1 Polynomial approximation and interpolation

1.1 Jackson theorems

1.1.1 Polynomials $\mathcal{P}_n$ and trigonometric polynomials $\mathcal{T}_n$

In order to state the approximation problem we define the functions which we want to approximate and the functions which we want to use for approximation:

Definition 1.1 *We denote by $C^k[a,b]$ for $k = 0, 1, 2, \ldots$ the space of functions which have derivatives $f^{(1)}, \ldots, f^{(k)}$ that are continuous on the closed interval $[a,b]$.*

We denote by

$$\mathcal{P}_n = \{ c_0 + c_1 x + \cdots + c_n x^n \mid c_k \in \mathbb{C} \}$$

the space of polynomials of degree less or equal to n ($n = 0, 1, 2, \ldots$).

Then the approximation problem is: Given $f \in C^k[a,b]$, what is the rate with which error of the best approximation

$$\inf_{p_n \in \mathcal{P}_n} \|f - p_n\|_\infty$$

converges to zero as n goes to infinity?

The so-called **Jackson theorems** shows that the decay rate of the error depends on the smoothness of the function f. E.g. for $f \in C^1[a,b]$ we will prove an approximation rate of $O(1/n)$, and for $f \in C^2[a,b]$ we will obtain an approximation rate of $O(1/n^2)$.

The problem of approximating functions on intervals by polynomials is closely related to the problem of approximating *periodic functions* by *trigonometric polynomials*:

Definition 1.2 *The space $C_{2\pi}$ of 2π-periodic functions consists of all functions $f \in C(\mathbb{R})$ which satisfy*

$$\forall x \in \mathbb{R} \qquad f(x) = f(x + 2\pi).$$

The space of k times continuously differentiable 2π-periodic functions is defined as $C^k_{2\pi} = C^k(\mathbb{R}) \cap C_{2\pi}$.

We denote by $\mathcal{T}_n$ the space of trigonometric polynomials of degree less or equal to n ($n = 0, 1, 2, \ldots$):

$$\mathcal{T}_n = \{ a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \mid a_k, b_k \in \mathbb{C} \}$$

1.1.2 Interpolation with trigonometric polynomials

Note that the space $\mathcal{T}_n$ has dimension $2n+1$ (whereas $\mathcal{P}_n$ has dimension $n+1$). Therefore we can ask whether we can always find a trigonometric interpolation polynomial $p_n \in \mathcal{T}_n$ which passes through $2n+1$ given points (x_j, y_j), $j = 0, \ldots, 2n$. We first note that $\mathcal{T}_n$ has a property similar to $\mathcal{P}_n$:

Lemma 1.3 *A nonzero function $f \in \mathcal{T}_n$ has at most $2n$ zeros in $[0, 2\pi)$.*

Proof: Assume that $f \in \mathcal{T}_n$ has $2n+1$ zeros $\theta_0, \ldots, \theta_{2n}$ in $[0, 2\pi)$. Writing $\sin k\theta$ and $\cos k\theta$ in terms of $e^{ik\theta}$ and $e^{-ik\theta}$ and with $z := e^{i\theta}$ we have that

$$f(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta} = \sum_{k=-n}^{n} c_k z^k$$

equals zero for $z_j := e^{i\theta_j}$. Multiplying by z^n we obtain that $\sum_{k=-n}^{n} c_k z^{k+n}$ is a polynomial of degree $\leq 2n$ in z which has at least $2n+1$ zeros $z_j \in \mathbb{C}$. Hence it must be the zero polynomial and all $c_k = 0$. ❏

Corollary 1.4 *Let $x_0, \ldots, x_{2n}$ be distinct values in $[0, 2\pi)$. Then for any given values $y_0, \ldots, y_{2n}$ there exists a unique interpolating trigonometric polynomial $p_n \in \mathcal{T}_n$ which satisfies $p_n(x_j) = y_j$, $j = 0, \ldots, 2n$.*

Proof: The interpolation problem leads to a linear system of $2n+1$ equations for the $2n+1$ unknowns $a_0, \ldots, a_n$, $b_1, \ldots, b_n$ with the right hand side vector $(y_0, \ldots, y_{2n})^\top$. This system has a unique solution for every right hand side if the matrix is nonsingular. To show that the matrix is nonsingular consider the problem with the zero right hand side vector. Any solution of this linear system corresponds to a function $p_n \in \mathcal{T}_n$ which is zero in $x_0, \ldots, x_{2n}$. By Lemma 1.3 p_n must be zero. Hence the homogeneous linear sytem has only the zero solution and the matrix is nonsingular. ❏

1.1.3 An auxiliary approximation problem

As a first step toward proving the Jackson theorems let us consider the 2π periodic function f with $f(x) = x$ for $x \in (-\pi, \pi]$. In order to approximate it by a function in $\mathcal{T}_n$ we can use the interpolation p_n through the $2n+1$ nodes $k\frac{\pi}{n+1}$, $k = -n, \ldots, n$ which exists and is unique due to Corollary 1.4. Since the function f is odd, i.e., $f(-x) = -f(x)$ (for $x \neq k\pi$), the function $-p_n(-x) \in \mathcal{T}_n$ is also an interpolation. By the uniqueness of the interpolation, $p_n(x)$ and $-p_n(-x)$ must have the same coefficients and so we have that

$$p_n(x) = \sum_{k=1}^{n} b_k \sin kx. \tag{1.1}$$

Now we consider the interpolation error $e(x) := f(x) - p_n(x)$. Since p_n interpolates f in $2n+1$ points in $(-\pi, \pi)$, $e(x)$ has at least $2n+1$ simple zeros in $(-\pi, \pi)$. There cannot be more zeros in $(-\pi, \pi)$: If $e(x)$ has $2n+2$ zeros in $(-\pi, \pi)$, then $e'(x) = 1 - p_n'(x) \in \mathcal{T}_n$ has at least $2n+1$ zeros by Rolle's theorem. By Corollary 1.3 we then have $e'(x) = 0$ which is a contradiction. The same argument also shows that the $2n+1$ zeros of e in $(-\pi, \pi)$ are simple, i.e., $e'(x) \neq 0$ in those points. Hence the function $e(x)$ changes its sign in $(-\pi, \pi)$ only at the interpolation points $k\frac{\pi}{n+1}$, $k = -n, \ldots, n$. Let $s(x)$ denote the function which is alternatingly 1 and -1 between the nodes, i.e.,

$$s(x)\big|_{[k\frac{\pi}{n+1}, (k+1)\frac{\pi}{n+1})} = (-1)^k,$$

then we have

$$\int_{-\pi}^{\pi} |f(x) - p_n(x)| \, dx = \Big| \int_{-\pi}^{\pi} (f(x) - p_n(x)) s(x) \, dx \Big| = \Big| \int_{-\pi}^{\pi} f(x) s(x) \, dx - \int_{-\pi}^{\pi} p_n(x) s(x) \, dx \Big|.$$

Let us show that the second term on the right hand side must be 0: Consider

$$\begin{aligned} A &:= \int_{-\pi}^{\pi} s(x)\sin kx\,dx = -\int_{-\pi}^{\pi} s(x+\frac{\pi}{n+1})\sin kx\,dx = -\int_{-\pi}^{\pi} s(x)\sin k(x-\frac{\pi}{n+1})\,dx \\ &= -\int_{-\pi}^{\pi} s(x)\sin kx\cos(-k\frac{\pi}{n+1})\,dx - \int_{-\pi}^{\pi} \underbrace{s(x)}_{\text{odd}}\underbrace{\cos kx}_{\text{even}}\sin(-k\frac{\pi}{n+1})\,dx \\ &= -\cos(k\frac{\pi}{n+1})A + 0 \end{aligned}$$

Since $\cos(k\frac{\pi}{n+1}) \neq -1$ we obtain $A = 0$ and also $\int_{\pi}^{\pi} p_n(x)s(x)\,dx = 0$ because of (1.1). It remains to evaluate $\int_{\pi}^{\pi} f(x)s(x)\,dx$: It can be easily seen geometrically that we have

$$\Big|\int_{\pi}^{\pi} xs(x)\,dx\Big| = 2\Big|\int_{0}^{\pi} xs(x)\,dx\Big| = 2\left(\frac{\pi}{n+1}\right)^2\Big|-\frac{1}{2}+\frac{3}{2}-\cdots\pm(n+\frac{1}{2})\Big| = 2\left(\frac{\pi}{n+1}\right)^2\frac{n+1}{2} = \frac{\pi^2}{n+1}.$$

Hence we obtain the following result for the interpolation error:

$$\int_{\pi}^{\pi} |f(x)-p_n(x)|\,dx = \frac{\pi^2}{n+1}.$$

1.1.4 Jackson theorems for periodic functions

Now we can prove the Jackson theorem for $f \in C^1_{2\pi}$: Using integration by parts we see that

$$\int_{-\pi}^{\pi} \theta f'(\theta+x+\pi)\,d\theta = [\theta f(\theta+x+\pi)]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 1f(\theta+x+\pi)\,d\theta = 2\pi f(x) - \int_{-\pi}^{\pi} f(\theta)\,d\theta$$

yielding

$$f(x) = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta}_{=:a_0=\text{const.}} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta f'(\theta+x+\pi)\,d\theta \tag{1.2}$$

In order to approximate $f(x)$ we replace θ in the second integral by the interpolation $p_n \in \mathcal{T}_n$ from above and obtain

$$q(x) = a_0 + \frac{1}{2\pi}\int_{-\pi}^{\pi} p_n(\theta)f'(\theta+x+\pi)\,d\theta = a_0 + \frac{1}{2\pi}\int_{-\pi}^{\pi} p_n(\theta-\pi-x)f'(\theta)\,d\theta \tag{1.3}$$

where

$$p_n(\theta-\pi-x) = \sum_{k=1}^{n} b_k \sin k(\theta-\pi-x) = \sum_{k=1}^{n} b_k\Big(\sin k(\theta-\pi)\underbrace{\cos kx} - \cos k(\theta-\pi)\underbrace{\sin kx}\Big). \tag{1.4}$$

If we insert this expression for $p_n(\theta-\pi-x)$ into the integral in (1.3) we see that the underbraced terms can be pulled out of the integral and that $q \in \mathcal{T}_n$.

It remains to estimate the approximation error:

$$\begin{aligned} \|f-q\|_\infty &= \sup_x \Big| \frac{1}{2\pi} \int_{-\pi}^{\pi} (\theta - p_n(\theta)) f'(\theta + x + \pi)\, d\theta \Big| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |\theta - p_n(\theta)|\, |f'(\theta+x+\pi)|\, d\theta \\ &\le \frac{1}{2\pi} \|f'\|_\infty \int_{-\pi}^{\pi} |\theta - p_n(\theta)|\, d\theta = \frac{1}{2\pi} \|f'\|_\infty \frac{\pi^2}{n+1} \end{aligned}$$

We have therefore proved the following theorem:

Theorem 1.5 (Jackson theorem for $C^1_{2\pi}$) *For $f \in C^1_{2\pi}$ there holds*

$$\inf_{p \in \mathcal{T}_n} \|f-p\|_\infty \le \frac{\pi}{2(n+1)} \|f'\|_\infty \tag{1.5}$$

If the function $f(x)$ has mean 0, i.e., $\int_{-\pi}^{\pi} f(x)\,dx = 0$, then the approximating function $q(x)$ has the same property: The constant a_0 in (1.2) is 0, and (1.3), (1.4) show that $q(x)$ is of the form $\sum_{k=1}^n (a_k \cos kx + b_k \sin kx)$. Let us define $\mathcal{T}_N^0$ as the space of trigonometric polynomials of degree less or equal n with mean 0:

$$\mathcal{T}_N^0 := \{\, p \in \mathcal{T}_N \mid \int_{-\pi}^{\pi} p(x)\,dx = 0 \,\} = \{\, \sum_{k=1}^n (a_k \cos kx + b_k \sin kx) \mid a_k, b_k \in \mathbb{C} \,\}$$

We then have

Lemma 1.6 *Let $f \in C^1_{2\pi}$ with $\int_{-\pi}^{\pi} f(x)\,dx = 0$. Then*

$$\inf_{p \in \mathcal{T}_N^0} \|f-p\|_\infty \le \frac{\pi}{2(n+1)} \|f'\|_\infty\,. \tag{1.6}$$

Next, we would like to investigate the approximation rates for functions with more smoothness, e.g., $f \in C^3_{2\pi}$. We can use the following "bootstrap" argument: Assume that $f \in C^1_{2\pi}$. Let $\tilde p$ be an arbitrary trigonometric polynomial in $\mathcal{T}_n$. Then we have obviously

$$\inf_{p \in \mathcal{T}_n} \|f-p\|_\infty = \inf_{p \in \mathcal{T}_n} \|(f-\tilde p) - p\|_\infty \le \frac{\pi}{2(n+1)} \|f' - \tilde p'\|_\infty \tag{1.7}$$

using (1.5) for $(f - \tilde p)$. Note that (1.7) holds for any function $\tilde p \in \mathcal{T}_n$, so we can choose it in such a way that the right hand side becomes as small as possible. Since

$$\{\, \tilde p' \mid \tilde p \in \mathcal{T}_n \,\} = \mathcal{T}_n^0,$$

equation (1.7) implies

$$\inf_{p \in \mathcal{T}_n} \|f-p\|_\infty \le \frac{\pi}{2(n+1)} \inf_{p \in \mathcal{T}_n^0} \|f'-p\|_\infty \tag{1.8}$$

For approximation errors in $\mathcal{T}_n^0$ we can use a similar argument: Assume $f \in C^1_{2\pi}$ with $\int_{-\pi}^{\pi} f(x)\,dx = 0$. Using (1.6) and

$$\{\, \tilde p' \mid \tilde p \in \mathcal{T}_n^0 \,\} = \mathcal{T}_n^0,$$

we obtain analogously

$$\inf_{p \in \mathcal{T}_n^0} \|f-p\|_\infty \le \frac{\pi}{2(n+1)} \inf_{p \in \mathcal{T}_n^0} \|f'-p\|_\infty\,. \tag{1.9}$$

This allows us to prove

Theorem 1.7 (Jackson theorem for $C^k_{2\pi}$) *Let $k \in \{1, 2, \dots\}$ and $f \in C^k_{2\pi}$. Then*

$$\inf_{p\in\mathcal{T}_n} \|f-p\|_\infty \le \left(\frac{\pi}{2(n+1)}\right)^k \left\|f^{(k)}\right\|_\infty \tag{1.10}$$

Proof: Note that $f',\dots,f^{(k-1)}$ have mean 0 since f is periodic. Applying (1.8), (1.9) ($k-2$ times) and finally (1.6) we get

$$\begin{aligned}\inf_{p\in\mathcal{T}_n} \|f-p\|_\infty &\overset{(1.8)}{\le} \frac{\pi}{2(n+1)} \inf_{p\in\mathcal{T}_n^0} \|f'-p\|_\infty \overset{(1.9)}{\le} \left(\frac{\pi}{2(n+1)}\right)^2 \inf_{p\in\mathcal{T}_n^0} \|f''-p\|_\infty \\ &\overset{(1.9)}{\le} \cdots \overset{(1.9)}{\le} \left(\frac{\pi}{2(n+1)}\right)^{k-1} \inf_{p\in\mathcal{T}_n^0} \left\|f^{(k-1)}-p\right\|_\infty \overset{(1.6)}{\le} \left(\frac{\pi}{2(n+1)}\right)^k \left\|f^{(k)}\right\|_\infty .\end{aligned}$$

❑

1.1.5 Jackson theorems on an interval

For simplicity we first consider the interval $[-1, 1]$ and consider a function $f \in C^1[-1, 1]$. We can transform this function to a 2π-periodic function g by the change of variables

$$g(\theta) = f(\cos\theta) \tag{1.11}$$

Note that the function g is even. This transformation is useful since polynomials $f(x)$ of degree less or equal n are transformed into even trigonometric polynomials $g(\theta)$ and vice versa. If $g \in \mathcal{T}_m$ and $h \in \mathcal{T}_n$ then the product $g \cdot h$ is in $\mathcal{T}_{m+n}$ as can be seen, e.g., by writing $\cos x$ and $\sin x$ in terms of $e^{\pm ix}$. If $f \in \mathcal{P}_n$ then we see that g given by (1.11) is in $\mathcal{T}_n$ and even. It is also clear that g can only be the zero function if f is the zero function. Since $\dim \mathcal{T}_n^{\text{even}} = \dim \mathcal{P}_n = n+1$ it follows that the transformation (1.11) gives a one-to-one linear mapping between $\mathcal{P}_n$ and $\mathcal{T}_n^{\text{even}}$.

The functions $\cos kx$, $k = 0, \dots, n$ form a basis of $\mathcal{T}_n^{\text{even}}$. These functions are transformed by (1.11) to certain polynomials $T_k(x)$ with

$$\cos k\theta = T_k(\cos\theta). \tag{1.12}$$

These are the so-called *Chebyshev polynomials*. Obviously, $T_0(x) = 1$ and $T_1(x) = x$. If we add the formulae $\cos(k \pm 1)x = \cos kx \cos x \mp \sin kx \sin x$ we obtain $\cos(k+1)x = 2\cos kx \cos x - \cos(k-1)x$ which gives the recursion formula

$$T_{k+1}(x) = 2T_k(x)x - T_{k-1}(x). \tag{1.13}$$

We can now prove an approximation result for polynomials on an interval:

Theorem 1.8 (Jackson theorem for $C^1[-1, 1]$) *For $f \in C^1[-1, 1]$ there holds*

$$\inf_{p\in\mathcal{P}_n} \|f-p\|_\infty \le \frac{\pi}{2(n+1)} \|f'\|_\infty \tag{1.14}$$

Proof: Define g by (1.11). Then $g'(\theta) = -f'(\cos\theta)\sin\theta$. Therefore the limits $\lim_{\theta\to\pm\pi} g'(\theta)$ exist and are equal (both are 0), hence $g \in C^1_{2\pi}$. We also see that

$$\|g'\|_\infty \le \|f'\|_\infty . \tag{1.15}$$

Consider an approximation $p \in \mathcal{T}_n$ for g. Since p need not be even we consider the symmetrized function $\tilde{p}(\theta) = (p(\theta) + p(-\theta))/2$. Then we see

$$\begin{aligned}\|g(\theta) - \tilde{p}(\theta)\|_\infty &= \tfrac{1}{2}\|g(\theta) + g(-\theta) - p(\theta) - p(-\theta)\|_\infty \\ &\le \tfrac{1}{2}(\|g(\theta) - p(\theta)\|_\infty + \|g(-\theta) - p(-\theta)\|_\infty) = \|g(\theta) - p(\theta)\|_\infty\end{aligned}$$

and this implies

$$\inf_{p\in\mathcal{T}_n^{\text{even}}} \|g - p\|_\infty = \inf_{p\in\mathcal{T}_n} \|g - p\|_\infty \overset{(1.5)}{\le} \frac{\pi}{2(n+1)} \|g'\|_\infty \tag{1.16}$$

The change of variables $q(\cos\theta) = p(\theta)$ defines for $p \in \mathcal{T}_n^{\text{even}}$ a function $q \in \mathcal{P}_n$ such that

$$\max_{x\in[-1,1]} |f(x) - q(x)| = \max_{\theta\in[-\pi,\pi)} |g(\theta) - p(\theta)| . \tag{1.17}$$

Equations (1.17), (1.16), and (1.15) together yield (1.14). ❑

In the same way as we proved Theorem 1.7 using Theorem 1.5 we can use Theorem 1.8 to prove an approximation result for $f \in C^k[-1,1]$:

Theorem 1.9 (Jackson theorem for $C^k[-1,1]$) *Let n, k be integers with $n \ge k-1 \ge 0$ and $f \in C^k[-1,1]$. Then*

$$\inf_{p\in\mathcal{P}_n} \|f - p\|_\infty \le \left(\frac{\pi}{2}\right)^k \frac{1}{(n+1)n\cdots(n-k+2)} \left\|f^{(k)}\right\|_\infty \tag{1.18}$$

1.2 Polynomial interpolation

The proofs for the Jackson theorems are constructive. But the constructions of the approximating functions are of no practical value since they are very complicated, involving various differentiations and integrations (especially for higher values of k) and are therefore not very efficient.

In this section we will see that we can achieve an approximation which is almost as good as the optimal approximation using interpolation, provided the interpolation nodes are correctly chosen.

1.2.1 The error formula and Chebyshev nodes

Theorem 1.10 *Let $f \in C^{n+1}[a,b]$ and let $p_n \in \mathcal{P}_n$ be the interpolating polynomial through the distinct interpolation nodes $x_0, \ldots, x_n \in [a,b]$. Then for $t \in [a,b]$ there exists $\xi \in [a,b]$ such that*

$$f(t) - p_n(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(t - x_0)\cdots(t - x_n) \tag{1.19}$$

Proof: Let $\omega(x) := (x-x_0)\cdots(x-x_n)$. Fix $t \in [a,b]$ with $t \neq x_j$, $j = 0,\dots,n$ (otherwise (1.19) is trivial). Then $\omega(t) \neq 0$ and the definitions

$$A_t := (f(t) - p_n(t))/\omega(t), \qquad g(x) := f(x) - p_n(x) - A_t\omega(x)$$

imply that $g(t) = 0$. Since also $g(x_j) = 0$ (p_n interpolates f and $\omega(x_j) = 0$) we see that $g(x)$ has at least $n+2$ zeros in $[a,b]$. By Rolle's theorem, g' has at least $n+1$ zeros,..., $g^{(n+1)}$ has at least 1 zero which we will call ξ. Therefore

$$0 = g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - p_n^{(n+1)}(\xi) - A_t\omega^{(n+1)}(\xi) = f^{(n+1)}(\xi) - 0 - A_t(n+1)!$$

since $p_n \in \mathcal{P}_n$ and $\omega(x) = x^{n+1} + q(x)$, $q \in \mathcal{P}_n$. Therefore $A_t = f^{(n+1)}(\xi)/(n+1)!$ which together with $0 = g(t) = f(t) - p_n(t) - A_t\omega(t)$ yields (1.19). ❑

Estimate (1.19) implies

$$\|f - p_n\|_\infty \leq \frac{1}{(n+1)!}\left\|f^{(n+1)}\right\|_\infty \|\omega(x)\|_\infty \tag{1.20}$$

where $\omega(x)$ denotes the *node polynomial*

$$\omega(x) := (x - x_0)\cdots(x - x_n).$$

We see that only the last factor in (1.20) depends on the choice of nodes. Therefore it is a good idea to choose the $n+1$ nodes in $[a,b]$ in such a way that $\|\omega(x)\|_\infty$ becomes minimal.

Let us first consider the interval $[-1,1]$. The Chebyshev polynomial $T_k(x)$ has by its definition (1.12) the property that it oscillates between 1 and -1 and has all its k zeros

$$x_j = \cos\left(\frac{j+1/2}{k}\pi\right), \quad j = 0,\dots,k-1$$

in the interval $[-1,1]$. Furthermore, by the recursion formula (1.13) the polynomial T_k has the leading coefficient 2^{k-1} for $k \geq 1$. Hence

$$T_k(x) = 2^{k-1}(x - x_0)\cdots(x - x_k). \tag{1.21}$$

If we choose the $n+1$ interpolation nodes as the zeros of the polynomial T_{n+1} we have therefore

$$\omega(x) = 2^{-n}T_{n+1}(x) \tag{1.22}$$

and because of $\|T_k\|_\infty = 1$ we obtain

$$\|\omega\|_\infty = 2^{-n}.$$

One cannot achieve a smaller value of $\|\omega\|_\infty$ than 2^{-n} with any arrangement of nodes $x_0,\dots,x_n$: Assume $\|\tilde{\omega}\|_\infty < 2^{-n}$. Since ω alternatingly assumes $n+2$ extrema 2^{-n} and -2^{-n}, the polynomial $q := \omega - \tilde{\omega}$ must have a zero between two subsequent extrema of ω by the intermediate value theorem. Thus $q = \omega - \tilde{\omega}$ has at least $n+1$ zeros. But since both ω and $\tilde{\omega}$ are of the form x^{n+1} + lower order terms we see that $q \in \mathcal{P}_n$ and hence $q = 0$ which contradicts the assumption.

Therefore using the Chebyshev nodes $x_k = \cos\left(\frac{k+1/2}{n+1}\pi\right)$, $k = 0, \ldots, n$ leads to the best estimate in (1.20). Using a linear transformation, we see that the best choice of nodes for the interval $[a, b]$ is

$$\frac{a+b}{2} + \cos\left(\frac{k+1/2}{n+1}\pi\right)\frac{a-b}{2}, \quad k = 0, \ldots, n \tag{1.23}$$

yielding $\|\omega\|_\infty = 2^{-n}\left(\frac{b-a}{2}\right)^{n+1} = 2\left(\frac{b-a}{4}\right)^{n+1}$ and

$$\|f - p_n\|_\infty \leq 2\left(\frac{b-a}{4}\right)^{n+1}\frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!} \tag{1.24}$$

For an arbitrary choice of nodes we have using (1.20) and $|x - x_j| \leq (b-a)$ the estimate

$$\|f - p_n\|_\infty \leq (b-a)^{n+1}\frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!}. \tag{1.25}$$

These formulas show that the interpolation error converges to 0 if the derivatives do not grow too fast. For example, assume that the function f is *analytic* on $[a, b]$, i.e. the Taylor series converges for every $x \in [a, b]$ for all y with $|y - x| < \rho$ with some $\rho > 0$. Equivalently, the function f can be extended to a holomorphic function in all points in $\mathbb{C}$ which have a distance of less than ρ to $[a, b]$ (ρ can be chosen as the distance of the closest singularity of f in $\mathbb{C}$ to the interval $[a, b]$). Then we know (e.g., expressing $f^{(n)}$ with the Cauchy theorem) that for any $\tilde{\rho} < \rho$ there exists C such that

$$\text{for all } n \geq 0 \qquad \frac{1}{n!}\max_{x\in[a,b]}\left|f^{(n)}(x)\right| \leq C\frac{1}{\tilde{\rho}^n}.$$

If $(b-a) < \rho$ then we see that we have exponential convergence for any choice of nodes.

But the formulae (1.24) and (1.25) do not give any information about the convergence if f is less smooth. In fact, interpolation with equally spaced nodes can diverge even if f is analytic on $[a, b]$, as the *Runge example* $f(x) = \frac{1}{1+x^2}$ on $[-5, 5]$ shows. (Here we have singularities at $\pm i$ and hence $\rho = 1$.) On the other hand, interpolation at the Chebyshev nodes converges if f is only marginally smoother than a continuous function: The so-called Dini-Lipschitz condition

$$\log\varepsilon \max_{\substack{|x-y|\leq\varepsilon \\ x,y\in[a,b]}} |f(x) - f(y)| \to 0 \quad \text{for } \varepsilon \to 0$$

is sufficient. This includes continuous, piecewise differentiable functions, as well as functions like $\sqrt{|x|}$. However, there exist continuous functions for which the Chebyshev approximations do not uniformly converge. But one can show that this is true for *any* sequence of interpolation nodes.

1.2.2 The Lagrange form and the Lebesgue function

Let $x_0, \ldots, x_n$ be distinct nodes in the interval $[a, b]$. For a function $f \in C[a, b]$ let us denote the interpolating polynomial in $\mathcal{P}_n$ by $P_n f$. We can give an explicit formula for this polynomial: The *Lagrange polynomials*

$$l_k(x) := \prod_{\substack{j=0,\ldots,n \\ j\neq k}} \frac{x - x_j}{x_k - x_j}$$

obviously satisfy

$$l_k \in \mathcal{P}_n, \qquad l_k(x_j) = \begin{cases} 0 & \text{for } k \neq j \\ 1 & \text{for } k = j \end{cases}. \tag{1.26}$$

Therefore we can write the interpolating polynomial in the so-called *Lagrange form*

$$P_n f = \sum_{k=0}^{n} f(x_k) l_k.$$

Then we get the following estimate

$$|(P_n f)(x)| = \left|\sum_{k=0}^{n} f(x_k) l_k(x)\right| \leq (\max_{k=0,\ldots,n} |f(x_k)|) \sum_{k=0}^{n} |l_k(x)| \leq \|f\|_\infty \lambda_n(x)$$

where $\lambda_n(x) := \sum_{k=0}^{n} |l_k(x)|$ is the so-called *Lebesgue function*. We also have

$$\|P_n f\|_\infty \leq \|\lambda_n\|_\infty \|f\|_\infty \tag{1.27}$$

and this estimate is sharp, i.e., there exist $f \in C[a,b]$ where we have equality in (1.27). Since $\lambda_n(x_k) = 1$ by (1.26) there holds $\|\lambda_n\|_\infty \geq 1$. In order to have a good approximation we would like to choose the interpolation points in such a way that $\|\lambda_n\|_\infty$ does not become too large. Otherwise there exist functions with small values which have interpolating polynomials with very large values.

The size of the *Lebesgue constant* $\|\lambda_n\|_\infty$ also characterizes the relation between the interpolation error and the best possible approximation error: Let $q_n \in \mathcal{P}_n$ be an arbitrary polynomial, then $P_n q = q$ and

$$\|f - P_n f\|_\infty \leq \|f - q\|_\infty + \|q - P_n f\| = \|f - q\|_\infty + \|P_n(q - f)\|_\infty \leq \|f - q\|_\infty + \|\lambda_n\|_\infty \|f - q\|_\infty$$

Therefore

$$\|f - P_n f\|_\infty \leq (1 + \|\lambda_n\|_\infty) \inf_{q \in \mathcal{P}_n} \|f - q\|_\infty \tag{1.28}$$

1.2.3 Estimates for the Lebesgue constant

For uniformly spaced interpolation points one can prove

$$\|\lambda_n\|_\infty \geq C e^{n/2}$$

This is illustrated by the Runge example where $\|P_n f\|_\infty$ grows exponentially. Unfortunately one can never achieve a bounded Lebesgue constant: For *any* sequence of interpolation nodes there holds

$$\|\lambda_n\|_\infty \geq \frac{2}{\pi} \log n - c.$$

However, the Chebyshev nodes give a Lebesgue constant which has the same growth rate as this lower bound: For Chebyshev nodes we have

Theorem 1.11 *The Lebesgue function for Chebyshev nodes* (1.23) *satisfies*

$$\|\lambda_n\|_\infty \leq \frac{2}{\pi} \log(n+1) + \frac{4}{\pi} \tag{1.29}$$

Proof: Note that we can write the Lagrange polynomial l_k in terms of the node polynomial $\omega(x) = (x - x_0) \cdots (x - x_n)$ as

$$l_k(x) = \frac{\omega(x)}{(x - x_k)\omega'(x_k)}$$

Using (1.22) and (1.12) and the change of variables $x = \cos\theta$, $x_k = \cos\theta_k$ with $\theta, \theta_k \in [0, \pi]$ we have

$$l_k(\cos\theta) = \frac{\cos(n+1)\theta}{(\cos\theta - \cos\theta_k)(n+1)\sin(n+1)\theta_k \left.\frac{d\theta}{dx}\right|_{x=x_k}}$$

Because of $\sin(n+1)\theta_k = \pm 1$ and $\frac{dx}{d\theta} = -\sin\theta$ this yields

$$\mu(\theta) := \lambda_n(\cos\theta) = \frac{|\cos(n+1)\theta|}{n+1} \sum_{k=0}^{n} \left| \frac{\sin\theta_k}{\cos\theta - \cos\theta_k} \right| \tag{1.30}$$

or, using the trigonometric identity $\sin a/(\cos a - \cos b) = (\cot\frac{a-b}{2} - \cot\frac{a+b}{2})/2$

$$\mu(\theta) = \frac{|\cos(n+1)\theta|}{2(n+1)} \sum_{k=0}^{n} \left| \cot\frac{\theta+\theta_k}{2} - \cot\frac{\theta-\theta_k}{2} \right| \tag{1.31}$$

We want to show that $\max_{x\in[-1,1]} \lambda_n(x) = \max_{\theta\in[0,\pi]} \mu(\theta)$ attains its maximal value in the interval $[0, \theta_0] = [0, \frac{\pi}{2(n+1)}]$. Let $\phi \in [0, \pi]$ arbitrary, then we can write $\phi = \phi_0 + l\pi/(n+1)$ with $\phi_0 \in [-\frac{\pi}{2(n+1)}, \frac{\pi}{2(n+1)}]$ and integer l. Since $|\cos t|$ is a π-periodic function we have $|\cos(n+1)\phi| = |\cos(n+1)\phi_0|$. For the sum we claim

$$\begin{aligned} \sum_{k=0}^{n} \left| \cot\frac{\phi+\theta_k}{2} - \cot\frac{\phi-\theta_k}{2} \right| &\le \sum_{k=0}^{n} \left(\left|\cot\frac{\phi+\theta_k}{2}\right| + \left|\cot\frac{\phi-\theta_k}{2}\right| \right) \\ = \sum_{k=0}^{n} \left(\left|\cot\frac{\phi_0+\theta_k}{2}\right| + \left|\cot\frac{\phi_0-\theta_k}{2}\right| \right) &= \sum_{k=0}^{n} \left| \cot\frac{\phi_0+\theta_k}{2} - \cot\frac{\phi_0-\theta_k}{2} \right| \end{aligned} \tag{1.32}$$

The $2(n+1)$ summands in the second and third sum are the same, but in different combinations since cot is π-periodic and

$$\{\, \phi + \theta_k, \phi - \theta_k \mid k = 0, \ldots, n \,\} = \{\, \phi + \frac{\pi}{2(n+1)} + k\frac{\pi}{n+1} \mid k = -n-1, \ldots, n \,\}.$$

The last equality in (1.32) follows from

$$0 \le \frac{\phi_0+\theta_k}{2} \le \frac{\pi}{2}, \quad -\frac{\pi}{2} \le \frac{\phi_0-\theta_k}{2} \le 0 \quad \Longrightarrow \quad \cot\frac{\phi_0+\theta_k}{2} \ge 0, \quad \cot\frac{\phi_0-\theta_k}{2} \le 0$$

Therefore we have (using that μ is an even function)

$$\mu(\phi) \le \mu(\phi_0) = \mu(|\phi_0|) \quad \Longrightarrow \quad \max_{\phi\in[0,\pi]} \mu(\phi) = \max_{\phi_0\in[0,\frac{\pi}{2(n+1)}]} \mu(\phi_0).$$

Now we consider $\theta \in [0, \frac{\pi}{2(n+1)}]$. Then $\cos(n+1)\theta \ge 0$, $\cos\theta - \cos\theta_k \ge 0$ and hence we obtain

$$\mu(\theta) = \lambda_n(\cos\theta) = \frac{1}{n+1} \sum_{k=0}^{n} \sin\theta_k \frac{\cos(n+1)\theta}{\cos\theta - \cos\theta_k} = \frac{1}{n+1} \sum_{k=0}^{n} (\sin\theta_k)\, 2^n \prod_{\substack{j=0\ldots n \\ j\ne k}} (\cos\theta - \cos\theta_j)$$

using (1.21). Since $\cos\theta - \cos\theta_j \geq 0$, each term in the sum is a nonnegative decreasing function of θ, therefore

$$\|\lambda_n\|_\infty = \max_{\theta\in[0,\frac{\pi}{2(n+1)}]} \mu(\theta) = \mu(0) \stackrel{(1.31)}{=} \frac{1}{2(n+1)} \sum_{k=0}^{n} 2\cot\frac{\theta_k}{2} \tag{1.33}$$

We see that the last expression is exactly the composite midpoint rule applied to the integral $\frac{1}{\pi}\int_0^\pi \cot(t/2)\,dt$ (which is infinite). We split off the first term and note that the integrand is convex ($f'' \geq 0$), hence the integral is larger than the midpoint rule approximation:

$$\begin{aligned}\|\lambda_n\|_\infty &= \frac{1}{n+1}\cot\frac{\theta_0}{2} + \frac{1}{n+1}\sum_{k=1}^{n}\cot\frac{\theta_k}{2} \leq \frac{1}{n+1}\cot\frac{\pi}{4(n+1)} + \frac{1}{\pi}\int_{\pi/(n+1)}^{\pi}\cot\frac{t}{2}\,dt \\ &= \frac{1}{n+1}\cot\frac{\pi}{4(n+1)} + \frac{1}{\pi}\left[2\log\left(\sin\frac{t}{2}\right)\right]_{\pi/(n+1)}^{\pi} = \frac{1}{n+1}\cot\frac{\pi}{4(n+1)} - \frac{2}{\pi}\log\left(\sin\frac{\pi}{2(n+1)}\right)\end{aligned}$$

Finally the inequalities $\tan t \geq t$ $(0 \leq t < \pi/2)$ and $\sin t \geq \frac{2}{\pi}t$ $(0 \leq t \leq \pi/2)$ give

$$\|\lambda_n\|_\infty \leq \frac{4}{\pi} + \frac{2}{\pi}\log(n+1) \qquad \square$$

Table 1 shows the values of $\|\lambda_n\|_\infty$ computed directly from (1.33) and the upper bounds

n	5	10	15	20	25
$\|\lambda_n\|_\infty$	2.104	2.489	2.728	2.901	3.037
$\frac{2}{\pi}\log(n+1)+\frac{4}{\pi}$	2.414	2.800	3.038	3.211	3.347

Table 1: Lebesgue constants for Chebychev nodes

from (1.29).

Because of (1.28) the interpolation $P_n f$ at Chebyshev nodes satisfies for $n \leq 20$

$$\|f - P_n f\|_\infty \leq 4 \inf_{q\in\mathcal{P}_n} \|f - q\|_\infty .$$

Therefore the Chebyshev interpolation gives almost the best possible approximation. Trying to find the *best* possible approximation (which can only be done approximately by an iterative procedure) can at most decrease the error by a factor of 4.

One can do even better by using the *expanded Chebyshev nodes*

$$\tilde{x}_j := x_j / x_0$$

which stretch the Chebyshev nodes so that the endpoints $\tilde{x}_0 = 1$ and $\tilde{x}_n = -1$ are nodes. In this case we have $\|\lambda_n\|_\infty \leq 2.01$ for $n \leq 9$ and $\|\lambda_n\|_\infty \leq 3$ for $n \leq 47$. Furthermore one can show that $\|\lambda_n\|_\infty$ is never more than 0.02 larger than the best possible value of the Lebesgue constant with $n+1$ nodes.