# 1 Complex Manifolds, Almost Complex Structures, and Integrability

**Def**: A complex manifold is a smooth manifold with complex valued coordinate functions that depend on one another holomorphically on coordinate patch intersections.

In other words, if $z_j = x_j + iy_j$ and $w_k = u_k + iv_k$, the we have the Cauchy-Riemann equations:
$\frac{\partial u_k}{\partial x_j} = \frac{\partial v_k}{\partial y_j}$
$\frac{\partial v_k}{\partial x_j} = -\frac{\partial u_k}{\partial y_j}$.

A complex manifold of dimension $n$ is a real manifold of dimension $2n$ with extra structure. Let $M$ be a complex manifold of dimension $n$. The real tangent bundle of $M$ is a real vector bundle of dimension $2n$ with an extra structure, $J$ such that
$X \to JX, J^2 = -Id$, and
$J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}$
$J(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j}$.

It is essential to consider *all* complex valued differentiable functions on $M$, e.g. $\bar{z}_j = x_j - iy_j$.

**Def** : A holomorphic function, $f(z_i, \bar{z}_j)$, is a function such that $iXf = JXf$, $\forall$ tangent vectors $X$.

**Def** : An almost complex structure, $J$, is a fibrewise operator on the tangent bundle of an even dimensional smooth manifold with $J^2 \equiv 1$. Note that an almost complex structure does not imply that the manifold admits a complex structure. Let $J$ be an almost complex structure. Consider the following terms:
$[X, JY] \qquad [JX, Y] \qquad J[X, Y] \qquad J[JX, JY]$.

We want a quantity that involves these components and is tensorial. Using our definition of holomorphicity, we are able to derive the Newlander-Nirenberg Tensor:
$[X, JY] - [JX, Y] - J[X, Y] - J[JX, JY]$.

This tensor has the property that it vanishes on complex manifolds. We are then able to understand the Integrability Theorem.

**Theorem** : If an even dimensional manifold, $M$, admits an almost complex structure and its associated Newlander-Nirenberg tensor vanishes everywhere, then $M$ admits a complex manifold structure, $J$, with $J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}$ on any coordinate patch.

# 2 Riemann Surfaces

Let $X$ be an oriented Riemannian surface. Then there is a unique almost complex structure $J$ such that

$$\begin{aligned} < JX, JX > &= < X, X > \\ < X, JX > &= 0 \\ w(X, JX) &> 0 \end{aligned}$$

Where $w$ is the volume form.

- Let $s$, $t$ be local coordinates on $S$.
- Set $J\frac{\partial}{\partial s} = \alpha\frac{\partial}{\partial s} + \beta\frac{\partial}{\partial t}$
- Let $z = x + \imath y$ be a locally defined complex valued function on $S$.

We would like to count $z$ as a complex analytic coordinate provided

$$J\frac{\partial}{\partial s}z = \imath\frac{\partial}{\partial s}z$$

What we must know is that such a function exist, and that any two of them are complex analytic functions of one another.

$$\begin{aligned}
\alpha\frac{\partial z}{\partial s} + \beta\frac{\partial z}{\partial t} &= \imath\frac{\partial z}{\partial s} \\
\alpha\frac{\partial x}{\partial s} + \beta\frac{\partial x}{\partial t} &= -\frac{\partial y}{\partial s} \\
\alpha\frac{\partial y}{\partial s} + \beta\frac{\partial y}{\partial t} &= \frac{\partial x}{\partial s}
\end{aligned}$$

It is a generality of PDE theory that there is at least one nonconstant local solution. Suppose $w = u + \imath v$ is a function of $z = x + \imath y$.

$$\begin{aligned}
\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \\
\frac{\partial v}{\partial s} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s}
\end{aligned}$$

$$\begin{aligned}
\alpha\frac{\partial u}{\partial s} + \beta\frac{\partial u}{\partial t} &= -\frac{\partial v}{\partial s} \\
\alpha\frac{\partial u}{\partial s} + \beta\frac{\partial u}{\partial t} &= \alpha\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\right) + \beta\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\right) \\
&= \frac{\partial u}{\partial x}\left(\alpha\frac{\partial x}{\partial s} + \beta\frac{\partial x}{\partial t}\right) + \frac{\partial u}{\partial y}\left(\alpha\frac{\partial y}{\partial s} + \beta\frac{\partial y}{\partial t}\right) \\
&= -\frac{\partial u}{\partial x}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial x}{\partial s} \\
-\frac{\partial v}{\partial s} &= -\frac{\partial v}{\partial x}\frac{\partial x}{\partial s} - \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} \\
-\frac{\partial u}{\partial x}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial x}{\partial s} &= -\frac{\partial v}{\partial x}\frac{\partial x}{\partial s} - \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} \\
0 &= \frac{\partial x}{\partial s}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \frac{\partial y}{\partial s}\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)
\end{aligned}$$

$$
\begin{aligned}
\alpha\frac{\partial v}{\partial s}+\beta\frac{\partial v}{\partial t} &= \frac{\partial u}{\partial s}\\
\alpha\frac{\partial v}{\partial s}+\beta\frac{\partial v}{\partial t} &= \alpha\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial s}\right)+\beta\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial t}\right)\\
&= \frac{\partial v}{\partial x}\left(\alpha\frac{\partial x}{\partial s}+\beta\frac{\partial x}{\partial t}\right)+\frac{\partial v}{\partial y}\left(\alpha\frac{\partial y}{\partial s}+\beta\frac{\partial y}{\partial t}\right)\\
&= -\frac{\partial v}{\partial x}\frac{\partial y}{\partial s}+\frac{\partial v}{\partial y}\frac{\partial x}{\partial s}\\
\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\\
-\frac{\partial v}{\partial x}\frac{\partial y}{\partial s}+\frac{\partial v}{\partial y}\frac{\partial x}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\\
0 &= \frac{\partial x}{\partial s}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\frac{\partial y}{\partial s}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial x}{\partial s}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) &- \frac{\partial y}{\partial s}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)=0\\
\frac{\partial x}{\partial s}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) &+ \frac{\partial y}{\partial s}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=0
\end{aligned}
$$

Either $\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}$ are zero, else $\frac{\partial x}{\partial s}$ and $\frac{\partial y}{\partial s}$ are zero, but the latter would imply that $x$ and $y$ are independent of $s$ so that $\frac{\partial z}{\partial s}=0$. But then $J\frac{\partial}{\partial s}z$ would have to vanish identically as well, implying that $z$ is constant.

## 3 Holomorphic vector bundles and holomorphic connections

A holomorphic vector bundle over a complex manifold is a smooth vector bundle locally generated by holomorphic sections. More explicitly, over any sufficiently small open subset

1. there exists a basis $\eta_1, ..., \eta_n$ consisting of holomorphic sections
2. the holomorphic sections are a module over the ring of holomorphic functions
3. the matrix converting any two holomorphic bases has holomorphic entries

Examples:

a. the tangent bundle of a complex manifold

b. the canonical complex line bundle over $\mathbb{C}\mathrm{P}^n$

A unitary metric on a complex line bundle is a metric $<,>$ on each fibre so that $<\eta,\chi>$ depends smoothly on the components of $\eta$ and $\chi$ with respect to any local trivialization and,

(i) $<\chi,\eta> = \overline{<\eta,\chi>}$

(ii) $<\eta,\eta> > 0$ if $\eta \neq 0$

(iii) $<\alpha\eta,\beta\chi> = \alpha\overline{\beta} <\eta,\chi>$

**Theorem 1** *Let $M$ be a complex manifold, and let $E$ be a holomorphic vector bundle over $M$ with inner product $<\,,\,>$. Then there is a unique connection $\nabla$ on $E$ such that*

1. *$X<\eta,\chi> = <\nabla_X\eta,\chi> + <\eta,\nabla_X\chi>$ where $X$ is a real vector field on $M$ and $\eta$ and $\chi$ are any sections (not necessarily holomorphic) of $E$.*

2. *Extending $\nabla$ by complex linearity to take "subscripts" in $T(M)\otimes\mathbb{C}$, then $\nabla_{\overline{z}}\eta = 0$ where $z \in T_{\mathbb{C}}(M)$ and $\eta$ is a holomorphic section of $E$.*

**Proof:** Suppose existence, then $\nabla$ is determined by $\nabla_z$ for $z \in T_{\mathbb{C}}(M)$, since it suffices to compute $\nabla$ for a basis of holomorphic sections and $\nabla_{\overline{z}}$ vanishes on such a basis. For a connection satisfying 1. and 2., letting $z = X + iJX$, we have $z<\eta_i,\eta_j> = <\nabla_z\eta_i,\eta_j> + <\eta_i,\nabla_{\overline{z}}\eta_j>$. But since $<\nabla_z\eta_i,\eta_j>$ for all $i$ precisely determine $\nabla_z\eta_i$, we have uniqueness. To show existence define by above relations to achieve local existence, which in turn implies global existence by the uniqueness already established. $\square$

We call such a connection a *holomorphic connection.*

## 4 Kähler manifolds

Let $<\,,\,>$ be a Riemannian metric on $T(M)$ satisfying $<X,Y> = <JX,JY>$ and $<X,JX> = 0$, then we can extend $<\,,\,>$ as an unitary metric on $T(M)\otimes\mathbb{C}$, which restricts to a unitary metric on the holomorphic subbundle $T_{\mathbb{C}}(M)$.

We have both the Riemannian connection and the holomorphic connection whose existence and uniqueness we proved above.

They do not, in general, coincide. If they do then $<,>$ is called a *Kähler metric* and $M$ is called a *Kähler manifold.*

Both the Riemannian connection and the holomorphic connection respect $<\,,\,>$. In addition, the Riemannian connection is symmetric and the holomorphic connection respects $J$.

**Theorem 2** *Let $M$ be a manifold with metric $<\,,\,>$ and almost complex structure $J$ such that $<JX,JY> = <X,Y>$ and $<JX,X> = 0$. Let $\nabla$ be the Riemannian connection. Then if $\nabla$ respects $J$ in the sense that $\nabla_X JY = J\nabla_X Y$, then*

1. *$J$ is integrable*

*2. the 2-form $\omega$ given by $\omega(X,Y) =< X, JY >$ is closed.*

**Proof:** $d\omega(X,Y,Z) = X\omega(Y,Z) - \omega([X,Y],Z) - \omega(Y,[X,Z]) - Y\omega(X,Z) + \omega(X,[Y,Z]) + Z\omega(X,Y) = X < Y, JZ > - < \nabla_X Y, JZ > + < \nabla_Y X, JZ > - < Y, \nabla_X JZ > + < Y, \nabla_Z JX > -Y < X, JZ > + < X, \nabla_Y JZ > - < X, \nabla_Z JY > +Z < X, JY >= 0$, which establishes 2.

Let $N$ be the Newlander-Nirenberg tensor, then $< N(X,Y), Z >=< [JX,Y], Z > + < [X,JY], Z > - < J[X,Y], Z > + < J[JX,JY], Z >= 0$, since the Riemannian metric is invariant over $J$. Thus we have the integrability of $J$. $\Box$

**Remark 1** *All Kähler manifolds have nonvanishing real cohomology in dimension 2.*