

# CONTENTS

**Preface xi**

**Preliminaries 1**

## 1

---

**TOOLS FOR ANALYSIS 5**

- 1.1 The Completeness Axiom and Some of Its Consequences 5
- 1.2 The Distribution of the Integers and the Rational Numbers 12
- 1.3 Inequalities and Identities 16

## 2

---

**CONVERGENT SEQUENCES 23**

- 2.1 The Convergence of Sequences 23
- 2.2 Sequences and Sets 35
- 2.3 The Monotone Convergence Theorem 38
- 2.4 The Sequential Compactness Theorem 43
- 2.5 Covering Properties of Sets\* 47

## 3

---

**CONTINUOUS FUNCTIONS 53**

- 3.1 Continuity 53
- 3.2 The Extreme Value Theorem 58
- 3.3 The Intermediate Value Theorem 62
- 3.4 Uniform Continuity 66
- 3.5 The  $\epsilon$ - $\delta$  Criterion for Continuity 70
- 3.6 Images and Inverses; Monotone Functions 74
- 3.7 Limits 81

## 4

---

**DIFFERENTIATION 87**

- 4.1 The Algebra of Derivatives 87
- 4.2 Differentiating Inverses and Compositions 96
- 4.3 The Mean Value Theorem and Its Geometric Consequences 101
- 4.4 The Cauchy Mean Value Theorem and Its Analytic Consequences 111
- 4.5 The Notation of Leibnitz 113

## 5\*

---

### ELEMENTARY FUNCTIONS AS SOLUTIONS OF DIFFERENTIAL EQUATIONS 116

- 5.1 Solutions of Differential Equations 116
- 5.2 The Natural Logarithm and Exponential Functions 118
- 5.3 The Trigonometric Functions 125
- 5.4 The Inverse Trigonometric Functions 132

## 6

---

### INTEGRATION: TWO FUNDAMENTAL THEOREMS 135

- 6.1 Darboux Sums; Upper and Lower Integrals 135
- 6.2 The Archimedes–Riemann Theorem 142
- 6.3 Additivity, Monotonicity, and Linearity 150
- 6.4 Continuity and Integrability 155
- 6.5 The First Fundamental Theorem: Integrating Derivatives 160
- 6.6 The Second Fundamental Theorem: Differentiating Integrals 165

## 7\*

---

### INTEGRATION: FURTHER TOPICS 175

- 7.1 Solutions of Differential Equations 175
- 7.2 Integration by Parts and by Substitution 178
- 7.3 The Convergence of Darboux and Riemann Sums 183
- 7.4 The Approximation of Integrals 190

## 8

---

### APPROXIMATION BY TAYLOR POLYNOMIALS 199

- 8.1 Taylor Polynomials 199
- 8.2 The Lagrange Remainder Theorem 203
- 8.3 The Convergence of Taylor Polynomials 209
- 8.4 A Power Series for the Logarithm 212
- 8.5 The Cauchy Integral Remainder Theorem 215
- 8.6 A Nonanalytic, Infinitely Differentiable Function 221
- 8.7 The Weierstrass Approximation Theorem 223

## 9

---

### SEQUENCES AND SERIES OF FUNCTIONS 228

- 9.1 Sequences and Series of Numbers 228
- 9.2 Pointwise Convergence of Sequences of Functions 241

- 9.3 Uniform Convergence of Sequences of Functions 245
- 9.4 The Uniform Limit of Functions 249
- 9.5 Power Series 255
- 9.6 A Continuous Nowhere Differentiable Function 264

## 10

---

### THE EUCLIDEAN SPACE $\mathbb{R}^n$ 269

- 10.1 The Linear Structure of  $\mathbb{R}^n$  and the Scalar Product 269
- 10.2 Convergence of Sequences in  $\mathbb{R}^n$  277
- 10.3 Open Sets and Closed Sets in  $\mathbb{R}^n$  282

## 11

---

### CONTINUITY, COMPACTNESS, AND CONNECTEDNESS 290

- 11.1 Continuous Functions and Mappings 290
- 11.2 Sequential Compactness, Extreme Values,  
and Uniform Continuity 298
- 11.3 Pathwise Connectedness and the Intermediate Value Theorem\* 304
- 11.4 Connectedness and the Intermediate Value Property\* 310

## 12\*

---

### METRIC SPACES 314

- 12.1 Open Sets, Closed Sets, and Sequential Convergence 314
- 12.2 Completeness and the Contraction Mapping Principle 322
- 12.3 The Existence Theorem for Nonlinear Differential Equations 328
- 12.4 Continuous Mappings between Metric Spaces 337
- 12.5 Sequential Compactness and Connectedness 342

## 13

---

### DIFFERENTIATING FUNCTIONS OF SEVERAL VARIABLES 348

- 13.1 Limits 348
- 13.2 Partial Derivatives 353
- 13.3 The Mean Value Theorem and Directional Derivatives 364

## 14

---

### LOCAL APPROXIMATION OF REAL-VALUED FUNCTIONS 372

- 14.1 First-Order Approximation, Tangent Planes, and Affine Functions 372
- 14.2 Quadratic Functions, Hessian Matrices, and Second Derivatives\* 380
- 14.3 Second-Order Approximation and the Second-Derivative Test\* 387

## 15

---

### APPROXIMATING NONLINEAR MAPPINGS BY LINEAR MAPPINGS 394

- 15.1 Linear Mappings and Matrices 394
- 15.2 The Derivative Matrix and the Differential 407
- 15.3 The Chain Rule 414

## 16

---

### IMAGES AND INVERSES: THE INVERSE FUNCTION THEOREM 421

- 16.1 Functions of a Single Variable and Maps in the Plane 421
- 16.2 Stability of Nonlinear Mappings 429
- 16.3 A Minimization Principle and the General Inverse Function Theorem 433

## 17

---

### THE IMPLICIT FUNCTION THEOREM AND ITS APPLICATIONS 440

- 17.1 A Scalar Equation in Two Unknowns: Dini's Theorem 440
- 17.2 The General Implicit Function Theorem 449
- 17.3 Equations of Surfaces and Paths in  $\mathbb{R}^3$  454
- 17.4 Constrained Extrema Problems and Lagrange Multipliers 460

## 18

---

### INTEGRATING FUNCTIONS OF SEVERAL VARIABLES 470

- 18.1 Integration of Functions on Generalized Rectangles 470
- 18.2 Continuity and Integrability 482
- 18.3 Integration of Functions on Jordan Domains 489

## 19

---

### ITERATED INTEGRATION AND CHANGES OF VARIABLES 498

- 19.1 Fubini's Theorem 498
- 19.2 The Change of Variables Theorem: Statements and Examples 505
- 19.3 Proof of the Change of Variables Theorem 510

## 20

---

### LINE AND SURFACE INTEGRALS 520

- 20.1 Arclength and Line Integrals 520
- 20.2 Surface Area and Surface Integrals 533
- 20.3 The Integral Formulas of Green and Stokes 543

## **A**

---

### **CONSEQUENCES OF THE FIELD AND POSITIVITY AXIOMS 559**

A.1 The Field Axioms and Their Consequences 559

A.2 The Positivity Axioms and Their Consequences 563

## **B**

---

### **LINEAR ALGEBRA 565**

**Index 581**