A Convex Separation Theorems

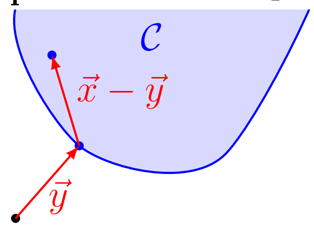
Notation: For vectors $\vec{x}, \vec{y} \in \mathbb{R}^M$ let $\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots + x_M y_M$ denote the *inner product*; let $\|\vec{x}\| := (\vec{x} \cdot \vec{x})^{1/2}$ denote the *Euclidian norm*.

We first state the basic convex separation theorem:

Theorem 6. *Assume that the set $\mathcal{C} \subset \mathbb{R}^M$ is closed, convex, and does not contain the origin $\vec{0}$. Then there exists $\vec{y} \in \mathbb{R}^M$ and $\alpha > 0$ such that*

$$\vec{y} \cdot \vec{x} \geq \alpha \qquad \text{for all } \vec{x} \in \mathcal{C}$$

Proof. **Idea: Step 1:** Choose $\vec{y}$ as the point in $\mathcal{C}$ which is closest to the origin.
Step 2: Show: For $\vec{x} \in \mathcal{C}$ we have $\vec{y} \cdot (\vec{x} - \vec{y}) \geq 0$ and hence $\vec{y} \cdot \vec{x} \geq \|\vec{y}\|^2 > 0$.



Step 1: Consider the ball $B_r := \{\vec{x} \in \mathbb{R}^M \mid \|\vec{x}\| \leq r\}$ and pick r such that $X := \mathcal{C} \cap B_r$ is nonempty. This set is closed and bounded, hence compact. Therefore the function $\vec{x} \mapsto \|\vec{x}\|$ attains its minimum at a point $\vec{y} \in X$. For $\vec{x} \notin B_r$ we have $\|\vec{x}\| \geq r \geq \|\vec{y}\|$. For $\vec{x} \in \mathcal{C} \cap B_r$ we also have $\|\vec{x}\| \geq \|\vec{y}\|$. Hence

$$\forall \vec{x} \in \mathcal{C} \qquad \|\vec{x}\| \geq \|\vec{y}\| \,. \tag{66}$$

Step 2: For $\vec{x} \in \mathcal{C}$ all the points on the line segment connecting $\vec{x}$ and $\vec{y}$ are in $\mathcal{C}$ since it is convex set:

$$\forall t \in [0,1] \qquad \vec{y} + t(\vec{x} - \vec{y}) \in \mathcal{C} \tag{67}$$

From (66), (67) we get by multiplying out

$$\begin{aligned} \|\vec{y} + t(\vec{x} - \vec{y})\|^2 &\geq \|\vec{y}\|^2 \\ \|\vec{y}\|^2 + 2t\vec{y} \cdot (\vec{x} - \vec{y}) + t^2 \|\vec{x} - \vec{y}\|^2 &\geq \|\vec{y}\|^2 \\ \forall t \in (0,1) \qquad 2\vec{y} \cdot (\vec{x} - \vec{y}) + t \|\vec{x} - \vec{y}\|^2 &\geq 0 \end{aligned}$$

Hence $\vec{y} \cdot (\vec{x} - \vec{y}) \geq 0$, i.e.,

$$\forall \vec{x} \in \mathcal{C} : \qquad \vec{y} \cdot \vec{x} \geq \|\vec{y}\|^2 > 0$$

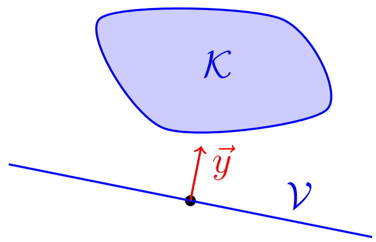
□

We will use the following corollary:

Theorem 7. *Assume that $\mathcal{V}$ is a subspace of $\mathbb{R}^M$, and that the set $\mathcal{K} \subset \mathbb{R}^M$ is convex, closed, and bounded. If $\mathcal{V} \cap \mathcal{K} = \emptyset$ there exists $\vec{y} \in \mathbb{R}^M$ and $\alpha > 0$ such that*

$$\vec{y} \cdot \vec{z} = 0 \qquad \text{for all } \vec{z} \in \mathcal{V} \tag{68}$$

$$\vec{y} \cdot \vec{z} \geq \alpha \qquad \text{for all } \vec{z} \in \mathcal{K} \tag{69}$$



Proof. Let $\mathcal{C} = \mathcal{K} - \mathcal{V} = \{x - y \mid x \in \mathcal{K},\, y \in \mathcal{V}\}$. This set is convex. Since $\mathcal{V}$ is closed and $\mathcal{K}$ is closed and bounded the set $\mathcal{C}$ is closed.

[Note: We need that $\mathcal{K}$ is bounded. The sum of two closed sets is not always closed! Consider e.g. $\mathcal{K} = \{(x, y) \in \mathbb{R}^2 \mid y \geq e^x\}$ and $\mathcal{V} = \{(x, 0) \mid x \in \mathbb{R}\}$.]

Since $\mathcal{V} \cap \mathcal{K} = \emptyset$ we have $\vec{0} \notin \mathcal{C}$. By Theorem 6 there exists $\vec{y} \in \mathbb{R}^M$ and $\alpha > 0$ such that $\vec{y} \cdot \vec{z} \geq \alpha$ for all $\vec{z} \in \mathcal{C}$. Hence

$$\forall \vec{x} \in \mathcal{K}, \quad \forall \vec{z} \in \mathcal{V}: \qquad \vec{y} \cdot (\vec{x} - \vec{z}) \geq \alpha$$

By taking $\vec{z} = \vec{0}$ we obtain (69). Now let $\vec{x} \in K$ and use $\lambda \vec{z}$ instead of $\vec{z}$ with $\lambda \in \mathbb{R}$. This yields

$$\forall \lambda \in \mathbb{R}: \qquad \lambda(\vec{y} \cdot \vec{z}) \leq \vec{y} \cdot \vec{x} - \alpha$$

Therefore we must have $\vec{y} \cdot \vec{z} = 0$ and we obtain (68). $\square$