# Solutions of the problems

## Central Limit Theorem

### Problems:

**1.** We have independent random variables $X_1, \ldots, X_N$ with $X_j = \begin{cases} .03 & \text{with prob. } \frac{1}{2} \\ -.01 & \text{with prob. } \frac{1}{2} \end{cases}$.
Let $Y := X_1 + \cdots + X_{100}$.

- **(a)** Find an approximation for $P(Y \le 1.3)$ using $\Phi(\cdots)$.
- **(b)** Assume we have stock prices $S_0, \ldots, S_{100}$ with $S_0 = 15$ and $\log(S_j/S_{j-1}) = X_j$. Find an approximation for $E[S_{100}]$ as an integral with $\phi(z)$. *Hint:* $\log(S_{100}/S_0) = X_1 + \cdots + X_{100}$.
- **(c)** The antiderivative of $e^{az}\phi(z)$ is $e^{a^2/2}\Phi(z-a)$. Use this to evaluate the integral from (b).

**Solution:** (a) $\mu_0 := E[X_j] = (.03 - .01)\frac{1}{2} = .01$, $\sigma_0^2 := \operatorname{Var}[X_j] = E\left[X_j^2\right] - E[X_j]^2 = (9 \cdot 10^{-4} + 1 \cdot 10^{-4})\frac{1}{2} - 10^{-4} = 4 \cdot 10^{-4}$
$\mu := E[Y] = 100\mu_0 = 1$, $\sigma^2 := \operatorname{Var}[Y] = 100\sigma_0^2 = 4 \cdot 10^{-2}$, $\sigma = .2$
$P(Y \le 1.3) = 1 - \Phi\left(\frac{1.3-\mu}{\sigma}\right) = \Phi\left(\frac{1.3-1}{.2}\right) = \Phi(1.5)$
(b),(c): We have $S_{100} = S_0 e^Y$ and $Y = \mu + \sigma Z = 1 + .2Z$ with a random variable $Z$ which has approximately $N(0,1)$ distribution. Therefore we have with $S_0 = 15$, $\mu = 1$, $\sigma = .2$

$$\begin{aligned} E[S_{100}] &= E\left[S_0 e^{\mu+\sigma Z}\right] = \int_{z=-\infty}^{\infty} S_0 e^{\mu+\sigma z}\phi(z)dz = S_0 e^{\mu}\int_{-\infty}^{\infty} e^{\sigma z}\phi(z)dz \\ &= S_0 e^{\mu}\left[e^{\sigma^2/2}\Phi(z-\sigma)\right]_{z=-\infty}^{\infty} = S_0 e^{\mu} e^{\sigma^2/2}(1-0) = 15e^{1.02} \end{aligned}$$

## Using geometric Brownian motion for stock prices

### Problems:

**1.** The interest rate with continuous compounding is $r = 10\%$. Under the real-world measure $P$ the stock price is given by

$$S(t) = S_0 e^{\mu t + \sigma B(t)}$$

with $S_0 = 15$, drift $\mu = .4$ and volatility $\sigma = .2$.

- **(a)** Find $P(S(4) \le 15e)$ using $\Phi(\cdots)$.
- **(b)** We consider a European option with maturity $T = 4$ and payoff function $H(S) = \begin{cases} 1 & \text{if } S > 15e \\ 0 & \text{if } S \le 15e \end{cases}$ (a so-called binary option). Write the option price $V_0$ using an integral with $\phi(z)$.
- **(c)** Evaluate the integral from (b) using $\Phi(\cdots)$.

**Solution:** (a) We have $S(T) = S_0 e^{\mu T + \sigma B(T)}$. Since $B(T) \sim N(0,T)$ we can write $B(T) = T^{1/2}Z$ with $Z \sim N(0,1)$. Hence

$$S(T) = S_0 e^{\mu T + \sigma T^{1/2} Z}$$

and $S(t) \leq b$ is equivalent to

$$\begin{aligned} S_0 e^{\mu T + \sigma T^{1/2} Z} &\leq b \\ \mu T + \sigma T^{1/2} Z &\leq \log(b/S_0) \\ Z &\leq \frac{\log(b/S_0) - \mu T}{\sigma T^{1/2}} = \frac{1 - .4 \times 4}{.2 \times 2} = -1.5 \end{aligned}$$

Hence

$$P\left(S(4) \leq 15e\right) = P\left(Z \leq 1.5\right) = \Phi(-1.5)$$

(b),(c) We have

$$V_0 = e^{-rT} E^Q \left[H(S(T))\right]$$

Under the risk-neutral measure $Q$ the stock price is $S(t) = S_0 e^{\mu_Q t + \sigma B(t)}$ with $\mu_Q = r - \frac{1}{2}\sigma^2 = .1 - \frac{1}{2}0.04 = .08$. We have $B(T) = T^{1/2} Z$ with $Z \sim N(0,1)$. Hence

$$E^Q\left[H(S(T))\right] = E\left[H\left(S_0 e^{\mu_Q T + \sigma T^{1/2} Z}\right)\right] = \int_{-\infty}^{\infty} H\left(S_0 e^{\mu_Q T + \sigma T^{1/2} z}\right) \phi(z) dz$$

Note that $H(S) = 0$ if $S \leq 15e =: b$. The condition $S = S_0 e^{\mu_Q T + \sigma T^{1/2} z} \leq b$ is equivalent to

$$z \leq \frac{\log(b/S_0) - \mu_Q T}{\sigma T^{1/2}} = \frac{1 - .08 \times 4}{.2 \times 2} = \frac{.68}{.4} = 1.7$$

Therefore

$$H\left(S_0 e^{\mu_Q T + \sigma T^{1/2} z}\right) = \begin{cases} 1 & \text{if } z > 1.7 \\ 0 & \text{if } z \leq 1.7 \end{cases}$$

and

$$\int_{-\infty}^{\infty} H\left(S_0 e^{\mu_Q T + \sigma T^{1/2} z}\right) \phi(z) dz = \int_{1.7}^{\infty} \phi(z) dz = 1 - \Phi(1.7)$$

Hence the option price is

$$V_0 = e^{-rT} E^Q\left[H(S(T))\right] = e^{-.1 \times 4}\left(1 - \Phi(1.7)\right)$$

## Ito calculus

### Problems:

**1.** Consider $Y(t) = B(t)^2$.

**(a)** Use the Ito Lemma to find a formula for $Y(T) - Y(0)$.

**(b)** Determine $c$ such that $B(t)^2 - ct$ is a martingale.

**Solution:** $F(t,x) = x^2$, $\dfrac{\partial F}{\partial x} = 2x$, $\dfrac{\partial^2 F}{\partial x^2} = 2$. Hence

$$\Delta Y = 2B \cdot \Delta B + \tfrac{1}{2} \cdot 2 \cdot \underbrace{\Delta B^2}_{\Delta t}$$

$$\underbrace{Y(T) - Y(0)}_{0} = \underbrace{\int_{t=0}^{T} 2B(t) dB}_{\text{martingale}} + \underbrace{\int_{t=0}^{T} 1\, dt}_{T}$$

$$B(T)^2 = \text{martingale} + T$$

We obtain that $B(t)^2 - t$ is a martingale, i.e. we need $c = 1$.

**2.** Consider geometric Brownian motion $S(t) = S_0 e^{\mu t + \sigma B(t)}$.

**(a)** Use the Ito Lemma to find a formula for $S(T) - S(0)$.

**(b)** Consider the discounted stock price process $\tilde{S}(t) = e^{-rt} S(t)$. Determine $\mu$ such that $\tilde{S}(t)$ is a martingale.

**Solution:** $F(t,x) = S_0 e^{\mu t + \sigma x}$, $\dfrac{\partial F}{\partial t} = S_0 e^{\mu t + \sigma x} \mu$, $\dfrac{\partial F}{\partial x} = S_0 e^{\mu t + \sigma x} \sigma$, $\dfrac{\partial^2 F}{\partial x^2} = S_0 e^{\mu t + \sigma x} \sigma^2$

$$\Delta S = S_0 e^{\mu t + \sigma B} \mu \cdot \Delta t + S_0 e^{\mu t + \sigma B} \sigma \cdot \Delta B + \tfrac{1}{2} S_0 e^{\mu t + \sigma B} \sigma^2 \underbrace{\Delta B^2}_{\Delta t}$$

$$S(T) - S(0) = \underbrace{\int_{t=0}^{T} S_0 e^{\mu t + \sigma B(t)} \sigma dB}_{\text{martingale}} + \int_{t=0}^{T} S_0 e^{\mu t + \sigma B(t)} \left(\mu + \tfrac{1}{2}\sigma^2\right) dt$$

For $\tilde{S}(t) = S_0 e^{(\mu - r)t + \sigma B(t)}$ we obtain with $(\mu - r)$ instead of $\mu$

$$\tilde{S}(T) - \tilde{S}(0) = \underbrace{\int_{t=0}^{T} S_0 e^{(\mu - r)t + \sigma B(t)} \sigma dB}_{\text{martingale}} + \int_{t=0}^{T} S_0 e^{(\mu - r)t + \sigma B(t)} \left((\mu - r) + \tfrac{1}{2}\sigma^2\right) dt$$

The process $\tilde{S}(t)$ is a martingale if the second integral is zero: We need $(\mu - r) + \frac{1}{2}\sigma^2 = 0$ or

$$\mu = r - \tfrac{1}{2}\sigma^2$$

**3.** Consider $Y(t) = B(t)^4 + atB(t)^2 + bt^2$ with constants $a, b$.

**(a)** Use the Ito Lemma to find a formula for $Y(T) - Y(0)$.

**(b)** Determine $a, b$ such that $Y(t)$ is a martingale.

**(c)** Use 3(b) and 1(b) to find a formula for $E\left[B(t)^4\right]$.

**Solution:** (a) $F(t,x) = x^4 + atx^2 + bt^2$, $\dfrac{\partial F}{\partial t} = ax^2 + 2bt$, $\dfrac{\partial F}{\partial x} = 4x^3 + 2atx$, $\dfrac{\partial^2 F}{\partial x^2} = 12x^2 + 2at$

$$\Delta Y = \left(aB^2 + 2bt\right) \Delta t + \left(4B^3 + 2atB\right) \Delta B + \tfrac{1}{2}\left(12B^2 + 2at\right) \underbrace{\Delta B^2}_{\Delta t}$$

$$Y(T) - Y(0) = \underbrace{\int_{t=0}^{T} \left[4B(t)^3 + 2atB(t)\right] dB}_{\text{martingale}} + \int_{t=0}^{T} \left[(a+6)B(t)^2 + (2b + a)t\right] dt$$

(b) $Y(t)$ is a martingale if the integrand in the second integral is zero: We need $a + 6 = 0$ and $2b + a = 0$, hence

$$a = -6, \qquad b = 3$$

(c) $Y(t) = B(t)^4 - 6tB(t)^2 + 3t^2$ is a martingale. Hence

$$E[Y(t)] = Y(0)$$

$$E\left[B(t)^4\right] - 6t \cdot \underbrace{E\left[B(t)^2\right]}_{t} + 3t^2 = 0$$

since by 1(b) $E\left[B(t)^2\right] = t$. We obtain $E\left[B(t)^4\right] = 3t^2$.