Assignment #2, due Wed, Oct. 12

1. Consider stock prices S_0, S_1, S_2, S_3 described by a binomial tree with $S_0 = 8$, $u = \frac{3}{2}$, $d = \frac{1}{2}$. Let $\rho = 20\%$. We consider options with maturity $T = t_N$ with $N = 3$.

For (a)–(g): We have $\boxed{\beta = (1+\rho)^{-1} = \frac{5}{6}, \qquad q = \frac{1+\rho-d}{u-d} = 0.7}$

(a) Find the price of a "**lookback put option**" with payoff $V_3 = (7 - \min\{S_0, \ldots, S_3\})^+$. *Hint:* This payoff is **path dependent**, hence you have to use the full tree with $1, 2, 4, 8$ possibilities at times t_0, t_1, t_2, t_3.

$$S_0 = 8,\ V_0 = .9080 \begin{cases} S_1 = 12,\ V_1 = 0.3958 \begin{cases} S_2 = 18,\ V_2 = 0 \begin{cases} S_3 = 27, & V_3 = 0 \\ S_3 = 9, & V_3 = 0 \end{cases} \\ S_2 = 6,\ V_2 = 1.5833 \begin{cases} S_3 = 9, & V_3 = 1 \\ S_3 = 3, & V_3 = 4 \end{cases} \end{cases} \\ S_1 = 4,\ V_1 = 2.7083 \begin{cases} S_2 = 6,\ V_2 = 2.75 \begin{cases} S_3 = 9, & V_3 = 3 \\ S_3 = 3, & V_3 = 4 \end{cases} \\ S_2 = 2,\ V_2 = 4.4167 \begin{cases} S_3 = 3, & V_3 = 5 \\ S_3 = 1, & V_3 = 6 \end{cases} \end{cases} \end{cases}$$

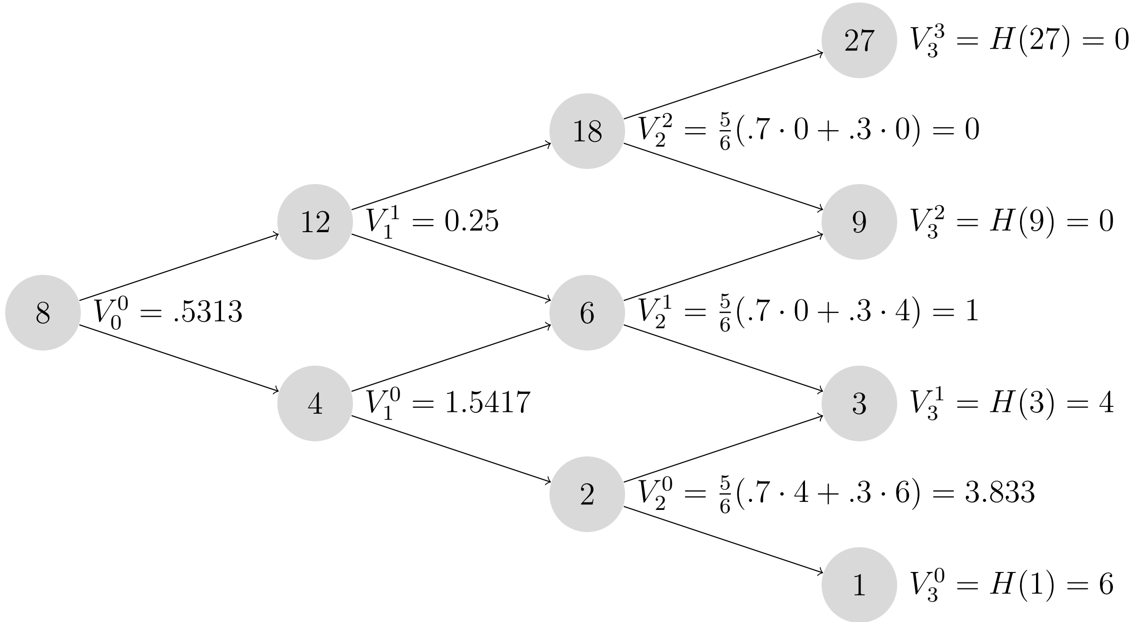
(see m-file `a2p1ab.m`)

(b) Show that you can **replicate the payoff** V_3 from (a) with an **investment strategy** (x_0, x_1, x_2): At time t_j (after we know the value of S_j) we choose a portfolio with x_j stocks and an amount y_j in the bank account, for $j = 0, 1, 2$. Label each node in the tree with the number x_j (see "Delta-hedging" in the notes). For the case $(S_0, S_1, S_2, S_3) = (8, 4, 2, 1)$ give the values x_j, y_j for $j = 0, 1, 2$ and the value of the portfolio U_j for $j = 0, 1, 2, 3$.
The investment strategy is given by $U_0 = .9080$ and x_0, x_1, x_2, x_3 given by

$$S_0 = 8,\ x_0 = -.2891 \begin{cases} S_1 = 12,\ x_1 = -.1319 \begin{cases} S_2 = 18,\ x_2 = 0 \begin{cases} S_3 = 27 \\ S_3 = 9 \end{cases} \\ S_2 = 6,\ x_2 = -.5 \begin{cases} S_3 = 9 \\ S_3 = 3 \end{cases} \end{cases} \\ S_1 = 4,\ x_1 = -.4167 \begin{cases} S_2 = 6,\ x_2 = -.1667 \begin{cases} S_3 = 9 \\ S_3 = 3 \end{cases} \\ S_2 = 2,\ x_2 = -.5 \begin{cases} S_3 = 3 \\ S_3 = 1 \end{cases} \end{cases} \end{cases}$$

For the path $(S_0, S_1, S_2, S_3) = (8, 4, 2, 1)$ our portfolio develops as follows:
time t_0: initial fortune is $U_0 = .9080$, buy $x_0 = -.2891$ stocks, put remaining amount $y_0 := U_0 - x_0 S_0 = 3.2205$ in bank account.
time t_1: fortune is now $U_1 := (1+\rho)y_0 + x_0 S_1 = 2.7083$. Change number of stocks to $x_1 = -.4167$, put remaining amount $y_1 := U_1 - x_1 S_1 = 4.3750$ in bank account.
time t_2: fortune is now $U_2 := (1+\rho)y_1 + x_1 S_2 = 4.4167$. Change number of stocks to $x_2 = -.5$, put remaining amount $y_2 := U_2 - x_2 S_2 = 5.4167$ in bank account.
time t_3: fortune is now $U_3 := (1+\rho)y_2 + x_2 S_3 = 6$.
(see m-file `a2p1ab.m`)

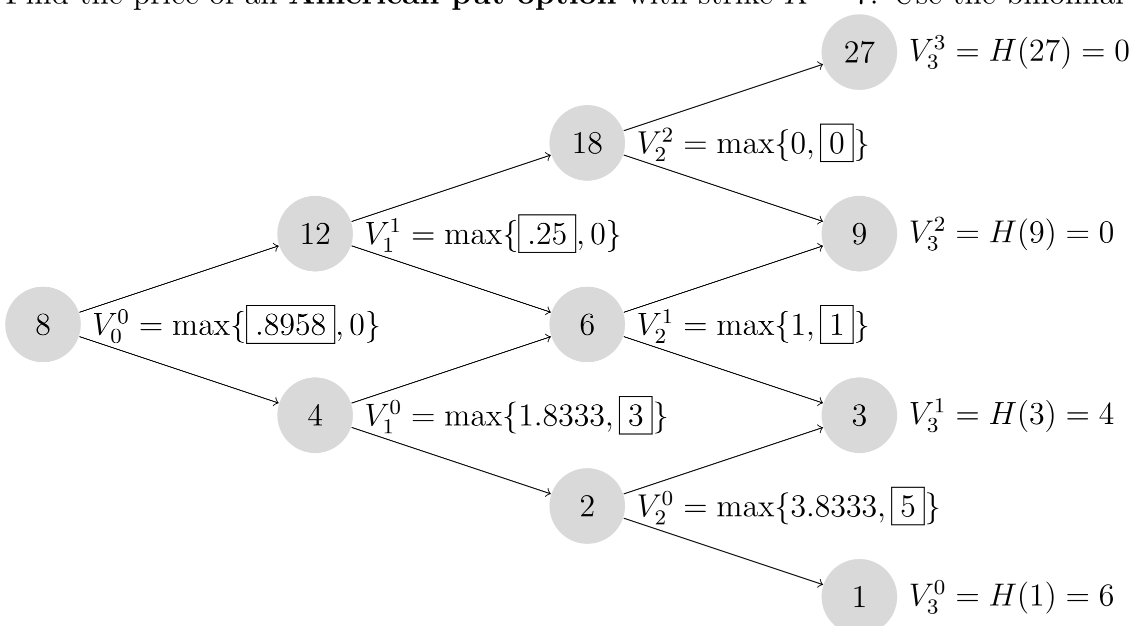
(c) Find the price of a **European put option** with strike $K = 7$. *Hint:* The payoff in (c), (d) depends only on S_3 (path independent). Use the binomial tree with $1, 2, 3, 4$ possibilities at times t_0, t_1, t_2, t_3 (as shown in the class notes).



Compute option price with Matlab:

```
H = @(s) max(7-s,0)      % payoff for put option with strike K=7
V0 = binom_eu(.2,8,1.5,.5,3,H)
```

(d) Find the price of an **American put option** with strike $K = 7$. Use the binomial tree..



Compute option price with Matlab:

```
H = @(s) max(7-s,0)      % payoff for put option with strike K=7
V0 = binom_am(.2,8,1.5,.5,3,H)
```

(e) Compare the option prices in (a),(c),(d): Which is smallest, which is largest? Explain the reason for this.
We have $V_0^{\text{lookback}} > V_0^{EP}$ and $V_0^{EP} < V_0^{AP}$.
The payoff in (a) is $V_3^{\text{lookback}} = (7 - \min\{S_0, S_1, S_2, S_3\})_+$, the payoff in (c) is $V_3^{EP} = (7 - S_3)_+$. We have $V_3^{\text{lookback}}(\omega_j) \geq V_3^{EP}(\omega_j)$ for all 8 outcomes $\omega_j \in \Omega$, and $V_3^{\text{lookback}}(\omega_j) > V_3^{EP}(\omega_j)$ for some of the outcomes. By the comparison principle we therefore must have $V_0^{\text{lookback}} > V_0^{EP}$.
If we have an American option we can choose the exercise strategy "always exercise at maturity t_N. Therefore we must have $V_0^{AP} \geq V_0^{EP}$. We saw in (d) that in some cases it is advantageous to exercise early. Therefore we have $V_0^{AP} > V_0^{EP}$.

(f) Extend the tree from (d) by using one additional V_3^4 and one additional value V_3^{-1} below. Explain how V_0^1 and V_0^{-1} give option prices for t_0 with $S_0 = 8$, but with different strikes.
We obtain

$$V_3^{-1} = 6.67,\ V_2^{-1} = \max\{5.167, \boxed{6.333}\},\ V_1^{-1} = \max\{4.5, \boxed{5.667}\},\ V_0^{-1} = \max\{3.167, \boxed{4.333}\}$$
$$V_3^1 = 0,\ V_2^1 = \max\{0, \boxed{0}\},\ V_1^1 = \max\{0, \boxed{0}\},\ V_0^1 = \max\{\boxed{.0625}, 0\}$$

Here $a = u/d = 3$. As explained in section 4.20 in the course notes:
For $S_0 = 8$ and strike $K = 3 \cdot 7 = 21$ the American put option price is $V_0^{AP} = 3 \cdot V_0^{-1} = 13$
For $S_0 = 8$ and strike $K = \frac{1}{3} \cdot 7 = \frac{7}{3}$ the American put option price is $V_0^{AP} = \frac{1}{3} \cdot V_0^1 = .020833$

(g) Find the price of a **European call option** with strike $K = 7$ for all nodes in the tree from (c) by using the **put-call parity**.
Put-call-parity gives $V_j^{EC} = S_j - \tilde{K} + V_j^{EP}$ where $\tilde{K} := (1+\rho)^{j-N} K$ is the discounted strike price.
For t_3 we have $\tilde{K} = K = 7$, hence $V_3^{k,EC} = S_3^k - 7 + V_3^{k,EP}$, $k = 0, 1, 2, 3$.
For t_2 we have $\tilde{K} = K = \beta \cdot 7 = \frac{5}{6} \cdot 7$, hence $V_2^{k,EC} = S_2^k - \frac{5}{6} \cdot 7 + V_2^{k,EP}$, $k = 0, 1, 2$
For t_1 we have $\tilde{K} = K = \beta^2 \cdot 7 = \left(\frac{5}{6}\right)^2 \cdot 7$, hence $V_1^{k,EC} = S_1^k - \left(\frac{5}{6}\right)^2 \cdot 7 + V_1^{k,EP}$, $k = 0, 1$
For t_0 we have $\tilde{K} = K = \beta^3 \cdot 7 = \left(\frac{5}{6}\right)^3 \cdot 7$, hence $V_0^{0,EC} = S_0^0 - \left(\frac{5}{6}\right)^3 \cdot 7 + V_0^{0,EP}$