SOLUTION OF EXAM 1 AMSC 424, Fall 2016

NO CALCULATORS allowed.
You need to SHOW ALL YOUR WORK in order to get credit.

1. (35 pts) We throw an ideal dice. If you get "1" you lose 1 point, if you get "5" or "6" you win 2 points. You play 80 rounds of this game. Let Y denote the total number of points.

(a) (20 pts) What is the probability that $Y \geq 60$? Give an approximate answer using $\Phi(\cdots).\Phi(\cdots)$.
$X = \begin{cases} -1 & \text{with prob. } \frac{1}{6} \\ 0 & \text{with prob. } \frac{3}{6}, \\ 2 & \text{with prob. } \frac{2}{6} \end{cases}$
$\mu_0 := E[X] = (-1+2\cdot 2)/6 = \frac{1}{2}$, $E[X^2] = (1+2\cdot 2^2)/6 = \frac{3}{2}$, $\sigma_0^2 = \text{Var}[X] = \frac{3}{2} - \frac{1}{4} = \frac{5}{4}$
$\mu = 80\mu_0 = 40$, $\sigma^2 = 80\sigma_0 = 100$.
By CLT Y has approx. $N(300, 250)$ distribution. Hence with

$$Z = \frac{Y-40}{10}, \qquad \tilde{b} = \frac{60-40}{10} = 2$$

$$P(Y \geq 260) = P(Z \geq \tilde{b}) \approx 1 - \Phi(\tilde{b}) = \Phi(2)$$

(b) If (15 pts) If $Y \geq 60$ you win Y\$, otherwise you get nothing. Find the expected value for your winnings, give an approximate answer as an integral using ϕ. Do NOT evaluate the integral.
$Y = 40 + 10Z$, expected winnings: $\int_{\tilde{b}}^{\infty}(40+10z)\phi(z)dz$ with $\tilde{b} = 2$.

2. (35 pts) The stock price $S(t)$ is given by geometric Brownian motion, with initial stock price $S(0) = 10$, yearly drift 0.2 and yearly volatility 0.4.

(a) (15 pts) Find the probability that $S(\frac{1}{4}) \leq 10$ using $\Phi(\cdots)$.
$S(T) = 10\exp(0.2T + 0.4B(T))$, $B(T) = T^{1/2}Z$ with $Z \sim N(0,1)$
$S(T) \leq 10 \iff \log\left(\frac{S(T)}{10}\right) \leq 0$
$\log\left(\frac{S(T)}{10}\right) = \mu T + \sigma T^{1/2} Z \leq 0 \iff Z \leq \dfrac{-\mu T}{\sigma T^{1/2}} = \dfrac{-\mu T^{1/2}}{\sigma} = -\dfrac{1}{4}$
probability $= \Phi(-\frac{1}{4})$

(b) (20 pts) A European option pays at maturity $T = \frac{1}{4}$ the amount 1 if $S(T) \leq 10$ and pays zero otherwise. The annual interest rate with continuous compounding is 10% . Find the option price V_0 using $\Phi(\cdots)$.
$\tilde{\mu} = r - \frac{1}{2}\sigma^2 = 0.1 - 0.08 = 0.02$

$V_0 = e^{-rT} Q\left(S(T) \leq 10\right) = e^{-.025}\Phi\left(-\frac{1}{40}\right)$

3. (30 pts) Let $B(t)$ denote standard Brownian motion. Let $Y(t) = B(t)^3 + a\cdot t\cdot B(t)$ where a is a constant.

(a) (20 pts) Use the Ito Lemma to find a formula for $Y(T) - Y(0)$.
$F(t,x) = x^3 + atx$, $\frac{\partial F}{\partial t} = ax$, $\frac{\partial F}{\partial x} = 3x^2 + at$, $\frac{\partial^2 F}{\partial x^2} = 6x$
$\Delta Y = aB\Delta t + (3B^2 + at)\Delta B + \frac{1}{2}6B\Delta B^2$
$Y(T) - Y(0) = \int_{t=0}^{T}(a+3)Bdt + \int_{t=0}^{T}\left(3B(t)^2 + at\right)dB$

(b) (10 pts) For which value of a is $Y(t)$ a martingale?
The integral $\int_{t=0}^{T}(\cdots)dB$ is a martingale. We need that the first integral is zero, i.e., $a = -3$.