Summary of topics for exam 2 and final exam

Exam 2 covered the following topics:

- Central Limit Theorem
- Using geometric Brownian motion for stock prices
- Ito calculus

The **final exam** will cover the following topics:

- topics of exam 1
- topics of exam 2
- option price $V(t) = v(\tau, s)$, Ito formula applied to $V(t)$, hedging

Central Limit Theorem

We consider random variables $X_1, X_2, \ldots, X_N$ where

- $X_1, \ldots, X_N$ are independent
- $X_1, \ldots, X_N$ have the same distribution
- $\mu_0 := E[X_j]$ and $\sigma_0^2 := \mathrm{Var}[X_j]$ exist.

Then the sum $Y := X_1 + \cdots + X_N$ has the expectation $\mu := E[Y] = N\mu_0$ and the variance $\sigma^2 := \mathrm{Var}[Y] = N\sigma_0^2$.

The **central limit theorem** states that Y has approximately normal distribution $N(\mu, \sigma^2)$. Equivalently, the normalized random variable

$$Z := \frac{Y-\mu}{\sigma}, \qquad Y = \mu + \sigma Z$$

has approximately standard normal distribution $N(0,1)$. Recall that the density function (pdf) is $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$ and the distribution function (cdf) is $\Phi(x)$.

This means that

$$P(a \le Y \le b) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right) \approx \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

and the difference between the left hand side and the right hand side goes to zero as $N \to \infty$.

We also have for the random variable $g(Y)$ the expectation

$$E[g(Y)] = E[g(\mu + \sigma Z)] \approx \int_{z=-\infty}^{\infty} g(\mu + \sigma z)\phi(z)dz$$

Problems:

1. We have independent random variables $X_1, \ldots, X_N$ with $X_j = \begin{cases} .03 & \text{with prob. } \frac{1}{2} \\ -.01 & \text{with prob. } \frac{1}{2} \end{cases}$.

Let $Y := X_1 + \cdots + X_{100}$.

(a) Find an approximation for $P(Y \le 1.3)$ using $\Phi(\cdots)$.

(b) Assume we have stock prices $S_0, \ldots, S_{100}$ with $S_0 = 15$ and $\log(S_j/S_{j-1}) = X_j$. Find an approximation for $E[S_{100}]$ as an integral with $\phi(z)$. *Hint:* $\log(S_{100}/S_0) = X_1 + \cdots + X_{100}$.

(c) The antiderivative of $e^{az}\phi(z)$ is $e^{a^2/2}\Phi(z-a)$. Use this to evaluate the integral from (b).

Using geometric Brownian motion for stock prices

Recall that standard Brownian motion gives for a given time T the random variable $B(T)$ which has distribution $N(0,T)$. Therefore $Z = \dfrac{B(T)}{T^{1/2}}$ has $N(0,1)$ distribution, and $B(T) = T^{1/2}Z$.

In the Black-Scholes model stock prices are modeled by geometric Brownian motion: **Under the real-world measure** P we have

$$\boxed{S(t) = S_0 e^{\mu t + \sigma B(t)}}$$

where μ is the drift and σ is the volatility.

An important result is: **Under the risk-neutral measure** Q we have

$$\boxed{S(t) = S_0 e^{\mu_Q t + \sigma B(t)}}$$

where the volatility σ is the same as under P, but the drift is now

$$\boxed{\mu_Q = r_c - \tfrac{1}{2}\sigma^2}$$

Note that this does not depend on the original drift μ.

It is important to understand whether to use P or Q:

- For probabilities and expectations corresponding to actual frequencies use P
- For option prices use Q:

  Consider a **European option with maturity** T **and payoff function** $H(S)$. Then the price of the option at time 0 is given by

  $$\boxed{V_0 = e^{-r_c T} E^Q\left[H(S(T))\right]} \tag{1}$$

Problems:

1. The interest rate with continuous compounding is $r = 10\%$. Under the real-world measure P the stock price is given by

$$S(t) = S_0 e^{\mu t + \sigma B(t)}$$

with $S_0 = 15$, drift $\mu = .4$ and volatility $\sigma = .2$.

(a) Find $P(S(4) \le 15e)$ using $\Phi(\cdots)$.

(b) We consider a European option with maturity $T = 4$ and payoff function $H(S) = \begin{cases} 1 & \text{if } S > 15e \\ 0 & \text{if } S \le 15e \end{cases}$ (a so-called binary option). Write the option price V_0 using an integral with $\phi(z)$.

(c) Evaluate the integral from (b) using $\Phi(\cdots)$.

Ito calculus

Let $F(t,x)$ be a function of two variables t, x. We can then plug in $B(t)$ for x and obtain the stochastic process

$$Y(t) := F(t, B(t))$$

For example, geometric Brownian motion $S_0 e^{\mu t + \sigma B(t)}$ is obtained with $F(t,x) := S_0 e^{\mu t + \sigma x}$.

Unfortunately, the function $B(t)$ is not differentiable. Hence also $Y(t)$ is not differentiable, and the usual fundamental theorem of calculus $Y(T) - Y(0) = \int_{t=0}^{T} Y'(t)dt$ does NOT make sense.

The problem is that the increment $\Delta B_j := B(t_{j+1}) - B(t_j)$ has distribution $N(0, \Delta t_j)$ where $\Delta t_j := t_{j+1} - t_j$. Therefore the standard deviation is $(\Delta t_j)^{1/2}$, and $\Delta B_j / \Delta t_j$ is of order $(\Delta t_j)^{-1/2}$ which blows up as $\Delta t_j \to 0$.

In this situation one has to use **"Ito calculus"** instead. This is based on **two key facts**:

1. Let $X(t)$ denote a stochastic process which depends only on past values of $B(t)$ and not on future values. We can interprete $X(t)$ as a betting strategy where we bet on increments of $B(t)$. Our fortune $U(T)$ at time T is given by the limit of

$$\sum_{j=0}^{N-1} X(t_j)\Delta B_j \tag{2}$$

as the partition on the interval $[0, T]$ gets finer and finer. One can show: **the limit of the sum exists and gives the so-called "Ito integral"**

$$\boxed{\sum_{j=0}^{N-1} X(t_j)\Delta B_j \to \underbrace{\int_{t=0}^{T} X(t)dB}_{\text{Ito integral}}}$$

and the **process** $U(T) = \int_{t=0}^{T} X(t)dB$ **is a martingale**. This makes sense since we are betting on the increments of the martingale $B(t)$, and we know that the discrete version (2) is a martingale.

2. Let $f(t)$ be continuous. Then the sum of $f(t_j)(\Delta B_j)^2$ converges to a limit:

$$\boxed{\sum_{j=0}^{N-1} f(t_j)(\Delta B_j)^2 \to \int_{t=0}^{T} f(t)dt}$$

Note that we can take the limit of $\sum_{j=0}^{N-1} f(t_j)\Delta t_j$ and obtain the same limit. Therefore we can use the following recipe:

$$\boxed{\text{Replace } (\Delta B_j)^2 \text{by } \Delta t_j. \text{ Then take the limit } \Delta t_j \to 0.}$$

Now we consider the function

$$Y(t) := F(t, B(t))$$

and want to obtain a result of the form

$$Y(T) - Y(0) = \int_{t=0}^{T} (???)$$

We use a partition $0 = t_0 < t_1 < \cdots < t_N = T$ of the interval $[0, T]$ and have with the increments $\Delta Y_j := Y(t_{j+1}) - Y(t_j)$

$$Y(T) - Y(0) = \Delta Y_0 + \cdots + \Delta Y_{N-1}$$

For the increment $\Delta Y_j := Y(t_{j+1}) - Y(t_j) = F(t_{j+1}, B_{j+1}) - F(t_j, B_j)$ we can use the Taylor series:

$$\Delta Y_j = \frac{\partial F}{\partial t}(t_j, B_j) \cdot \Delta t_j + \frac{\partial F}{\partial x}(t_j, B_j) \cdot \Delta B_j + \frac{1}{2} \cdot \frac{\partial^2 F}{\partial x^2}(t_j, B_j) \cdot (\Delta B_j)^2 + \text{higher order terms}$$

Now we take the sum $\Delta Y_0 + \cdots + \Delta Y_{N-1}$ and obtain in the limit

$$Y(T) - Y(0) = \underbrace{\int_{t=0}^{T} \frac{\partial F}{\partial x}(t, B(t))\, dB}_{\text{Ito integral (martingale)}} + \underbrace{\int_{t=0}^{T} \left[\frac{\partial F}{\partial t}(t, B(t)) + \frac{1}{2} \cdot \frac{\partial^2 F}{\partial x^2}(t, B(t)) \right] dt}_{\text{normal integral}}$$

This is the **Ito Lemma**. We obtain that **the process $Y(t)$ is a martingale if and only if**

$$\frac{\partial F}{\partial t}(t, B(t)) + \frac{1}{2} \cdot \frac{\partial^2 F}{\partial x^2}(t, B(t)) = 0 \quad \text{for all}\, t$$

We need that the partial derivatives $\frac{\partial F}{\partial t}(t,x)$, $\frac{\partial F}{\partial x}(t,x)$, $\frac{\partial^2 F}{\partial x^2}(t,x)$ exist and are continuous (I omit some additional technical assumptions here).

RECIPE: How to use the Ito Lemma for $Y(t) = F(t, B(t))$

1. Find the partial derivatives $\frac{\partial F}{\partial t}(t,x)$, $\frac{\partial F}{\partial x}(t,x)$, $\frac{\partial^2 F}{\partial x^2}(t,x)$.
2. Use the Taylor expansion for the increment ΔY with terms of order Δt, ΔB, ΔB^2:

$$\Delta Y = \frac{\partial F}{\partial t}(t,B) \cdot \Delta t + \frac{\partial F}{\partial x}(t,B) \cdot \Delta B + \frac{1}{2} \cdot \frac{\partial^2 F}{\partial x^2}(t,B) \cdot \Delta B^2 + \text{h.o.t.}$$

3. Replace ΔB^2 by Δt. The term $(\cdots)\,\Delta B$ gives an Ito integral $\int(\cdots)dB$, the terms $(\cdots)\,\Delta t$ give a normal integral $\int(\cdots)dt$.
4. The process $Y(t)$ is a martingale if and only if in the integral $\int(\cdots)dt$ the integrand is zero.

Problems:

1. Consider $Y(t) = B(t)^2$.
   - **(a)** Use the Ito Lemma to find a formula for $Y(T) - Y(0)$.
   - **(b)** Determine c such that $B(t)^2 - ct$ is a martingale.
2. Consider geometric Brownian motion $S(t) = S_0 e^{\mu t + \sigma B(t)}$.
   - **(a)** Use the Ito Lemma to find a formula for $S(T) - S(0)$.
   - **(b)** Consider the discounted stock price process $\tilde{S}(t) = e^{-rt} S(t)$. Determine μ such that $\tilde{S}(t)$ is a martingale.
3. Consider $Y(t) = B(t)^4 + atB(t)^2 + bt^2$ with constants a, b.
   - **(a)** Use the Ito Lemma to find a formula for $Y(T) - Y(0)$.
   - **(b)** Determine a, b such that $Y(t)$ is a martingale.
   - **(c)** Use 3(b) and 1(b) to find a formula for $E\left[B(t)^4\right]$.

Option price $V(t) = v(\tau, s)$

If the maturity of a European option is T the option price $V_0 = V(0)$ at time $t = 0$ is given by 1.

Now we consider a time $t \le T$. Let $\tau := T - t$ denote the time to maturity, and $s := S(t)$ denote the current stock price.

$$\boxed{\tau := T - t, \qquad s := S(t)}$$

Assume that the stock price $S(t)$ is given by geometric Brownian motion. Then we have under the risk-neutral measure Q

$$S(T) = S(t)e^{\mu_Q \tau + \sigma B(\tau)} = s e^{\mu_Q \tau + \sigma \tau^{1/2} Z} \qquad \text{with a random variable } Z \sim N(0,1)$$

Hence we obtain for the option price $V(t)$ at time t

$$\boxed{V(t) = e^{-r_c \tau} E^Q \left[H(S(T)) \mid S(t) = s \right] = e^{-r_c \tau} \int_{z=-\infty}^{\infty} H\left(s e^{\mu_Q \tau + \sigma \tau^{1/2} Z} \right) \phi(z) dz}$$

The option price $V(t)$ is a function $v(\tau, s)$ which depends on the time to maturity τ and the current stock price s.

Ito formula applied to $V(t)$

If the stock price $S(t)$ is given by geometric Brownian motion we have under the real-world measure P

$$S(t) = S_0 e^{\mu t + \sigma B(t)}$$

We saw that the Ito lemma gives

$$\boxed{\Delta S = \left(\mu + \tfrac{1}{2}\sigma^2 \right) S(t) \Delta t + \sigma S(t) \Delta B} \tag{3}$$

For $(\Delta S)^2$ we obtain

$$\Delta S^2 = (\cdots)\Delta t^2 + (\cdots)\, \Delta t \Delta B + \sigma^2 S(t)^2 \Delta B^2$$

In a sum $\sum_{j=0}^{N-1} \Delta S^2$ the terms Δt^2 and $\Delta t \Delta B$ will go to zero as the partition gets finer. Hence we obtain

$$\boxed{\Delta S^2 = \sigma^2 S(t)^2 \Delta t}$$

Now we consider a process $Y(t) = F(t, S(t))$ where $F(t, x)$ is a function of two variables. Then the Ito lemma gives

$$\Delta Y = \frac{\partial F}{\partial t}(t, S)\Delta t + \frac{\partial F}{\partial x}(t, S)\Delta S + \tfrac{1}{2} \frac{\partial^2 F}{\partial x^2}(t, S) \underbrace{\Delta S^2}_{\sigma^2 S^2 \Delta t}$$

$$Y(T) - Y(0) = \int_{t=0}^{T} \left[\frac{\partial F}{\partial t}(t, S(t)) + \tfrac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 F}{\partial x^2}(t, S(t)) \right] dt + \int_{t=0}^{T} \frac{\partial F}{\partial x}(t, S(t)) dS$$

where the second integral is an Ito integral with respect to dS. Note that by 3 we have $dS = \left(\mu + \frac{1}{2}\sigma^2 \right) S(t)dt + \sigma S(t)dB$, so the second integral can be written as a sum of a dt integral and a dB integral.

Now we apply this Ito formula to the option price $V(t) = v(\tau, s)$ where $\tau := T - t$ and $s := S(t)$. By the chain rule we have $\frac{\partial}{\partial t} = -\frac{\partial}{\partial \tau}$ yielding

$$\Delta V = \left[-\frac{\partial v}{\partial \tau}(\tau, s) + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial s^2}(\tau, s) \right] \Delta t + \frac{\partial v}{\partial s}(\tau, s) \Delta S$$

Hedging

We now assume that the interest rate r_c is zero. We consider an investment strategy where we have $x(t)$ stocks in our portfolio at time t (all the remaining money is always in the bank account). Let $U(t)$ denote the value of our portfolio at time t. For interest rate 0 the change in value of $U(t)$ is only due to the change of stock price and we have

$$\boxed{\Delta U = x(t)\Delta S}$$

We have an option with option price $V(t) = v(\tau, s)$ and we want to construct a replicating investment strategy with $U(t) = V(t)$ for all times $t \in [0, T]$. Then we need to have (all this is under the real-world measure P)

$$\begin{aligned} \Delta U &= \Delta V \\ x(t)\Delta S &= \left[-\frac{\partial v}{\partial \tau}(\tau, s) + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial s^2}(\tau, s)\right] \Delta t + \frac{\partial v}{\partial s}(\tau, s)\Delta S \end{aligned}$$

In order to achieve this we must match the ΔS terms, and have the Δt terms equal to zero:

$$\boxed{x(t) = \frac{\partial v}{\partial s}(\tau, s)}, \qquad \boxed{-\frac{\partial v}{\partial \tau}(\tau, s) + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial s^2}(\tau, s)}$$

The resulting hedging strategy $x(t) = \frac{\partial v}{\partial s}(\tau, s)$ with $\tau := T - t$ and $s := S(t)$ is called **Delta hedging**. We obtain that a European option with any payoff can be replicated, hence we have a complete market.

Problems:

1. The stock price is given by geometric Brownian motion with $S_0 = 10$, $\mu = 1$ and $\sigma = 2$, the interest rate is $r_c = 0$. Consider an option with payoff $H(S) = S^{-1}$.

- **(a)** Find a formula for the option price $V(t) = v(\tau, s)$ as an integral with ϕ. Then evaluate this integral.
  Hint: the antiderivative of $e^{ct}\phi(t)$ is $e^{c^2/2}\Phi(t - c)$.
- **(b)** We conjecture that the option price has the form $v(\tau, s) = e^{a\tau}s^{-1}$. What is the investment strategy $x(t)$ which replicates the option price?
- **(c)** Apply the Ito Lemma to the option price and find a formula $\Delta V = (\cdots)\,\Delta t + (\cdots)\,\Delta S$.
- **(d)** Let $U(t)$ denote the value of our portfolio with investment strategy $x(t)$. Compare ΔU with ΔV and use this to find the value of a.

2. Now consider an option with payoff $H(S) = S^{1/2}$. Answer the same questions as for the previous problem. For (b), (c), (d) we conjecture that the option price has the form $v(\tau, s) = e^{a\tau}s^{1/2}$ with some constant a.