EXAM 1 AMSC 424, Fall 2016

NO CALCULATORS allowed.
You need to SHOW ALL YOUR WORK in order to get credit.

1. (30 pts) You get a loan of 2000\$ now. You make a payment P at the end of month 5, month 6, ..., month 20.
   We have
$$\beta^{-12} = 1 + r_{\text{eff}}, \qquad 2000 = P(\beta^5 + \cdots + \beta^{20}) = P\beta^5 \frac{1-\beta^{16}}{1-\beta}$$

   (a) (15 pts) Assume the yearly effective interest rate is $r_{\text{eff}} = 5\%$. Write a Matlab program which prints out the payment P.

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b = 1.05^(-1/12); P = 2000*(1-b)/b^5/(1-b^16)
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   (b) (15 pts) Assume the payment is $P = 130\$$. Write a Matlab program which prints out the yearly effective interest rate r_{eff}.

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g = @(b) 130*b^5*(1-b^16)/(1-b)-2000;
b = fzero(g,[0,.9999]);
reff = b^-12-1
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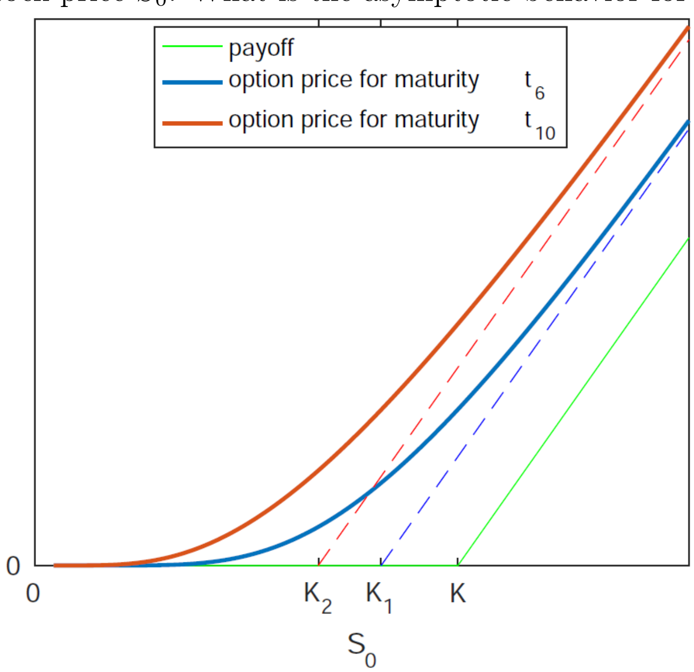
2. (30 pts) The interest rate per period Δt is ρ. At time t_0 we have the following prices for **European call options with strike** K:

   - for maturity $T = t_6$ the price of the option is V_0
   - for maturity $\hat{T} = t_{10}$ the prince of the option is $\hat{V}_0$

   (a) (10 pts) Make a sketch showing the two graphs for the option prices V_0, $\hat{V}_0$ as a function of the stock price S_0. What is the asymptotic behavior for large values of S_0?



Here $K_1 := (1+\rho)^{-6}K$, $K_2 := (1+\rho)^{-10}K$.
For large values of S_0 we have $V_0 \approx S_0 - K_1$ (blue dashed line), $\hat{V}_0 \approx S_0 - K_2$ (red dashed line).

   (b) (20 pts) Use the comparison principle to prove the inequality $V_0 \le \hat{V}_0$. *Hint:* Consider the two investment strategies
   strategy 1 with values $\hat{U}_j$: at time t_0 buy 1 option with maturity $\hat{T} = t_{10}$.
   strategy 2 with values U_j: at time t_0 buy 1 option with maturity $T = t_6$. **At time** t_6 you

get the payoff from the option. If the payoff is > 0 buy 1 stock and put the remaining money in the bank account. If the payoff at time t_6 is 0 your final amount at time t_{10} is zero.
Find the values $\hat{U}_0$, $\hat{U}_{10}$, U_0, U_{10}. Then apply the comparison principle.
strategy 1: $\hat{U}_0 = \hat{V}_0$, $\boxed{\hat{U}_{10} = (S_{10} - K)_+}$
strategy 2: $U_0 = V_0$, $U_6 = (S_6 - K)_+$
case 1: $S_6 > K$. Then $U_6 = S_6 - K$, we buy 1 stock for S_6 and put $-K$ into the bank account. At time t_{10} we have $\boxed{U_{10} = S_{10} - K(1+\rho)^4}$.
case 2: $S_6 \leq K$. Then $U_6 = 0$ and $\boxed{U_{10} = 0}$.
We have

$$\hat{U}_{10} = (S_{10} - K)_+ \geq S_{10} - K \geq S_{10} - K(1+\rho)^4$$
$$\hat{U}_{10} = (S_{10} - K)_+ \geq 0$$

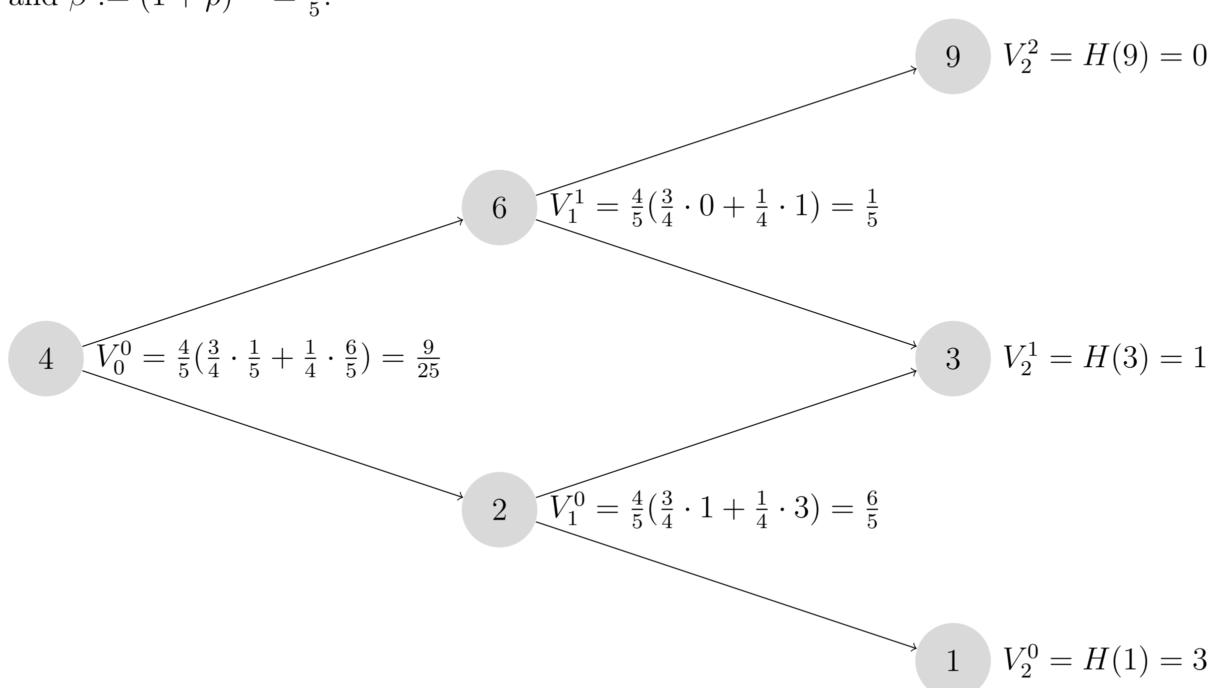
Hence in both case 1 and case 2 we have $\hat{U}_{10} \geq U_{10}$. Now the comparison principle implies $\hat{U}_0 \geq U_0$, i.e., $\hat{V}_0 \geq V_0$.

3. (40 pts) The interest rate is $\rho = 25\%$ per period Δt. A stock has at time t_0 the price $S_0 = 4$ and follows a binomial tree model with $u = \frac{3}{2}$ and $d = \frac{1}{2}$. We consider options with maturity at t_2 and strike $K = 4$.

(a) (20 pts) Find the initial price V_0^{EP} of a **European put option**. Give the answer as a fraction. We have the payoff function $H(S) = (4 - S)_+$. The risk-neutral measure Q is given by

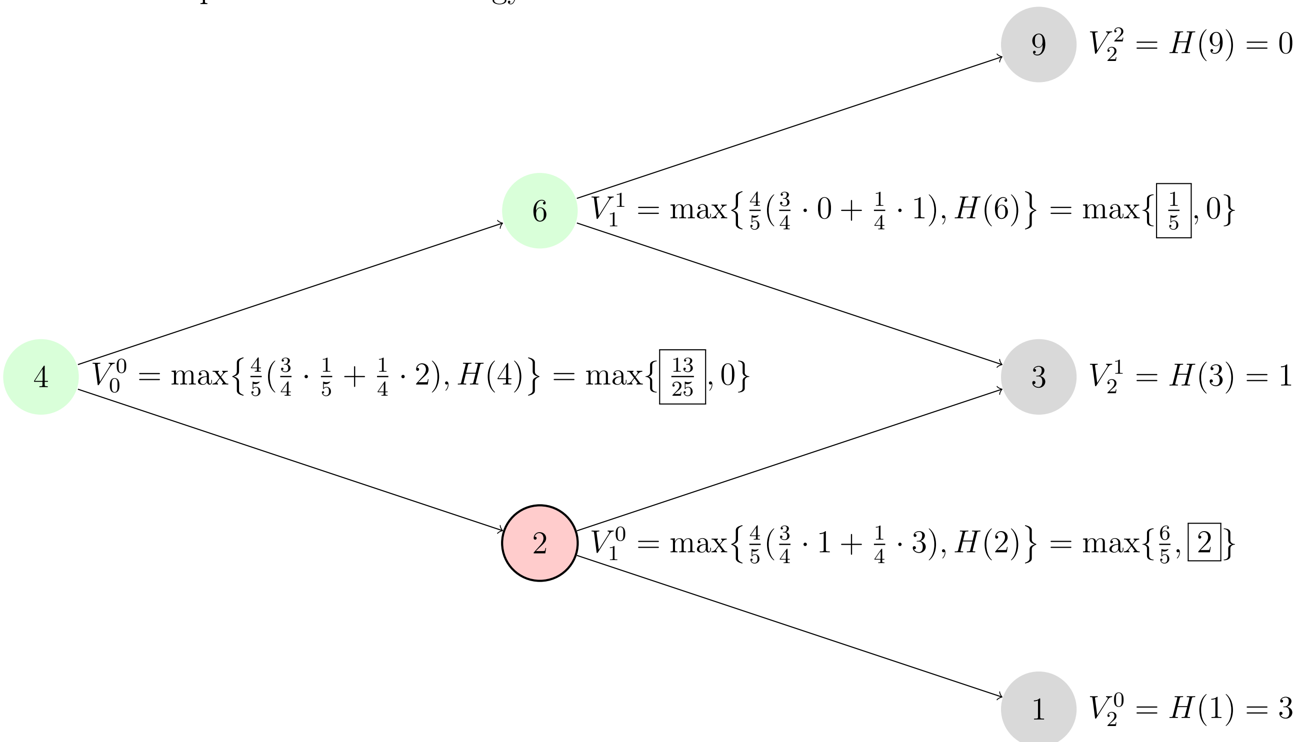
$$q = \frac{1+\rho-d}{u-d} = \frac{1+\frac{1}{4}-\frac{1}{2}}{\frac{3}{2}-\frac{1}{2}} = \frac{3}{4}$$

and $\beta := (1+\rho)^{-1} = \frac{4}{5}$.



We obtain the initial option price $\boxed{V_0^{EP} = \frac{9}{25}}$.

(b) (20 pts) Find the initial price V_0^{AP} of an **American put option**. Give the answer as a fraction. Describe the optimal exercise strategy.



We obtain the initial option price $\boxed{V_0^{AP} = \frac{13}{25}}$. The optimal strategy is: if $S_1 = 2$ exercise at time t_1, otherwise exercise at time t_2.