1 Polynomial approximation and interpolation

1.1 Jackson theorems

1.1.1 Polynomials $\mathcal{P}_n$ and trigonometric polynomials $\mathcal{T}_n$

In order to state the approximation problem we define the functions which we want to approximate and the functions which we want to use for approximation:

Definition 1.1 *We denote by $C^k[a,b]$ for $k = 0, 1, 2, \ldots$ the space of functions which have derivatives $f^{(1)}, \ldots, f^{(k)}$ that are continuous on the closed interval $[a,b]$.*

We denote by

$$\mathcal{P}_n = \{\, c_0 + c_1 x + \cdots + c_n x^n \mid c_k \in \mathbb{C} \,\}$$

the space of polynomials of degree less or equal to n ($n = 0, 1, 2, \ldots$).

Then the approximation problem is: Given $f \in C^k[a,b]$, what is the rate with which error of the best approximation

$$\inf_{p_n \in \mathcal{P}_n} \|f - p_n\|_\infty$$

converges to zero as n goes to infinity?

The so-called **Jackson theorems** shows that the decay rate of the error depends on the smoothness of the function f. E.g. for $f \in C^1[a,b]$ we will prove an approximation rate of $O(1/n)$, and for $f \in C^2[a,b]$ we will obtain an approximation rate of $O(1/n^2)$.

The problem of approximating functions on intervals by polynomials is closely related to the problem of approximating *periodic functions* by *trigonometric polynomials*:

Definition 1.2 *The space $C_{2\pi}$ of 2π-periodic functions consists of all functions $f \in C(\mathbb{R})$ which satisfy*

$$\forall x \in \mathbb{R} \qquad f(x) = f(x + 2\pi).$$

The space of k times continuously differentiable 2π-periodic functions is defined as $C^k_{2\pi} = C^k(\mathbb{R}) \cap C_{2\pi}$.

We denote by $\mathcal{T}_n$ the space of trigonometric polynomials of degree less or equal to n ($n = 0, 1, 2, \ldots$):

$$\mathcal{T}_n = \{\, a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \mid a_k, b_k \in \mathbb{C} \,\}$$

1.1.2 Interpolation with trigonometric polynomials

Note that the space $\mathcal{T}_n$ has dimension $2n+1$ (whereas $\mathcal{P}_n$ has dimension $n+1$). Therefore we can ask whether we can always find a trigonometric interpolation polynomial $p_n \in \mathcal{T}_n$ which passes through $2n+1$ given points (x_j, y_j), $j = 0, \ldots, 2n$. We first note that $\mathcal{T}_n$ has a property similar to $\mathcal{P}_n$:

Lemma 1.3 *A nonzero function $f \in \mathcal{T}_n$ has at most $2n$ zeros in $[0, 2\pi)$.*

Proof: Assume that $f \in \mathcal{T}_n$ has $2n+1$ zeros $\theta_0, \ldots, \theta_{2n}$ in $[0, 2\pi)$. Writing $\sin k\theta$ and $\cos k\theta$ in terms of $e^{ik\theta}$ and $e^{-ik\theta}$ and with $z := e^{i\theta}$ we have that

$$f(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta} = \sum_{k=-n}^{n} c_k z^k$$

equals zero for $z_j := e^{i\theta_j}$. Multiplying by z^n we obtain that $\sum_{k=-n}^{n} c_k z^{k+n}$ is a polynomial of degree $\leq 2n$ in z which has at least $2n+1$ zeros $z_j \in \mathbb{C}$. Hence it must be the zero polynomial and all $c_k = 0$. ❑

Corollary 1.4 *Let $x_0, \ldots, x_{2n}$ be distinct values in $[0, 2\pi)$. Then for any given values $y_0, \ldots, y_{2n}$ there exists a unique interpolating trigonometric polynomial $p_n \in \mathcal{T}_n$ which satisfies $p_n(x_j) = y_j$, $j = 0, \ldots, 2n$.*

Proof: The interpolation problem leads to a linear system of $2n+1$ equations for the $2n+1$ unknowns $a_0, \ldots, a_n$, $b_1, \ldots, b_n$ with the right hand side vector $(y_0, \ldots, y_{2n})^\top$. This system has a unique solution for every right hand side if the matrix is nonsingular. To show that the matrix is nonsingular consider the problem with the zero right hand side vector. Any solution of this linear system corresponds to a function $p_n \in \mathcal{T}_n$ which is zero in $x_0, \ldots, x_{2n}$. By Lemma 1.3 p_n must be zero. Hence the homogeneous linear sytem has only the zero solution and the matrix is nonsingular. ❑

1.1.3 An auxiliary approximation problem

As a first step toward proving the Jackson theorems let us consider the 2π periodic function f with $f(x) = x$ for $x \in (-\pi, \pi]$. In order to approximate it by a function in $\mathcal{T}_n$ we can use the interpolation p_n through the $2n+1$ nodes $k\frac{\pi}{n+1}$, $k = -n, \ldots, n$ which exists and is unique due to Corollary 1.4. Since the function f is odd, i.e., $f(-x) = -f(x)$ (for $x \neq k\pi$), the function $-p_n(-x) \in \mathcal{T}_n$ is also an interpolation. By the uniqueness of the interpolation, $p_n(x)$ and $-p_n(-x)$ must have the same coefficients and so we have that

$$p_n(x) = \sum_{k=1}^{n} b_k \sin kx. \tag{1.1}$$

Now we consider the interpolation error $e(x) := f(x) - p_n(x)$. Since p_n interpolates f in $2n+1$ points in $(-\pi, \pi)$, $e(x)$ has at least $2n+1$ simple zeros in $(-\pi, \pi)$. There cannot be more zeros in $(-\pi, \pi)$: If $e(x)$ has $2n+2$ zeros in $(-\pi, \pi)$, then $e'(x) = 1 - p_n'(x) \in \mathcal{T}_n$ has at least $2n+1$ zeros by Rolle's theorem. By Corollary 1.3 we then have $e'(x) = 0$ which is a contradiction. The same argument also shows that the $2n+1$ zeros of e in $(-\pi, \pi)$ are simple, i.e., $e'(x) \neq 0$ in those points. Hence the function $e(x)$ changes its sign in $(-\pi, \pi)$ only at the interpolation points $k\frac{\pi}{n+1}$, $k = -n, \ldots, n$. Let $s(x)$ denote the function which is alternatingly 1 and -1 between the nodes, i.e.,

$$s(x)|_{[k\frac{\pi}{n+1}, (k+1)\frac{\pi}{n+1})} = (-1)^k,$$

then we have

$$\int_{-\pi}^{\pi} |f(x) - p_n(x)| \, dx = \Big| \int_{-\pi}^{\pi} (f(x) - p_n(x)) s(x) \, dx \Big| = \Big| \int_{-\pi}^{\pi} f(x) s(x) \, dx - \int_{-\pi}^{\pi} p_n(x) s(x) \, dx \Big|.$$

Let us show that the second term on the right hand side must be 0: Consider

$$\begin{aligned} A &:= \int_{-\pi}^{\pi} s(x)\sin kx\,dx = -\int_{-\pi}^{\pi} s(x+\frac{\pi}{n+1})\sin kx\,dx = -\int_{-\pi}^{\pi} s(x)\sin k(x-\frac{\pi}{n+1})\,dx \\ &= -\int_{-\pi}^{\pi} s(x)\sin kx\cos(-k\frac{\pi}{n+1})\,dx - \int_{-\pi}^{\pi} \underbrace{s(x)}_{\text{odd}}\underbrace{\cos kx}_{\text{even}}\sin(-k\frac{\pi}{n+1})\,dx \\ &= -\cos(k\frac{\pi}{n+1})A + 0 \end{aligned}$$

Since $\cos(k\frac{\pi}{n+1}) \neq -1$ we obtain $A = 0$ and also $\int_{\pi}^{\pi} p_n(x)s(x)\,dx = 0$ because of (1.1). It remains to evaluate $\int_{\pi}^{\pi} f(x)s(x)\,dx$: It can be easily seen geometrically that we have

$$\Big|\int_{\pi}^{\pi} xs(x)\,dx\Big| = 2\Big|\int_{0}^{\pi} xs(x)\,dx\Big| = 2\left(\frac{\pi}{n+1}\right)^2\Big|-\frac{1}{2}+\frac{3}{2}-\cdots\pm(n+\frac{1}{2})\Big| = 2\left(\frac{\pi}{n+1}\right)^2\frac{n+1}{2} = \frac{\pi^2}{n+1}.$$

Hence we obtain the following result for the interpolation error:

$$\int_{\pi}^{\pi} |f(x)-p_n(x)|\,dx = \frac{\pi^2}{n+1}.$$

1.1.4 Jackson theorems for periodic functions

Now we can prove the Jackson theorem for $f \in C^1_{2\pi}$: Using integration by parts we see that

$$\int_{-\pi}^{\pi} \theta f'(\theta+x+\pi)\,d\theta = [\theta f(\theta+x+\pi)]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 1 f(\theta+x+\pi)\,d\theta = 2\pi f(x) - \int_{-\pi}^{\pi} f(\theta)\,d\theta$$

yielding

$$f(x) = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta}_{=:a_0=\text{const.}} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta f'(\theta+x+\pi)\,d\theta \tag{1.2}$$

In order to approximate $f(x)$ we replace θ in the second integral by the interpolation $p_n \in \mathcal{T}_n$ from above and obtain

$$q(x) = a_0 + \frac{1}{2\pi}\int_{-\pi}^{\pi} p_n(\theta)f'(\theta+x+\pi)\,d\theta = a_0 + \frac{1}{2\pi}\int_{-\pi}^{\pi} p_n(\theta-\pi-x)f'(\theta)\,d\theta \tag{1.3}$$

where

$$p_n(\theta-\pi-x) = \sum_{k=1}^{n} b_k \sin k(\theta-\pi-x) = \sum_{k=1}^{n} b_k\Big(\sin k(\theta-\pi)\underbrace{\cos kx} - \cos k(\theta-\pi)\underbrace{\sin kx}\Big). \tag{1.4}$$

If we insert this expression for $p_n(\theta-\pi-x)$ into the integral in (1.3) we see that the underbraced terms can be pulled out of the integral and that $q \in \mathcal{T}_n$.

It remains to estimate the approximation error:

$$\begin{aligned}\|f-q\|_\infty &= \sup_x\Big|\frac{1}{2\pi}\int_{-\pi}^{\pi}(\theta-p_n(\theta))f'(\theta+x+\pi)\,d\theta\Big| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|\theta-p_n(\theta)|\,|f'(\theta+x+\pi)|\,d\theta \\ &\le \frac{1}{2\pi}\|f'\|_\infty\int_{-\pi}^{\pi}|\theta-p_n(\theta)|\,d\theta = \frac{1}{2\pi}\|f'\|_\infty\frac{\pi^2}{n+1}\end{aligned}$$

We have therefore proved the following theorem:

Theorem 1.5 (Jackson theorem for $C^1_{2\pi}$) *For $f\in C^1_{2\pi}$ there holds*

$$\inf_{p\in\mathcal{T}_n}\|f-p\|_\infty \le \frac{\pi}{2(n+1)}\|f'\|_\infty \tag{1.5}$$

If the function $f(x)$ has mean 0, i.e., $\int_{-\pi}^{\pi}f(x)\,dx=0$, then the approximating function $q(x)$ has the same property: The constant a_0 in (1.2) is 0, and (1.3), (1.4) show that $q(x)$ is of the form $\sum_{k=1}^n(a_k\cos kx+b_k\sin kx)$. Let us define $\mathcal{T}_N^0$ as the space of trigonometric polynomials of degree less or equal n with mean 0:

$$\mathcal{T}_N^0 := \{\, p\in\mathcal{T}_N \mid \int_{-\pi}^{\pi}p(x)\,dx=0\,\} = \{\,\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx) \mid a_k,b_k\in\mathbb{C}\,\}$$

We then have

Lemma 1.6 *Let $f\in C^1_{2\pi}$ with $\int_{-\pi}^{\pi}f(x)\,dx=0$. Then*

$$\inf_{p\in\mathcal{T}_N^0}\|f-p\|_\infty \le \frac{\pi}{2(n+1)}\|f'\|_\infty\,. \tag{1.6}$$

Next, we would like to investigate the approximation rates for functions with more smoothness, e.g., $f\in C^3_{2\pi}$. We can use the following "bootstrap" argument: Assume that $f\in C^1_{2\pi}$. Let $\tilde p$ be an arbitrary trigonometric polynomial in $\mathcal{T}_n$. Then we have obviously

$$\inf_{p\in\mathcal{T}_n}\|f-p\|_\infty = \inf_{p\in\mathcal{T}_n}\|(f-\tilde p)-p\|_\infty \le \frac{\pi}{2(n+1)}\|f'-\tilde p'\|_\infty \tag{1.7}$$

using (1.5) for $(f-\tilde p)$. Note that (1.7) holds for any function $\tilde p\in\mathcal{T}_n$, so we can choose it in such a way that the right hand side becomes as small as possible. Since

$$\{\,\tilde p' \mid \tilde p\in\mathcal{T}_n\,\} = \mathcal{T}_n^0,$$

equation (1.7) implies

$$\inf_{p\in\mathcal{T}_n}\|f-p\|_\infty \le \frac{\pi}{2(n+1)}\inf_{p\in\mathcal{T}_n^0}\|f'-p\|_\infty \tag{1.8}$$

For approximation errors in $\mathcal{T}_n^0$ we can use a similar argument: Assume $f\in C^1_{2\pi}$ with $\int_{-\pi}^{\pi}f(x)\,dx=0$. Using (1.6) and

$$\{\,\tilde p' \mid \tilde p\in\mathcal{T}_n^0\,\} = \mathcal{T}_n^0,$$

we obtain analogously

$$\inf_{p\in\mathcal{T}_n^0}\|f-p\|_\infty \le \frac{\pi}{2(n+1)}\inf_{p\in\mathcal{T}_n^0}\|f'-p\|_\infty\,. \tag{1.9}$$

This allows us to prove

Theorem 1.7 (Jackson theorem for $C^k_{2\pi}$) *Let $k \in \{1, 2, \dots\}$ and $f \in C^k_{2\pi}$. Then*

$$\inf_{p\in\mathcal{T}_n} \|f - p\|_\infty \le \left(\frac{\pi}{2(n+1)}\right)^k \left\|f^{(k)}\right\|_\infty \tag{1.10}$$

Proof: Note that $f', \dots, f^{(k-1)}$ have mean 0 since f is periodic. Applying (1.8), (1.9) ($k-2$ times) and finally (1.6) we get

$$\begin{aligned}\inf_{p\in\mathcal{T}_n} \|f - p\|_\infty &\overset{(1.8)}{\le} \frac{\pi}{2(n+1)} \inf_{p\in\mathcal{T}_n^0} \|f' - p\|_\infty \overset{(1.9)}{\le} \left(\frac{\pi}{2(n+1)}\right)^2 \inf_{p\in\mathcal{T}_n^0} \|f'' - p\|_\infty \\ &\overset{(1.9)}{\le} \cdots \overset{(1.9)}{\le} \left(\frac{\pi}{2(n+1)}\right)^{k-1} \inf_{p\in\mathcal{T}_n^0} \left\|f^{(k-1)} - p\right\|_\infty \overset{(1.6)}{\le} \left(\frac{\pi}{2(n+1)}\right)^k \left\|f^{(k)}\right\|_\infty .\end{aligned}$$

❑

1.1.5 Jackson theorems on an interval

For simplicity we first consider the interval $[-1, 1]$ and consider a function $f \in C^1[-1, 1]$. We can transform this function to a 2π-periodic function g by the change of variables

$$g(\theta) = f(\cos\theta) \tag{1.11}$$

Note that the function g is even. This transformation is useful since polynomials $f(x)$ of degree less or equal n are transformed into even trigonometric polynomials $g(\theta)$ and vice versa. If $g \in \mathcal{T}_m$ and $h \in \mathcal{T}_n$ then the product $g \cdot h$ is in $\mathcal{T}_{m+n}$ as can be seen, e.g., by writing $\cos x$ and $\sin x$ in terms of $e^{\pm ix}$. If $f \in \mathcal{P}_n$ then we see that g given by (1.11) is in $\mathcal{T}_n$ and even. It is also clear that g can only be the zero function if f is the zero function. Since $\dim \mathcal{T}_n^{\text{even}} = \dim \mathcal{P}_n = n + 1$ it follows that the transformation (1.11) gives a one-to-one linear mapping between $\mathcal{P}_n$ and $\mathcal{T}_n^{\text{even}}$.

The functions $\cos kx$, $k = 0, \dots, n$ form a basis of $\mathcal{T}_n^{\text{even}}$. These functions are transformed by (1.11) to certain polynomials $T_k(x)$ with

$$\cos k\theta = T_k(\cos\theta). \tag{1.12}$$

These are the so-called *Chebyshev polynomials*. Obviously, $T_0(x) = 1$ and $T_1(x) = x$. If we add the formulae $\cos(k \pm 1)x = \cos kx \cos x \mp \sin kx \sin x$ we obtain $\cos(k + 1)x = 2\cos kx \cos x - \cos(k - 1)x$ which gives the recursion formula

$$T_{k+1}(x) = 2T_k(x)x - T_{k-1}(x). \tag{1.13}$$

We can now prove an approximation result for polynomials on an interval:

Theorem 1.8 (Jackson theorem for $C^1[-1, 1]$) *For $f \in C^1[-1, 1]$ there holds*

$$\inf_{p\in\mathcal{P}_n} \|f - p\|_\infty \le \frac{\pi}{2(n+1)} \|f'\|_\infty \tag{1.14}$$

Proof: Define g by (1.11). Then $g'(\theta) = -f'(\cos\theta)\sin\theta$. Therefore the limits $\lim_{\theta\to\pm\pi} g'(\theta)$ exist and are equal (both are 0), hence $g \in C^1_{2\pi}$. We also see that

$$\|g'\|_\infty \le \|f'\|_\infty \,. \tag{1.15}$$

Consider an approximation $p \in \mathcal{T}_n$ for g. Since p need not be even we consider the symmetrized function $\tilde{p}(\theta) = (p(\theta) + p(-\theta))/2$. Then we see

$$\begin{aligned} \|g(\theta) - \tilde{p}(\theta)\|_\infty &= \tfrac{1}{2} \|g(\theta) + g(-\theta) - p(\theta) - p(-\theta)\|_\infty \\ &\le \tfrac{1}{2}(\|g(\theta) - p(\theta)\|_\infty + \|g(-\theta) - p(-\theta)\|_\infty) = \|g(\theta) - p(\theta)\|_\infty \end{aligned}$$

and this implies

$$\inf_{p\in\mathcal{T}_n^{\text{even}}} \|g - p\|_\infty = \inf_{p\in\mathcal{T}_n} \|g - p\|_\infty \overset{(1.5)}{\le} \frac{\pi}{2(n+1)} \|g'\|_\infty \tag{1.16}$$

The change of variables $q(\cos\theta) = p(\theta)$ defines for $p \in \mathcal{T}_n^{\text{even}}$ a function $q \in \mathcal{P}_n$ such that

$$\max_{x\in[-1,1]} |f(x) - q(x)| = \max_{\theta\in[-\pi,\pi)} |g(\theta) - p(\theta)| \,. \tag{1.17}$$

Equations (1.17), (1.16), and (1.15) together yield (1.14). ❏

In the same way as we proved Theorem 1.7 using Theorem 1.5 we can use Theorem 1.8 to prove an approximation result for $f \in C^k[-1,1]$:

Theorem 1.9 (Jackson theorem for $C^k[-1,1]$) *Let n, k be integers with $n \ge k-1 \ge 0$ and $f \in C^k[-1,1]$. Then*

$$\inf_{p\in\mathcal{P}_n} \|f - p\|_\infty \le \left(\frac{\pi}{2}\right)^k \frac{1}{(n+1)n\cdots(n-k+2)} \left\|f^{(k)}\right\|_\infty \tag{1.18}$$