise definition.) As it is also a regular space, [vdD98, Theorem 10.1.8] implies that it is isomorphic (as a definable space) to an affine definable space, i.e., a definable subset of  $\mathbb{R}^n$ . However, the method used in [vdD98, Example 10.1.4] to show that  $\mathbb{P}^n(\mathbb{R})$  is affine, can be imitated to realize  $\text{Gr}(d, n)$  as a closed definable  $C^\infty$ -submanifold of  $\mathbb{R}^N$  for some suitable  $N$ .

To be explicit about this last point, endow  $\mathbb{R}^n$  with the usual dot product. For each  $L \in \text{Gr}(d, n)$ , choose an ordered basis  $\tilde{L} = (\ell_1, \dots, \ell_d)$ , and set  $\wedge^d \tilde{L} := \ell_1 \wedge \dots \wedge \ell_d$ . Then let  $\phi(L)$  denote the point

$$\phi(L) := \frac{1}{\|\wedge^d \tilde{L}\|^2} (\wedge^d \tilde{L}) \otimes (\wedge^d \tilde{L}) \in \text{Sym}^2(\wedge^d \mathbb{R}^n).$$

The resulting map  $\phi : \text{Gr}(d, n) \rightarrow \text{Sym}^2(\wedge^d \mathbb{R}^n)$  is a well-defined and definable smooth morphism, embedding  $\text{Gr}(d, n)$  as a closed, definable, smooth submanifold of  $\text{Sym}^2(\wedge^d \mathbb{R}^n)$ . So we can identify  $\text{Gr}(d, n)$  with the image of  $\phi$ .

We can also view  $\text{Gr}(d, n)$  as a quotient of the orthogonal group  $\mathbf{O}(n)$  in the usual way. Note that the orthogonal group  $\mathbf{O}(n)$  (of real  $n \times n$ -matrices which are orthogonal with respect to the standard inner product) is a closed definable and smooth submanifold of  $\mathbb{R}^{n^2}$ . Moreover, it is a group in the category of definable spaces. It acts definably, properly and transitively on the space  $\text{Gr}(d, n)$ . The stabilizer of  $L \in \text{Gr}(d, n)$  is the definable, closed subgroup  $\mathbf{O}(L) \times \mathbf{O}(L^\perp)$ , which is definably isomorphic to  $\mathbf{O}(d) \times \mathbf{O}(n-d)$ . From this, it is not hard to see that, in the language of [vdD98, p. 162],  $\text{Gr}(d, n)$  is a definably proper quotient of  $\mathbf{O}(n)$ , and, in fact,  $\text{Gr}(d, n)$  is definably isomorphic to the quotient  $\mathbf{O}(n)/(\mathbf{O}(d) \times \mathbf{O}(n-d))$ .

If  $L \in \text{Gr}(d, n)$  is a  $d$ -dimensional linear subspace, we write  $\pi_L : \mathbb{R}^n \rightarrow L$  for the orthogonal projection onto the subspace  $L$ . The map  $z \mapsto (\pi_L(z), \pi_{L^\perp}(z))$  is then an isometric (and, thus, volume-preserving) definable isomorphism from  $\mathbb{R}^n$  to  $L \times L^\perp$ .

**Lemma 6.** *Suppose  $L \in \text{Gr}(d, n)$ . There exists a definable open neighborhood  $U_L$  of  $L$  in  $\text{Gr}(d, n)$  such that the following two statements hold:*

- (i)  $\pi_L(L') = L$  for all  $L' \in U_L$ .
- (ii) Suppose  $C = \{(x, y) \in D \times L^\perp : y = f(x)\}$  where  $D \subset L$  is a definable  $d$ -dimensional cell and  $f : D \rightarrow L^\perp$  is a  $C^2$  definable function. Assume that the tangent space  $T_z C$  is in  $U_L$  for all  $z = (x, y) \in C$ . Then, for all  $r \in \mathbb{R}$ ,  $\text{vol}_d C(r) \leq 2 \text{vol}_d D(r)$ .

*Proof.* Since the group  $\mathbf{O}(n)$  acts transitively on  $\text{Gr}(d, n)$  and preserves the metric (and hence the volume form) on  $\mathbb{R}^n$ , we can assume  $L = \mathbb{R}^d = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i = 0 \text{ for } i = d+1, \dots, n\}$ . Write  $e_i$  for the tangent vector  $\partial/\partial x_i$ , and write  $\Phi : D \rightarrow \mathbb{R}^n$  for the map  $x \mapsto (x, f(x))$ . Let  $A(x) = Df(x)$  denote the derivative of  $f$  at  $x$ . For  $z = (x, f(x)) \in C$ , the tangent space  $T_z C$  is the  $d$ -dimensional space generated by the vectors  $v_i = e_i + A(x)e_i \in \mathbb{R}^n$ . For  $r \in \mathbb{R}$ ,  $\pi_L(C(r)) \subset D(r)$ . Therefore,  $\text{vol}_d C(r) \leq \int_{D(r)} \sqrt{\det g_{ij}(x)} dx$  where  $g_{ij}(x)$  is the matrix  $v_i \cdot v_j = \delta_{ij} + e_i \cdot A(x)^* A(x) e_j$ . (See [Fed69, §3.2.46] for the relevant formula computing the volume in terms of  $g_{ij}$ .) For  $T_z C$  sufficiently close to  $L$ , the matrix  $A(x)$  will be close to 0. Therefore,  $g_{ij}(x)$  will be close to the identity matrix. From these considerations, the lemma follows easily.  $\square$

For the rest of the section, pick definable neighborhoods  $U_L$  for each  $L \in \text{Gr}(d, n)$  once and for all. For each definable set  $X$ , we let  $\text{Reg}^2 X$  denote the locus in  $X$  consisting of all points  $x \in X$  such that there is an open subset  $U$  of  $\mathbb{R}^n$  such that  $U \cap X$  is a  $C^2$ -manifold. This is a definable subset of  $X$ , and the complement  $X \setminus \text{Reg}^2 X$  has dimension strictly less than the dimension of  $X$ . (This follows from the Cell Decomposition Theorem of [vdDM96, §4.2].)

**Corollary 7.** *Suppose  $L \in \text{Gr}(d, n)$  and  $M$  is a  $d$ -dimensional  $C^2$  definable submanifold of  $\mathbb{R}^n$  such that  $T_x M \in U_L$  for each  $x \in M$ . Then, assuming that  $P(k)$  holds for  $k < d$ , we have  $\text{vol}_d M(r) = O(r^d)$ .*

*Proof.* We can write  $M$  as a finite union of cells of dimension  $\leq d$ . Moreover, we can assume that  $L = \{x \in \mathbb{R}^n : x_i = 0 \text{ for } i > d\}$ . Then, using the assumption that  $P(k)$  holds for  $k < d$ , we can assume that  $M$  is, in fact, a single cell. Since  $D\pi_L(x) : T_x M \rightarrow L$  is onto for all  $x \in M$ , we see easily that  $M$  has the form of the subset  $C$  in Lemma 6. So  $M = \{(x, y) \in D \times L^\perp : y = f(x)\}$  with  $f$  and  $D$  as in Lemma 6. As  $D$  is a  $d$ -dimensional cell in  $\mathbb{R}^d$ , it is open. So it is obvious that  $\text{vol}_d D(r) = O(r^d)$ . Then the volume estimate for  $M$  follows from Lemma 6.  $\square$

*Proof of Theorem 4.* Suppose  $P(k)$  holds for  $k < d$  and that  $S$  is a  $d$ -dimensional definable set. We can write  $S$  as a finite union of  $C^2$ -cells, so, since we are assuming  $P(k)$  holds for  $k < d$ , we can assume that  $S$  is itself a  $C^2$  cell. In particular  $S$  is a  $C^2$  submanifold of  $\mathbb{R}^n$ .

Write  $\Gamma : S \rightarrow \text{Gr}(d, n)$  for the Gauss map sending  $x \in S$  to  $T_x S$ . Then  $\Gamma$  is definable and continuous. So cover  $\text{Gr}(d, n)$  with finitely many opens of the form  $U_{L_i}$  for  $i = 1, \dots, m$ . Then each set  $S_i := \Gamma^{-1}(U_{L_i})$  is definable open in  $S$ . In particular,  $S_i$  is a  $C^2$  submanifold of  $\mathbb{R}^n$  and  $T_x S_i \in U_{L_i}$  for all  $i$  and for each  $x \in S_i$ . The fact that  $\text{vol}_d S_i(r) = O(r^d)$  then follows from Corollary 7. Since  $m < \infty$ , this proves the theorem.  $\square$

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