AMSC 424, Fall 2016

Practice problems for Exam #1

(No calculators allowed for exam)

1.

(a) Let ρ denote the monthly interest rate and $\beta := \dfrac{1}{1+\rho}$. Give a formula for β in terms of **(i)** r_{eff}, **(ii)** r_c. Here r_{eff} denotes the yearly effective interest rate, and r_c denotes the yearly interest rate for continuous compounding.
Here we have $n = 12$ periods per year, hence $1 + r_{\text{eff}} = (1+\rho)^n = e^{r_c}$ and

$$\beta = (1 + r_{\text{eff}})^{-1/n} = e^{-r_c/n}$$

(b) You get a loan of 1000$ now. You make a payment P at the end of month 8, month 9, month 10. At the end of month 12 you make a final payment of 200$. Assume you know β and find the payment P in terms of β.
We get the equation

$$1000 = \left(\beta^8 + \beta^9 + \beta^{10}\right) P + \beta^{12} 200$$

yielding

$$P = \frac{1000 - \beta^{12} 200}{\beta^8 + \beta^9 + \beta^{10}}$$

2. We use a biased coin which gives "heads" with probability $\frac{2}{3}$ and "tails" with probability $\frac{1}{3}$. We toss the coin twice. You win the amount X where X is the number of "tails".

(a) Find $E[X]$ and $\mathrm{Var}[X]$.
We obtain (binomial distribution)

$$P(X=0) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}, \quad P(X=1) = 2 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{9}, \quad P(X=2) = \left(\frac{1}{3}\right)^2 = \frac{1}{9} \tag{1}$$

Hence

$$E[X] = 0 \cdot \frac{4}{9} + 1 \cdot \frac{4}{9} + 2 \cdot \frac{1}{9} = \frac{2}{3}, \quad E\left[X^2\right] = 0^2 \cdot \frac{4}{9} + 1^2 \cdot \frac{4}{9} + 2^2 \cdot \frac{1}{9} = \frac{8}{9}$$
$$\mathrm{Var}[X] = E\left[X^2\right] - E[X]^2 = \frac{8}{9} - \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

(b) Let A denote the event "at least one coin shows heads". Find the conditional expectation $E[X \mid A]$. Note that

$$\text{number of heads} \geq 1 \iff \text{number of tails} = 0 \text{or } 1$$

Hence we get using (1)

$$E[X \mid A] = \frac{E[X \cdot 1_A]}{P(A)} = \frac{0 \cdot \frac{4}{9} + 1 \cdot \frac{4}{9} + 0 \cdot \frac{1}{9}}{\frac{4}{9} + \frac{4}{9}} = \frac{1}{2}$$

3. At time t_0 the price of a European call option with strike 10 is V_0, and the price of a European call option with strike 15 is $\tilde{V}_0$. Both options have the same maturity $t_N = N\Delta t$, the interest rate is ρ per period of Δt. Prove an inequality $V_0 \leq \tilde{V}_0 + \cdots$ using the comparison principle.
Hint: *Investment 1:* At time t_0 buy a call option with strike 10. *Investment 2:* at time t_0 buy a call option with strike 15 and put a certain amount z in the bank account.
Investment strategy 1: We have $U_0 = V_0$ and $U_N = (S_N - 10)_+$
Investment strategey 2: At time t_0 we buy a call option with strike 15 and put $(1+\rho)^{-N}\,5$ in the bank account. Hence we have$\tilde{U}_0 = \tilde{V}_0 + (1+\rho)^{-N}\,5$. At time t_N we have $U_N = (S_N - 15)_+ + 5$.
Note that for all $s \in \mathbb{R}$

$$(s-10) \leq (s-15)_+ + 5$$

(draw the graphs). Hence we have $U_N \leq \tilde{U}_N$. By the comparison principle we must have

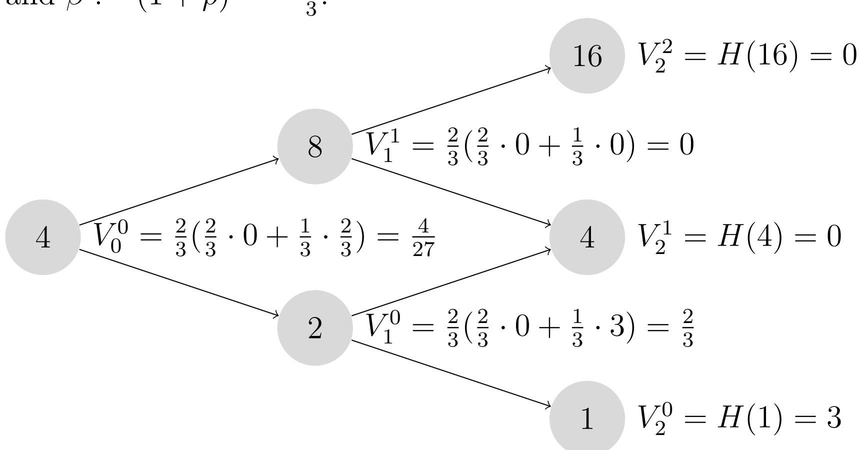
$$V_0 \leq \tilde{V}_0 + (1+\rho)^{-N}\,5$$

4. The interest rate is $\rho = 50\%$ per period Δt. A stock has at time t_0 the price $S_0 = 4$ and follows a binomial tree model with $u = 2$ and $d = \frac{1}{2}$. We consider options with maturity at t_2 and strike $K = 4$.

(a) Find the initial price V_0^{EP} of a European put option. Give the answer as a fraction.
We have the payoff function $H(S) = (4-S)_+$. The risk-neutral measure Q is given by

$$q = \frac{1+\rho-d}{u-d} = \frac{1+\frac{1}{2}-\frac{1}{2}}{2-\frac{1}{2}} = \frac{2}{3}$$

and $\beta := (1+\rho)^{-1} = \frac{2}{3}$.



We obtain the initial option price $V_0^{EP} = \frac{4}{27}$.

(b) Find the price V_0^{AP} of an American put option. Give the answer as a fraction. Describe the optimal exercise strategy.

$$16 \quad V_2^2 = H(16) = 0$$

$$8 \quad V_1^1 = \max\{\tfrac{2}{3}(\tfrac{2}{3}\cdot 0 + \tfrac{1}{3}\cdot 0,\ \underbrace{H(8))}_{0}\} = 0$$

$$4 \quad V_0^0 = \max\{\tfrac{2}{3}(\tfrac{2}{3}\cdot 0 + \tfrac{1}{3}\cdot 2),\ \underbrace{H(4)}_{0}\} = \tfrac{4}{9}$$

$$4 \quad V_2^1 = H(4) = 0$$

$$2 \quad V_1^0 = \max\{\tfrac{2}{3}(\tfrac{2}{3}\cdot 0 + \tfrac{1}{3}\cdot 3),\ \underbrace{H(2)}_{2}\} = 2$$

$$1 \quad V_2^0 = H(1) = 3$$

We obtain the initial option price $V_0^{AP} = \frac{4}{9}$. An optimal strategy is to always exercise at time t_1. (If $S_1 = 8$ we could also exercise at time t_2).