Solution of Problems

1. The stock price is given by geometric Brownian motion with $S_0 = 10$, $\mu = 1$ and $\sigma = 2$, the interest rate is $r_c = 0$. Consider an option with payoff $H(S) = S^{-1}$.

(a) Find a formula for the option price $V(t) = v(\tau, s)$ as an integral with ϕ. Then evaluate this integral.
Hint: the antiderivative of $e^{ct}\phi(t)$ is $e^{c^2/2}\Phi(t-c)$.
$\mu_Q = r_c - \sigma^2/2 = -2$, with $\tau = T - t$ and $s = S(t)$ we have

$$\begin{aligned} V(t) = v(\tau,s) &= e^{-r_c\tau}\int_{z=-\infty}^{\infty} H\left(se^{\mu_Q\tau+\sigma\tau^{1/2}z}\right)\phi(z)dz \\ &= \int_{z=-\infty}^{\infty}\left(se^{-2\tau+2\tau^{1/2}z}\right)^{-1}\phi(z)dz = s^{-1}e^{2\tau}\underbrace{\int_{z=-\infty}^{\infty} e^{-2\tau^{1/2}z}\phi(z)dz}_{\left[e^{c^2/2}\Phi(z-c)\right]_{z=-\infty}^{\infty} = e^{c^2/2}} \end{aligned}$$

using the hint with $c := -2\tau^{1/2}$, hence $e^{c^2/2} = e^{2\tau}$ yielding

$$V(t) = v(\tau,s) = s^{-1}e^{2\tau}e^{2\tau} = s^{-1}e^{4\tau}$$

(b) We conjecture that the option price has the form $v(\tau, s) = e^{a\tau}s^{-1}$. What is the investment strategy $x(t)$ which replicates the option price?
We use "Delta-hedging" with $x(t) = \dfrac{\partial v(\tau,s)}{\partial s} = \dfrac{\partial}{\partial s}\left(e^{a\tau}s^{-1}\right) = -e^{a\tau}s^{-2}$

(c) Apply the Ito Lemma to the option price and find a formula $\Delta V = (\cdots)\,\Delta t + (\cdots)\,\Delta S$.
We use that $(\Delta S)^2 = \sigma^2 S^2\Delta t$. For $v(\tau,s) = e^{a\tau}s^{-1}$ we have $\dfrac{\partial v}{\partial \tau} = ae^{a\tau}s^{-1}$, $\dfrac{\partial v}{\partial s} = e^{a\tau}(-s^{-2})$, $\dfrac{\partial^2 v}{\partial s^2} = e^{a\tau}2s^{-3}$ yielding

$$\begin{aligned} \Delta V &= -\frac{\partial v}{\partial \tau}(\tau,S)\Delta t + \frac{\partial v}{\partial s}(\tau,S)\Delta S + \tfrac{1}{2}\frac{\partial^2 v}{\partial s^2}(\tau,S)\underbrace{\Delta S^2}_{\sigma^2S^2\Delta t} \\ &= -ae^{a\tau}S^{-1}\Delta t - e^{a\tau}S^{-2}\Delta S + \tfrac{1}{2}e^{a\tau}2S^{-3}4S^2\Delta t \\ &= \left[-a+4\right]e^{a\tau}S^{-1}\Delta t - e^{a\tau}S^{-2}\Delta S \end{aligned}$$

(d) Let $U(t)$ denote the value of our portfolio with investment strategy $x(t)$. Compare ΔU with ΔV and use this to find the value of a.
Since $r_c = 0$ the change in $U(t)$ is only caused by changes in $S(t)$ and we have

$$\begin{aligned} \Delta U &= x(t)\Delta S \\ \Delta V &= -e^{a\tau}S^{-2}\Delta S + \left[-a+4\right]e^{a\tau}S^{-1}\Delta t \end{aligned}$$

For a replicating portfolio we want $U(t) = V(t)$ for all times, i.e., $\Delta U = \Delta V$. This holds if

$$x(t) = -e^{a\tau}S(t)^{-2}, \qquad -a+4 = 0$$

If we use the Delta-hedging strategy from (b) and let $a = 4$ we obtain that $U(t) = V(t) = e^{4\tau}s^{-1}$ where $\tau = T - t$ and $s = S(t)$. Since with our investment strategy we exactly replicate the payoff $S(T)^{-1}$ at maturity T, by the comparison principle the option price must be equal to $U(t)$ for all times $t \in [0, T]$. Note that this is the same option price we obtained in (a) by evaluating the integral.

2. Now consider an option with payoff $H(S) = S^{1/2}$. Answer the same questions as for the previous problem.

(a) We proceed exactly as in 1(a):

$$V(t) = v(\tau, s) = e^{-r_c\tau} \int_{z=-\infty}^{\infty} H\left(se^{\mu_Q \tau + \sigma\tau^{1/2}z}\right)\phi(z)dz$$
$$= \int_{z=-\infty}^{\infty} \left(se^{-2\tau+2\tau^{1/2}z}\right)^{1/2} \phi(z)dz = s^{1/2}e^{-\tau} \underbrace{\int_{z=-\infty}^{\infty} e^{\tau^{1/2}z}\phi(z)dz}_{\left[e^{c^2/2}\Phi(z-c)\right]_{z=-\infty}^{\infty} = e^{c^2/2}}$$

using the hint with $c := \tau^{1/2}$, hence $e^{c^2/2} = e^{\tau/2}$ yielding

$$V(t) = v(\tau, s) = s^{1/2}e^{-\tau}e^{\tau/2} = s^{1/2}e^{-\tau/2}$$

(b) We conjecture that the option price has the form $v(\tau, s) = e^{a\tau}s^{1/2}$. What is the investment strategy $x(t)$ which replicates the option price?

We use "Delta-hedging" with $x(t) = \dfrac{\partial v(\tau, s)}{\partial s} = \dfrac{\partial}{\partial s}\left(e^{a\tau}s^{1/2}\right) = e^{a\tau}\frac{1}{2}s^{-1/2}$

(c) For $v(\tau, s) = e^{a\tau}s^{1/2}$ we have $\dfrac{\partial v}{\partial \tau} = ae^{a\tau}s^{1/2}$, $\dfrac{\partial v}{\partial s} = e^{a\tau}\frac{1}{2}s^{-1/2}$, $\dfrac{\partial^2 v}{\partial s^2} = e^{a\tau}(-\frac{1}{4})s^{-3/2}$ yielding

$$\Delta V = -\frac{\partial v}{\partial \tau}(\tau, S)\Delta t + \frac{\partial v}{\partial s}(\tau, S)\Delta S + \frac{1}{2}\frac{\partial^2 v}{\partial s^2}(\tau, S) \underbrace{\Delta S^2}_{\sigma^2 S^2 \Delta t}$$
$$= -ae^{a\tau}S^{1/2}\Delta t + e^{a\tau}\tfrac{1}{2}S^{-1/2}\Delta S + \tfrac{1}{2}e^{a\tau}(-\tfrac{1}{4})S^{-3/2}4S^2\Delta t$$
$$= \left[-a - \tfrac{1}{2}\right]e^{a\tau}S^{1/2}\Delta t + e^{a\tau}\tfrac{1}{2}S^{-1/2}\Delta S$$

(d) We proceed as in 1(d):

$$\Delta U = x(t)\Delta S$$
$$\Delta V = e^{a\tau}\tfrac{1}{2}S^{-1/2}\Delta S + \left[-a - \tfrac{1}{2}\right]e^{a\tau}S^{-1}\Delta t$$

For a replicating portfolio we want $U(t) = V(t)$ for all times, i.e., $\Delta U = \Delta V$. This holds if

$$x(t) = e^{a\tau}\tfrac{1}{2}S(t)^{-1/2}, \qquad -a - \tfrac{1}{2} = 0$$

Hence we use Delta-hedging as in (b) and we use $a = -\frac{1}{2}$. As in 1(d) this implies that the option price is $V(t) = e^{-\tau/2}s^{1/2}$. Note that this is the same option price we obtained in (a) by evaluating the integral.