Pergamon

Topology Vol. 35, No. 3, pp. 731–748, 1996

0040-9383/96 $15.00 + 0.00

0040-9383(95)00041-0

BIRATIONAL EQUIVALENCES OF VORTEX MODULI

STEVEN B. BRADLOW†, GEORGIOS D. DASKALOPOULOS‡ and RICHARD A. WENTWORTH§

(*Received* 3 *May* 1994; *in revised form* 28 *March* 1995)

WE CONSTRUCT a finite-dimensional Kähler manifold with a holomorphic, symplectic circle action whose symplectic reduced spaces may be identified with the τ-vortex moduli spaces (or τ-stable pairs). The Morse theory of the circle action induces natural birational maps between the reduced spaces for different values of τ which in the case of rank two bundles can be canonically resolved in a sequence of blow-ups and blow-downs. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

For holomorphic bundles over a Riemann surface there is essentially one notion of stability, and hence a single moduli space for bundles of fixed rank and degree. This rigidity can disappear when one considers moduli of bundles over higher-dimensional varieties or when one considers bundles with additional structure, such as parabolic bundles. The concept of stability can then depend on parameters, and one can get families of moduli. In this paper we explore this phenomenon in the case of holomorphic bundles with prescribed global sections — the so-called holomorphic pairs. The point of view we take is inspired by Morse theory and symplectic geometry.

In [4, 5] we introduced a notion of stability for a pair (E, ϕ) consisting of a holomorphic bundle together with a holomorphic section. The definition involves a real valued parameter and can be stated as follows.

Definition 1.1. Let $E \to \Sigma$ be a holomorphic vector bundle over a compact Riemann surface Σ. Let $\phi \in H^0(\Sigma, E)$ be a holomorphic section, and let τ be a real number. We say that the pair (E, ϕ) is τ-stable (resp. τ-semistable) if the following two conditions hold:

(i) $\deg(F)/rk(F) < \tau$ (resp. $\leqslant \tau$), for every nontrivial holomorphic subbundle $F \subset E$;

(ii) $\deg(E/F)/rk(E/F) > \tau$ (resp. $\geqslant \tau$), for every proper holomorphic subbundle $F \subset E$ such that ϕ is a section of F.

Throughout the paper, we shall denote the rank of E by R, the degree of E by d, and the genus of Σ by g. We shall also assume that $g \geqslant 2$, $R \geqslant 2$, and that $d > R(2g - 2)$

†Supported in part by NSF grant DMS-9103950 and an NSF-NATO Postdoctoral Fellowship.
‡Supported in part by NSF grant DMS-9303494.
§Supported in part by NSF Mathematics Postdoctoral Fellowship DMS-9007255.

(cf. Assumption 2 of [5]). Definition 1.1, and specifically the origin of the parameter τ, is motivated by a correspondence between stability criteria and the existence of special bundle metrics. In the case of holomorphic bundles, this is the Hitchin–Kobayashi correspondence between stability and the Hermitian–Einstein condition [14]. For bundles over closed Riemann surfaces, the topology of the bundle admits no ambiguity in the definition of stability. Now the Hermitian–Einstein equations admit a natural modification which is appropriate when a global section is prescribed [4]. The new equations, called the vortex equations, are obtained by adding extra terms which involve only the global section. These terms are not subject to any topological constraint, and thus unlike the Hermitian–Einstein equations, the vortex equations involve a true parameter. Tracing back the Hitchin–Kobayashi correspondence one is led from the vortex equations to the above notion of stability. Since the equations have a parameter, so does the notion of stability.

The impact of the parameter τ is shaped primarily by two things. Firstly, for purely numerical reasons, at almost all values of τ, the strict inequalities in Definition 1.1 are equivalent to weak inequalities. At only a discrete set of values (specifically, rational numbers whose denominator is strictly between 0 and $\mathrm{Rank}(E)$) is equality possible. Let us call these values the critical values of τ. Secondly, for values of τ between any two successive critical values, the definitions of stability are entirely equivalent. Furthermore, for these "generic" values of τ we get good moduli spaces of τ-stable pairs. Specifically, we have:

THEOREM 1.2 (cf. [2, 4, 5, 8, 18]). *Let $\mathscr{B}_\tau$ denote the set of isomorphism classes of τ-stable pairs on E. If τ is not a positive rational number with denominator less than R, then $\mathscr{B}_\tau$ naturally has the structure of a compact Kähler manifold of dimension $d + (R^2 - R)(g - 1)$. Indeed, $\mathscr{B}_\tau$ is an algebraic variety (see Theorem 4.6), and the same result holds for $\mathscr{B}_\tau(L)$, where $L \to \Sigma$ is a degree d line bundle and $\mathscr{B}_\tau(L)$ denotes the space of τ-stable pairs with fixed determinant L.*

If we think of the parameter as a "height function" and the moduli spaces as "level sets", then these features strongly suggest a Morse theory interpretation. In this paper we will show that this is more than simply an analogy. We will give a precise way of realizing just such a picture. In fact, the parameter τ can be realized as the Morse function arising from a symplectic moment map. This is not too surprising since it is well known that the equations corresponding to stability criteria (i.e. the Hermitian–Einstein and Vortex equations) have moment map interpretations. By using this aspect of the problem, we relate the present situation to a phenomenon studied in symplectic geometry, i.e. the variation of symplectically reduced level sets of moment maps [11].

In essence what we do is to construct a single large "master space" (the terminology is due to Bertram) which contains the stable and semistable pairs for all values of the parameter τ. The space has a symplectic structure and a symplectic circle action. We detect τ as the value of the moment map for this circle action, and we recover the moduli spaces of τ-stable pairs as the Marsden–Weinstein reductions for different values of this moment map. Stated more precisely, we prove the following:

THEOREM 1.3. *Consider holomorphic pairs on E over Σ. There is a compact topological space $\hat{\mathscr{B}}$ whose points correspond to holomorphic pairs which are τ-semistable for at least one value of τ. Furthermore, there is an open set $\hat{\mathscr{B}}_0 \subset \hat{\mathscr{B}}$ which has a natural Kähler manifold structure. The spaces $\hat{\mathscr{B}}$ and $\hat{\mathscr{B}}_0$ have the following properties:*

(i) *There is a quasi-free holomorphic and symplectic $U(1)$-action on $\hat{\mathscr{B}}$, i.e. an action for which the isotropy subgroup is either trivial or the whole $U(1)$.*

(ii) *There is a moment map* $f: \hat{\mathscr{B}}_0 \to \mathbb{R}$ *for this* $U(1)$*-action which extends continuously to* $\hat{\mathscr{B}}$. *The critical values for* f *are precisely the critical values of the parameter* τ.

(iii) *The level sets* $f^{-1}(\tau)$ *are* $U(1)$*-invariant. At regular values, the orbit spaces* $f^{-1}(\tau)/U(1)$ *can be identified as Kähler manifolds with the moduli spaces* $\mathscr{B}_\tau$. *At critical values the orbit spaces correspond to the spaces of isomorphism classes of semistable pairs.*

In the case of rank two bundles we can use our construction to recover some of the beautiful results of Thaddeus [18] (see also Theorem 4.9 below). Using techniques from Geometric Invariant Theory, Thaddeus constructed and analyzed the moduli spaces of τ-stable pairs with fixed determinant and rank two. He showed that for values of τ separated by a single critical value, the moduli spaces are related by flip in the sense of Mori theory. That is, the spaces are birationally equivalent projective varieties with a common blow-up. In our master space construction this phenomenon has an explanation both from the symplectic point of view as well as in terms of the Morse theory. From the symplectic point of view it corresponds exactly to the relationship between reduced level sets of moment maps as described by Guillemin and Sternberg in [11]. In terms of the Morse theory, the birationality of the level sets comes from a map induced by flows along the gradient lines of the Morse function. The centers of the blow-up in the two spaces are given by the stable and unstable manifolds in the sense of Morse theory (i.e. the points on flow lines which terminate at critical points).

2. MOMENT MAPS AND MASTER SPACES

2.1. Review of stable pairs

We begin with a review of some notation and results from [4, 5]. As in the Introduction, let $E \to \Sigma$ be a fixed complex vector bundle of rank R and degree d. Also let $\mathscr{C}$ denote the space of $\bar{\partial}$-operators on E, and let $\Omega^0(E)$ denote the space of smooth sections of E. The space of holomorphic pairs is then given by

$$\mathscr{H} = \{(\bar{\partial}_E, \phi) \in \mathscr{C} \times \Omega^0(E) \colon \bar{\partial}_E \phi = 0\}.$$

The complex gauge group $\mathfrak{G}^{\mathbb{C}}$, i.e. the group of bundle automorphisms, acts on $\mathscr{C} \times \Omega^0(E)$ by $g(\bar{\partial}_E, \phi) = (g \circ \bar{\partial}_E \circ g^{-1}, g\phi)$, preserving $\mathscr{H}$. We recall

THEOREM 2.1. (cf. [4]) *Let* $E \to \Sigma$ *be a fixed complex vector bundle over a closed Riemann surface, and let* $(\bar{\partial}_E, \phi)$ *be a holomorphic pair as in Definition* 1.1. *Suppose that* $(\bar{\partial}_E, \phi)$ *is* τ*-stable for a given value of the parameter* τ. *Then the* τ*-vortex equation,*

$$\sqrt{-1}\Lambda F_{\bar{\partial}_E, H} + \tfrac{1}{2}\phi \otimes \phi^* = \frac{\tau}{2}\mathbf{I}$$

considered as an equation for the metric H, *has a unique smooth solution. Here* $F_{\bar{\partial}_E, H}$ *is the curvature of a metric connection,* $\Lambda F_{\bar{\partial}_E, H}$ *is a section in* $\Omega^0(End\, E)$ *and is obtained by contraction of* $F_{\bar{\partial}_E, H}$ *by the Kähler form on* Σ, $\phi \otimes \phi^*$ *is a section of* $\Omega^0(E \otimes E^*) \simeq \Omega^0(End\, E)$, *and* $\mathbf{I}$ *is the identity section in* $\Omega^0(End\, E)$. *Conversely, suppose that for a given value of* τ *there is a Hermitian metric* H *on* E *such that the* τ*-vortex equation is satisfied by* $(\bar{\partial}_E, \phi, H)$. *Then* E *splits holomorphically as* $E = E_\phi \oplus E_s$, *where*

(i) E_s, *if not zero, is a direct sum of stable bundles, each of slope* $\tau(\mathrm{Vol}(\Sigma)/4\pi)$;

(ii) E_ϕ *contains the section* ϕ *and* (E_ϕ, ϕ) *is* τ*-stable, where* E_ϕ *has the holomorphic structure induced from* $\bar{\partial}_E$.

Notice that the split case $E = E_\phi \oplus E_s$ cannot occur unless $\tau(\mathrm{Vol}(\Sigma)/4\pi)$ is a rational number with denominator less than the rank of E. Hence, for all other values of τ, henceforth called *generic*, the summand E_s must be empty, and the set of τ-stable pairs can be identified with the set

$$\mathscr{V}_\tau = \left\{(\bar\partial_E, \phi) \in \mathscr{H} : \Lambda F_{\bar\partial_E, H} - \frac{\sqrt{-1}}{2}\phi \otimes \phi^* = -\sqrt{-1}\frac{\tau}{2}\mathbf{I} \text{ for some metric } H\right\}.$$

In addition, it is shown in [5] that the space $\mathscr{V}_\tau$ is empty unless $\tau(\mathrm{Vol}(\Sigma)/4\pi)$ lies in a bounded interval. Normalizing the volume of Σ to be 4π, the range for τ is the interval $[d/R, d/R - 1]$.

If a Hermitian bundle metric H is fixed on E, then $\mathscr{C} \times \Omega^0(E)$ acquires a natural symplectic structure coming from the usual symplectic structures on $\mathscr{C}$ and $\Omega^0(E)$ (cf. [5]). Moreover, the unitary gauge group $\mathfrak{G}$ acts *symplectically* and a moment map for this action is given by

$$\Psi(\bar\partial_E, \phi) = \Lambda F_{\bar\partial_E, H} - \frac{\sqrt{-1}}{2}\phi \otimes \phi^*.$$

Theorem 2.1 thus gives an identification (for generic τ)

$$\mathscr{B}_\tau = \mathscr{V}_\tau/\mathfrak{G}^{\mathbb{C}} = \Psi^{-1}(-\sqrt{-1}\frac{\tau}{2}\mathbf{I})/\mathfrak{G}.$$

The first equality gives $\mathscr{B}_\tau$ a complex structure and the second gives a symplectic structure.

2.2. Construction of the master space

The master space $\hat{\mathscr{B}}$ has similar descriptions as both a symplectic and a complex space. A key element in the construction the choice of a subgroup of $\mathfrak{G}$ (or $\mathfrak{G}^{\mathbb{C}}$). This is defined as follows.

Let $\mathfrak{G}_1^{\mathbb{C}} \subset \mathfrak{G}^{\mathbb{C}}$ denote the connected component of the identity, and let Υ denote the quotient group of components. Then Υ is a free abelian group on $2g$ generators corresponding to $H_1(\Sigma, \mathbb{Z})$ (see [1, p. 542]). We can find a splitting of the exact sequence $1 \to \mathfrak{G}_1^{\mathbb{C}} \to \mathfrak{G}^{\mathbb{C}} \to \Upsilon \to 1$, and this realizes $\mathfrak{G}^{\mathbb{C}}$ as a direct product $\mathfrak{G}^{\mathbb{C}} \simeq \mathfrak{G}_1^{\mathbb{C}} \times \Upsilon$, with the isomorphism given by $(g, h) \mapsto gh$. For $g \in \mathfrak{G}_1^{\mathbb{C}}$, the map $\det g : \Sigma \to \mathbb{C}^*$ is in the identity component of $\mathrm{Map}(\Sigma, \mathbb{C}^*)$. It thus lifts to a map $u \in \mathrm{Map}(\Sigma, \mathbb{C})$, and we can define a character $\chi : \mathfrak{G}_1^{\mathbb{C}} \to \mathbb{C}^*$ by $\chi(g) = \exp(\int_\Sigma u)$. Then we extend χ to $\mathfrak{G}_1^{\mathbb{C}} \times \Upsilon$ by $\chi(g, h) = \chi(g)$. This defines a homomorphism $\mathfrak{G}^{\mathbb{C}} \to \mathbb{C}^*$.

Definition 2.2. Let $\mathfrak{G}_0^{\mathbb{C}}$ be the kernel of the character $\chi : \mathfrak{G}^{\mathbb{C}} \to \mathbb{C}^*$ defined as above, and let $\mathfrak{G}_0 \subset \mathfrak{G}$ be defined by $\mathfrak{G}_0 = \mathfrak{G}_0^{\mathbb{C}} \cap \mathfrak{G}$.

Note that a different choice of splitting will give rise to a group isomorphic to $\mathfrak{G}_0^{\mathbb{C}}$ with the same connected component of the identity. The following is immediate from the definition.

PROPOSITION 2.3. *The groups $\mathfrak{G}_0$ and $\mathfrak{G}_0^{\mathbb{C}}$ have the structure of Fréchet Lie groups with Lie algebras*

$$\mathrm{Lie}\,\mathfrak{G}_0^{\mathbb{C}} = \Omega^0(\mathrm{End}\,E)_0 = \left\{u \in \Omega^0(\mathrm{End}\,E) :, \int_\Sigma \mathrm{Tr}\,u = 0\right\}$$

$$\mathrm{Lie}\,\mathfrak{G}_0 = \Omega^0(\mathrm{ad}\,E)_0 = \left\{u \in \Omega^0(\mathrm{ad}\,E) :, \int_\Sigma \mathrm{Tr}\,u = 0\right\}.$$

The essential feature of the subgroup $\mathfrak{G}_0$ is the following:

PROPOSITION 2.4. *A moment map for the action of* $\mathfrak{G}_0$ *on* $\mathscr{C} \times \Omega^0(E)$ *is given by*

$$\Psi_0(\bar\partial_E, \phi) = \Psi(\bar\partial_E, \phi) - \frac{1}{4\pi R}\int_\Sigma \operatorname{Tr}\Psi(\bar\partial_E, \phi)\cdot \mathbf{I}.$$

Proof. Let $j: \mathfrak{G}_0 \to \mathfrak{G}$ denote the inclusion. Then a moment map for $\mathfrak{G}_0$ is given by $j^*\Psi$, where $j^*: (\operatorname{Lie}\mathfrak{G})^* \to (\operatorname{Lie}\mathfrak{G}_0)^*$ is the induced map on the duals of the Lie algebras. Using the L^2-inner product on $\Omega^0(\operatorname{End} E)$ to identify the Lie algebras with their duals, the map j^* becomes the orthogonal projection onto the Lie subalgebra, and $j^*\Psi$ is given by the above formula for Ψ_0.

The symplectic description of our master space is obtained from the quotient by this moment map, and the complex structure is obtained as a quotient of $\mathscr{H}$ by $\mathfrak{G}_0^{\mathbb{C}}$. The constructions are similar to those given in [5] for the spaces $\mathscr{B}_\tau$, and since the techniques used are by now quite familiar, we will keep our discussion brief.

The first step in the construction of the complex quotient is to restrict from $\mathscr{H}$ to the subspace

$$\mathscr{H}^* = \{(\bar\partial_E, \phi) \in \mathscr{H}: E^{\bar\partial_E} \text{ is semistable if } \phi = 0\}.$$

This technicality, which will be justified in Corollary 2.6, is required in order to avoid possible singularities in $\mathscr{H}$. The infinitesimal deformations of a point in the orbit space $\mathscr{H}^*/\mathfrak{G}_0^{\mathbb{C}}$ are described by the following complex:

$$(C_{\phi,0}^{\bar\partial_E}) \qquad 0 \to \Omega^0(\operatorname{End} E)_0 \xrightarrow{d_1} \Omega^{0,1}(\operatorname{End} E) \oplus \Omega^0(E) \xrightarrow{d_2} \Omega^{0,1}(E) \to 0.$$

Here the maps are given by $d_1(u) = (-\bar\partial_E u, u\phi)$ and $d_2(\alpha, \eta) = \bar\partial_E \eta + \alpha\phi$. The only difference between this complex and the one used in the construction of the moduli space of τ-stable pairs for a fixed τ in [5] is the restriction imposed on the elements in $\Omega^0(\operatorname{End} E)$. The relevant properties $C_{\phi,0}^{\bar\partial_E}$ are given in the next proposition.

PROPOSITION 2.5. *Let* $(\bar\partial_E, \phi)$ *be an element of* $\mathscr{H}^*$. *Then*

(i) $C_{\phi,0}^{\bar\partial_E}$ *is a Fredholm complex*;
(ii) $H^2(C_{\phi,0}^{\bar\partial_E}) = H^2(C_\phi^{\bar\partial_E}) = 0$;
(iii) *either*

$$\begin{cases} H^1(C_{\phi,0}^{\bar\partial_E}) \simeq H^1(C_\phi^{\bar\partial_E}) \oplus \mathbb{C} \\ H^0(C_{\phi,0}^{\bar\partial_E}) = H^0(C_\phi^{\bar\partial_E}) \end{cases} \quad \text{or} \quad \begin{cases} H^1(C_{\phi,0}^{\bar\partial_E}) = H^1(C_\phi^{\bar\partial_E}) \\ H^0(C_{\phi,0}^{\bar\partial_E}) \oplus \mathbb{C} \simeq H^0(C_\phi^{\bar\partial_E}) \end{cases}$$

(iv) $\chi(C_{\phi,0}^{\bar\partial_E}) = \chi(\operatorname{End} E) - \chi(E) - 1$.

Proof. (i), (ii) follow from the definition of $C_{\phi,0}^{\bar\partial_E}$, the corresponding properties of $C_\phi^{\bar\partial_E}$ (cf. [5, 18]), and the fact that either $\phi \neq 0$ or $E^{\bar\partial_E}$ is semistable for all pairs in $\mathscr{H}^*$. (iii) $H^1(C_\phi^{\bar\partial_E})$ and $H^1(C_{\phi,0}^{\bar\partial_E})$ are related by the exact sequence $0 \to H^1(C_\phi^{\bar\partial_E}) \to H^1(C_{\phi,0}^{\bar\partial_E}) \xrightarrow{d_1^*} \mathbb{C}$. Similarly, the zeroth cohomology groups are related by $0 \to H^0(C_{\phi,0}^{\bar\partial_E}) \to H^0(C_\phi^{\bar\partial_E}) \xrightarrow{\pi} \mathbb{C}$, where the map π is orthogonal projection in $\Omega^0(\operatorname{End} E)$. Here by $\mathbb{C}$ we mean the constant multiples of the identity in $\Omega^0(\operatorname{End} E)$. The desired conclusion now follows from the fact that the map $d_1^*: H^1(C_{\phi,0}^{\bar\partial_E}) \to \mathbb{C}$ is surjective if and only if $\pi: H^0(C_\phi^{\bar\partial_E}) \to \mathbb{C}$ is zero. (iv) follows from (ii) and (iii).

COROLLARY 2.6. *The space $\mathscr{H}^*$ is a smooth submanifold of $\mathscr{C}\times\Omega^0(E)$.*

Proof. Consider the map $F:\mathscr{C}\times\Omega^0(E)\to\Omega^1(E)$ defined by $F(\bar\partial_E,\phi)=\bar\partial_E(\phi)$. The derivative of F at $(\bar\partial_E,\phi)$ is given by $\delta F_{\bar\partial_E,\phi}(\alpha,\eta)=\bar\partial_E\eta+\alpha\phi=d_2(\alpha,\eta)$. Let $\mathscr{C}_{ss}$ denote the semistable holomorphic structures on E. Then provided that $(\bar\partial_E,\phi)$ does not belong to the closed subspace $(\mathscr{C}\backslash\mathscr{C}_{ss})\times\{0\}\subset\mathscr{C}\times\Omega^0(E)$, it follows from Proposition 2.5(ii) that $\delta F_{\bar\partial_E,\phi}$ is onto. Hence, by the Inverse Function Theorem, $\mathscr{H}^*=F^{-1}(0)\cap\{\mathscr{C}\times\Omega^0(E)\backslash(\mathscr{C}\backslash\mathscr{C}_{ss})\times\{0\}\}$ is a smooth submanifold of $\mathscr{C}\times\Omega^0(E)$.

It is convenient at this stage to restrict further the space $\mathscr{H}^*$ to

$$\mathscr{H}^{**}=\{(\bar\partial_E,\phi)\in\mathscr{H}:\ \phi\neq 0\}.$$

Since $\mathscr{H}^{**}$ is an open subspace of $\mathscr{H}^*$, it is also a smooth manifold.

Definition 2.7. A pair $(\bar\partial_E,\phi)\in\mathscr{H}^{**}$ is called simple if $H^0(C^{\bar\partial_E}_{\phi,0})=0$. Let $\mathscr{H}_\sigma$ denote the subspace of simple pairs in $\mathscr{H}^{**}$.

Clearly, $\mathscr{H}_\sigma$ is an open subset in $\mathscr{H}^{**}$ and is therefore a submanifold. Now by identifying $H^1(C^{\bar\partial_E}_{\phi,0})$ with the tangent space to the space of orbits of $\mathfrak{G}^{\mathbb{C}}_0$, we have the following theorem whose proof is in all essentials the same as the proof of the analogous result in Section 3 of [5]:

THEOREM 2.8. *$\mathscr{H}_\sigma/\mathfrak{G}^{\mathbb{C}}$ is a complex manifold (possibly non-Hausdorff) of complex dimension $d+1+(R^2-R)(g-1)$. Moreover, we have the identification $T_{[\bar\partial_E,\phi]}(\mathscr{H}_\sigma/\mathfrak{G}^{\mathbb{C}}_0)\simeq H^1(C^{\bar\partial_E}_{\phi,0})$.*

We now consider the symplectic (Kähler) structure on an open subspace of $\mathscr{H}_\sigma/\mathfrak{G}^{\mathbb{C}}_0$.

Definition 2.9. Define $\hat{\mathscr{B}}=\Psi_0^{-1}(0)\cap\mathscr{H}^*/\mathfrak{G}_0$, and $\hat{\mathscr{B}}_0=\Psi_0^{-1}(0)\cap\mathscr{H}_\sigma/\mathfrak{G}_0$.

PROPOSITION 2.10 *The quotient $\hat{\mathscr{B}}$ is a compact, Hausdorff topological space. The quotient $\hat{\mathscr{B}}_0$ is a Hausdorff symplectic manifold.*

Proof. The Hausdorff property and compactness follows as in [5, Propositions 5.1 and 5.4]. The symplectic manifold structure follows from the Marsden–Weinstein reduction theorem for Banach spaces, cf. Theorem 4.5 in [5] and also Theorem 5.8 in [14, Ch. VII].

To complete the construction of the master space as a complex manifold we make the following:

Definition 2.11. Let $\mathscr{V}_0\subset\mathscr{H}^{**}$ denote the subset of $\mathfrak{G}^{\mathbb{C}}_0$-orbits through points in $\Psi_0^{-1}(0)$, i.e. $\mathscr{V}_0=\{(\bar\partial_E,\phi)\in\mathscr{H}^{**}:\ \Psi_0(g(\bar\partial_E,\phi))=0$ for some $g\in\mathfrak{G}^{\mathbb{C}}_0\}$.

It is easily seen that $\mathscr{V}_0\cap\mathscr{H}_\sigma$ is an open subset of $\mathscr{H}_\sigma$, and thus is a submanifold. Also, using the same techniques as those applied to $\mathscr{V}_\tau$ in [6] it can be shown that $\mathscr{V}_0$ and $\mathscr{V}_0\cap\mathscr{H}_\sigma$ are connected. Finally, there is clearly a bijective correspondence between $\hat{\mathscr{B}}_0$ and $\mathscr{V}_0\cap\mathscr{H}_\sigma/\mathfrak{G}^{\mathbb{C}}_0$. Combining Theorem 2.8 and Proposition 2.10 we obtain:

THEOREM 2.12. *$\hat{\mathscr{B}}_0=\mathscr{V}_0\cap\mathscr{H}_\sigma/\mathfrak{G}^{\mathbb{C}}_0$ is a smooth, Hausdorff, Kähler manifold of dimension $d+1+(R^2-R)(g-1)$.*

2.3. S^1-action, Morse function, and reduced level sets

The most important feature of the master space $\hat{\mathscr{B}}$ is the fact that it carries an S^1 action. This comes from the quotient $U(1) \simeq \mathfrak{G}/\mathfrak{G}_0$, and the action on $\hat{\mathscr{B}}$ is given by $e^{i\theta}[\bar{\partial}_E, \phi] = [\bar{\partial}_E, g_\theta \phi]$. Here g_θ denotes the gauge transformation $e^{i\theta/R} \cdot \mathbf{I}$. Notice that g_θ itself depends on the choice of an Rth root of unity but that the action is well-defined and independent of this choice, since if $h = e^{2\pi i/R} \cdot \mathbf{I}$, then $h \in \mathfrak{G}_0$ and $[\bar{\partial}_E, h\phi] = [h^{-1}(\bar{\partial}_E), \phi] = [\bar{\partial}_E, \phi]$. The following summarizes the basic properties of this action.

PROPOSITION 2.13. (i) *The action of $U(1)$ on $\hat{\mathscr{B}}_0$ is quasi-free, holomorphic and symplectic; the moment map for the action is given by*

$$\hat{f}[\bar{\partial}_E, \phi] = -2\pi i\left(\frac{\|\phi\|^2}{4\pi R} + \mu(E)\right).$$

(iii) *The action extends continuously to $\hat{\mathscr{B}}$ as does the moment map $\hat{f}$.*

Proof. The only statements needing verification are that the action is quasi-free and that the moment map is as claimed. To check the former, note that the stabilizer of $[\bar{\partial}_E, \phi]$ *in* $\hat{\mathscr{B}}_0$ is trivial unless E splits as $E = E_\phi \oplus E_s$, in which case the stabilizer is $U(1)$. In order to compute the moment map, observe that on the level of Lie algebras there is a splitting $\lambda: \operatorname{Lie} U(1) \to \operatorname{Lie} \mathfrak{G}$ given by $\lambda(\xi) = \xi/R$, where $\xi \in \operatorname{Lie} U(1)$ is identified with the constant infinitesimal gauge transformations. Under the identification with dual Lie algebras given by the L^2-metric, the dual map $\lambda^*(g) = \int_\Sigma \operatorname{Tr} g$. Choose a representative $(\bar{\partial}_E, \phi)$ satisfying $\Psi_0(\bar{\partial}_E, \phi) = 0$. It is then straightforward to compute that

$$\hat{f}[\bar{\partial}_E, \phi] = \frac{1}{R}\lambda^* \circ \Psi_0(\bar{\partial}_E, \phi) = -2\pi i\left(\frac{\|\phi\|^2}{4\pi R} + \mu(E)\right).$$

For convenience we let $f: \hat{\mathscr{B}} \to \mathbb{R}$ denote the function $f = -\hat{f}/2\pi i$. We now have

PROPOSITION 2.14. (i) *The image of f is the interval $[d/R, d/(R-1)]$.* (ii) *The critical points of f on $\hat{\mathscr{B}}_0$ are precisely the fixed points of the $U(1)$ action, and the critical values coincide with the nongeneric values of τ.* (iii) *Let τ be a regular value of f. Then the reduced space $f^{-1}(\tau)/U(1)$ is $\mathscr{B}_\tau$, the moduli space of τ-stable pairs.*

Proof. (i) Clearly, τ is in the image of f if and only if the equation $\Psi(\bar{\partial}_E, \phi) = -\sqrt{-1}\tau/2\mathbf{I}$ has a solution. Hence, by Theorem 2.1 the range for τ is in $[d/R, d/(R-1)]$. Now the endpoints of this interval are included in the image, since explicit elements of the pre-image can be constructed (see Proposition 2.17 below). The result then follows from the connectedness of $\mathscr{V}_0 \cap \mathscr{H}_\sigma$ (see the remark following Definition 2.11). (ii) This follows from the fact that $\hat{f}$ is a moment map for the $U(1)$ action. (iii) This follows from the principle of reduction in steps applied to the full gauge group $\mathfrak{G}$ and to the sub- and quotient groups $\mathfrak{G}_0$ and $U(1)$.

In the special case where the rank of the bundle is two, we have

PROPOSITION 2.15. *Suppose $R = 2$. Then $\hat{\mathscr{B}}_0 = \hat{\mathscr{B}} \setminus f^{-1}(d/2)$. Moreover, the $U(1)$ action on $\hat{\mathscr{B}}_0$ is quasi-free and the moment map is proper.*

Proof. Suppose $(\bar{\partial}_E, \phi) \in \Psi_0^{-1}(0)$. Then $(\bar{\partial}_E, \phi) \in \Psi^{-1}(-\sqrt{-1}\tau/2\mathbf{I})$ for some τ. By Theorem 2.1, $\phi \equiv 0$ if and only if $\tau = d/2$. Now suppose that $g \in \mathfrak{G}_0$, $g \neq I$, and

$g(\bar{\partial}_E, \phi) = (\bar{\partial}_E, \phi)$. If $\phi \not\equiv 0$, then E must split holomorphically as $E = E_\phi \oplus E_s$ with $\phi \in H^0(E_\phi)$ and $g = (1, \tilde{g})$. But $\text{rank}(E_s) = 1$ implies $\tilde{g}$ is constant, and since $g \in \mathfrak{G}_0$ we must have $\det(g) = \tilde{g} = 1$. This proves that the stabilizer for points away from $f^{-1}(d/2)$ is trivial, and therefore $\hat{\mathscr{B}} \backslash f^{-1}(d/2) = \hat{\mathscr{B}}_0$. The other statements follow from Proposition 2.14.

2.4. Critical sets

Definition 2.16. Let $\text{Fix}(\hat{\mathscr{B}})$ denote the $U(1)$ fixed point set in $\hat{\mathscr{B}}$. For a critical value of τ, let $\mathscr{Z}_\tau = f^{-1}(\tau) \cap \text{Fix}(\hat{\mathscr{B}})$.

The following is immediate from Theorem 2.1.

PROPOSITION 2.17. *$\mathscr{Z}_{d/R} = \mathscr{M}(R,d)$ and $\mathscr{Z}_{d/R-1} = \mathscr{M}(R-1,d)$ where $\mathscr{M}$ denotes the moduli space of semi-stable bundles of given rank and degree.*

Next, we describe the fixed point sets for the intermediate values of τ. Fix a critical value $\tau = p/q \in (d/R, d/(R-1))$. Then in any pair $(\bar{\partial}_E, \phi) \in \mathscr{Z}_\tau$ the bundle splits holomorphically as $E^{\bar{\partial}_E} = E_\phi \oplus E_{ss}$, where $\phi \in H^0(E_\phi)$, the pair (E_ϕ, ϕ) is a τ-stable, and E_{ss} is a direct sum $\bigoplus_i E_i$ of stable bundles of slope τ. Using the fact that $\Psi_0(\bar{\partial}_E, \phi) = p/q$, we get

LEMMA 2.18. *For $\tau = p/q \in (d/R, d/(R-1))$, let $E^{\bar{\partial}_E} = E_\phi \oplus E_{ss}$ be as above. Let the degree and rank of E_ϕ be (R_ϕ, d_ϕ) and those of E_i be (R_i, d_i). Then we have the following constraints:*

(i) $d_i/R_i = (d - d_\phi)/(R - R_\phi) = p/q$;
(ii) $\sum_i R_i = R - R_\phi$;
(iii) $d_\phi/R_\phi < p/q < d_\phi/(R_\phi - 1)$.

Conversely, given any stable pair (E_ϕ, ϕ) and set of stable bundles E_i such that the conditions above are satisfied, we obtain a representative for a fixed point $(E_\phi \oplus \bigoplus_i E_i, \phi)$ in the critical level set corresponding to p/q.

We see that to a fixed point $(\bar{\partial}_E, \phi) \in \mathscr{Z}_\tau$ we can assign an $(n+2)$-tuple of integers $\vec{\rho} = (d_\phi, R_\phi, R_1, \ldots, R_n)$. We will refer to $\vec{\rho}$ as the *type* of the fixed point. The degrees d_i are then determined by condition (i) of Lemma 2.18.

Definition 2.19. For $\tau = p/q \in (d/R, d/(R-1))$, let I_τ consist of all $(n+2)$-tuples $\vec{\rho} = (d_\phi, R_\phi, R_1, \ldots, R_n)$ in $\mathbb{Z}^{n+2}$ such that $n \geqslant 1$, $R > R_\phi > 0$, $R_i > 0$ for $1 \leqslant i \leqslant n$, $\tau = (d - d_\phi)/(R - R_\phi)$, and conditions (ii) and (iii) of Lemma 2.18 are satisfied. Also denote by $\mathscr{Z}_\tau(\vec{\rho})$ the set of critical points of type $\vec{\rho}$ in $\mathscr{Z}_\tau$.

We can extend this notation to include the extreme values $\tau = d/R$ and $\tau = d/(R-1)$. This requires the convention that

(i) if $\tau = d/R$, then $d_\phi = R_\phi = 0$ and $\sum_i R_i = R$,
(ii) if $\tau = d/(R-1)$, then $d_\phi = 0$, $R_\phi = 1$ and conditions (ii) and (iii) of Lemma 2.18 are satisfied with $p/q = \tau = d/(R-1)$.

Then Lemma 2.18 can be rephrased as

PROPOSITION 2.20. *Let τ be as in Lemma 2.18. Then $\mathscr{Z}_\tau = \bigcup_{\vec{\rho} \in I_\tau} \mathscr{Z}_\tau(\vec{\rho})$.*

Example 2.21. In the case of rank two the only possibility for split bundles is $E_\phi \oplus E_s$, where E_s is a line bundle of degree τ and $\phi \in H^0(E_\phi)$, where E_ϕ is a line bundle of degree $d - \tau$. The pair (E_ϕ, ϕ) is determined up to equivalence by the divisor class of ϕ; hence, the space of equivalence classes of pairs (E_ϕ, ϕ) is simply the $d - \tau$ symmetric product $\mathrm{Sym}^{d-\tau}\Sigma$. Since E_s is arbitrary, we have $\mathscr{Z}_\tau \simeq \mathrm{Sym}^{d-\tau}\Sigma \times \mathscr{J}_\tau$, where $\mathscr{J}_\tau$ denotes the component of the Jacobian variety of Σ corresponding to line bundles of degree τ.

3. THE ALGEBRAIC STRATIFICATION OF $\hat{\mathscr{B}}$

We examine two stratifications of the master space $\hat{\mathscr{B}}$. From one point of view these are a consequence of the stability property, and as such holomorphic pairs admit two natural filtrations analogous to the Seshadri filtration for semistable bundles. From a different perspective, the stratifications can be understood in terms of the Morse theory of the moment map f on $\hat{\mathscr{B}}$, where they are given by the stable (or unstable) manifolds in the sense of Morse theory. This will be explained in the next section. We first give a purely algebraic description. Given a holomorphic pair (E, ϕ) we set $\mu_+(E)$ and $\mu_-(E)$, where

$$\mu_+(E) = \sup\{\mu(E') \colon E' \subset E \text{ a non-trivial holomorphic subbundle}\}$$

$$\mu_-(E, \phi) = \inf\{\mu(E/E'') \colon E'' \subset E \text{ a proper holomorphic subbundle } \textit{and } \phi \in H^0(E'')\},$$

with the convention $\inf(\emptyset) = \infty$ Generalizing 1.1, we call the pair (E, ϕ) *stable* if $\mu_+(E) < \mu_-(E, \phi)$. This is clearly equivalent to the pair being τ-stable for all $\mu_+(E) < \tau < \mu_-(E, \phi)$.

PROPOSITION 3.1 (The μ_--filtration). *Let (E, ϕ) be a stable pair. There is a filtration of E by subbundles $0 \subset E_\phi = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = E$ such that the following properties hold*:

(i) *$\phi \in H^0(E_\phi)$, the pair (E_ϕ, ϕ) is a stable pair, and $\mu_+(E_\phi) < \mu_-(E, \phi) < \mu_-(E_\phi, \phi)$,*

(ii) *for $i = 1, \ldots, n$ the quotients F_i/F_{i-1} are stable bundles each of slope $\mu(F_i/F_{i-1}) = \mu_-(E, \phi)$,*

(iii) *E_ϕ has minimal rank among filtrations satisfying* (i) *and* (ii).

The subbundle E_ϕ is uniquely determined, and the graded object $E_\phi \oplus F_1/F_0 \oplus F_2/F_1 \oplus \cdots \oplus E/F_{n-1}$ is unique up to isomorphism of $F_1/F_0 \oplus F_2/F_1 \oplus \cdots \oplus E/F_{n-1}$.

Using this result, whose proof is provided below, we make the following:

Definition 3.2. The μ_--grading for a stable pair (E, ϕ) is given by

$$gr^-(E, \phi) = (E_\phi \oplus F_1/F_0 \oplus F_2/F_1 \oplus \cdots \oplus E/F_{n-1}, \phi).$$

Similarly, we have

PROPOSITION 3.3 (The μ_+-filtration). *Let (E, ϕ) be a stable pair. There is a filtration of E by subbundles $0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset F_{n+1} = E$ such that the following properties hold: If E is semistable then this is a Seshadri filtration and the quotients F_i/F_{i-1} are all stable bundles of slope $\mu_+(E) = \mu(E)$. Otherwise,*

(i) *for $i = 1, \ldots, n$ the quotients F_1/F_{i-1} are stable bundles each of slope $\mu(F_i/F_{i-1}) = \mu_+(E)$,*

(ii) *ϕ has a nonzero projection, φ, into $H^0(E/F_n)$, and the pair $(E/F_n, \varphi)$ is a stable pair with $\mu_+(E/F_n) < \mu_+(E) < \mu_-(E/F_n)$,*

(iii) *E/F_n has minimal rank among filtrations satisfying* (i) *and* (ii).

In the case where $\mu_+(E) > \mu(E)$, the quotient $Q = E/F_n$ is uniquely determined, and the graded object $F_1/F_0 \oplus F_2/F_1 \oplus \cdots \oplus F_n/F_{n-1} \oplus Q$ is unique up to isomorphism of $F_1/F_0 \oplus F_2/F_1 \oplus \cdots \oplus F_n/F_{n-1}$.

Definition 3.4. For a stable pair (E, ϕ) for which $\mu_+(E) > \mu(E)$, the $\mu_+(E)$-grading is defined to be $gr^+(E, \phi) = (F_1/F_0 \oplus F_2/F_1 \oplus \cdots \oplus F_n/F_{n-1} \oplus Q, \varphi)$. For $\mu_+(E) = \mu(E)$, we set $gr^+(E, \phi) = (\mathrm{Gr}(E), 0)$, where $\mathrm{Gr}(E)$ is the grading for E coming from the Seshadri filtration.

The proof of Proposition 3.3 follows immediately from the following:

PROPOSITION 3.5. *Given a stable pair (E, ϕ) there is a unique quotient Q of E arising from an exact sequence $0 \to F \to E \to Q \to 0$, with the properties*

(i) *F is a semi-stable bundle,*

(ii) *$\mu(F) = \mu_+(E)$,*

(iii) *if $Q \neq 0$, then under projection of E onto Q the section ϕ has a nontrivial image, φ, and the holomorphic pair (Q, φ) is stable*

(iv) *if $Q \neq 0$, then $\mu_+(Q) < \mu_+(E) < \mu_-(Q)$,*

(v) *Q has minimal rank among quotients satisfying* (i)–(iv).

Proof. If E is a semistable bundle then $\mu_+(E) = \mu(E)$, and we take $F = E$, $Q = 0$. Otherwise, for F we take the unique maximal semistable subbundle of E. Properties (i), (ii) and (v) follow immediately from this choice of F. Properties (iii) and (iv) are consequences of the following:

LEMMA 3.6. *Let (E, ϕ) be a stable pair with $\mu_+(E) > \mu(E)$. Let F be the unique maximal semistable subbundle of E, and let Q be the quotient E/F. Let $\varphi \in H^0(Q)$ be the image of ϕ under the projection of E onto Q. Then*

(i) $\varphi \neq 0$,

(ii) $\mu_+(Q) < \mu_+(E)$,

(iii) $\mu_-(Q, \phi) \geqslant \mu_-(E, \phi)$.

Proof. (i) If $\varphi = 0$ then $\phi \in H^0(F)$. But then, as (E, ϕ) is stable, $\mu(E/F) \geqslant \mu_-(E, \phi) > \mu_+(E) = \mu(F)$, i.e. $\mu(Q) > \mu(F)$. This is incompatible with $\mu(F) = \mu_+(E) > \mu(E)$. Thus, $\varphi \neq 0$.

(ii) Let $Q' \subset Q$ be any holomorphic subbundle. Lift Q' to a subbundle $E' \subset E$. This gives a short exact sequence $0 \to F \to E' \to Q' \to 0$. By definition, $\mu(E') \leqslant \mu_+(E) = \mu(F)$, but in fact the inequality must be strict since $\mathrm{rank}(E') > \mathrm{rank}(F)$. It follows from this and the above short exact sequence that $\mu(Q') < \mu_+(E)$. Thus $\mu_+(Q) < \mu_+(E)$.

(iii) Suppose in addition that $\varphi \in Q'$. Then $\phi \in E'$, and thus $\mu(E/E') \geqslant \mu_-(E, \phi)$. But $\mu(E/E') = \mu(Q/Q')$, and thus it follows that $\mu_-(Q, \phi) \geqslant \mu_-(E, \phi)$.

Proof of Proposition 3.1. We begin with the following:

LEMMA 3.7. *Let (E, ϕ) be a stable pair. Let $E_\phi \subset E$ be a holomorphic subbundle such that $\phi \in H^0(E_\phi)$ and $\mu(E/E_\phi) = \mu_-(E, \phi)$. Then*

(i) $\mu_+(E_\phi) \leqslant \mu_+(E)$,

(ii) $\mu_-(E_\phi, \phi) \geqslant \mu_-(E, \phi)$, *and the inequality is strict if E_ϕ has minimal rank among all subbundles satisfying the hypotheses of the lemma,*

(iii) (E_ϕ, ϕ) *is a stable pair,*

(iv) E/E_ϕ *is a semi-stable bundle,*

(v) $\mu(E_\phi) < \mu_-(E, \phi)$

(vi) *Suppose that E_ϕ has minimal rank among all subbundles satisfying the hypotheses of the lemma, and that E'_ϕ is any other subbundle such that $\phi \in H^0(E'_\phi)$ and $\mu(E/E'_\phi) = \mu_-$. Then $E_\phi \subseteq E'_\phi$.*

Proof. The first inequality is clear, since E_ϕ is a subbundle of E. (ii) Let E'' be such that $E'' \subset E_\phi \subset E$, and $\phi \in H^0(E'')$. Denote the ranks by R'', R_ϕ, and R, a computation yields

$$\mu(E_\phi/E'') - \mu(E/E'') = \left(\frac{R - R_\phi}{R_\phi - R''}\right)(\mu(E/E'') - \mu(E/E_\phi)).$$

The right-hand side is nonnegative by the definition of $\mu_-(E, \phi)$, and it is strictly positive if E_ϕ has minimal rank among all subbundles satisfying the hypotheses of the lemma. The result now follows from the fact that $\mu(E/E_\phi) = \mu_-(E, \phi)$. (iii) This follows immediately from (i) and (ii), and the fact that (E, ϕ) is stable. (iv) Suppose that E/E_ϕ is not semistable. Pick a subbundle $F \subset E/E_\phi$ such that $\mu(F) = \mu_+(E/E_\phi)$. Let $E' \subset E$ be the lift of F to E, i.e. such that $0 \to E_\phi \to E' \to F \to 0$. Now $\mu(E/E') = \mu((E/E_\phi)/F)$, and if $\mu(F) > \mu(E/E_\phi)$ then $\mu((E/E_\phi)/F) < \mu(E/E_\phi)$. Hence $\mu(E/E') < \mu(E/E_\phi)$. However, since $\phi \in H^0(E')$, we have $\mu(E/E') \geqslant \mu_-(E, \phi) = \mu(E/E_\phi)$. Thus, E/E_ϕ must be semistable. (v) Since (E, ϕ) is stable, we have $\mu(E_\phi) \leqslant \mu_+(E) < \mu_-(E, \phi)$. (vi) Let E_ϕ and E'_ϕ satisfy the hypotheses, and suppose that E_ϕ is of minimal rank. Now consider the map $E_\phi \to E/E'_\phi$, and let K and L be its kernel and image respectively. We thus have $0 \to K \to E_\phi \to L \to 0$. Suppose that $K \neq E_\phi$. Since ϕ is a section of K, we have $\mu(E_\phi/K) \geqslant \mu_-(E_\phi, \phi)$. Also, since by (iv) E/E'_ϕ is semistable, we have $\mu(L) \leqslant \mu(E/E'_\phi) = \mu_-(E, \phi)$. By (ii) we have $\mu_-(E, \phi) < \mu_-(E_\phi, \phi)$ and $\mu(E_\phi/K) = \mu(L)$. Thus, $\mu(L) \leqslant \mu_-(E, \phi) < \mu_-(E_\phi, \phi) \leqslant \mu(E_\phi/K) = \mu(L)$, which is impossible. We conclude that $K = E_\phi$, i.e. $E_\phi \subset E'_\phi$.

PROPOSITION 3.8. *Given a stable pair (E, ϕ) there is a unique subbundle $E_\phi \subset E$ such that*

(i) $\phi \in H^0(E_\phi)$ *and (E_ϕ, ϕ) is a stable pair,*

(ii) E/E_ϕ *is a semi-stable bundle,*

(iii) $\mu(E/E_\phi) = \mu_-(E, \phi)$,

(iv) $\mu_+(E_\phi) < \mu_-(E, \phi) < \mu_-(E_\phi, \phi)$,

(v) E_ϕ *has minimal rank among all subbundles satisfying* (i)–(iv).

Proof. By parts (i)–(v) of Lemma 3.7, any subbundle $E_\phi \subset E$ such that $\phi \in H^0(E_\phi)$ and $\mu(E/E_\phi) = \mu_-$ will satisfy (i)–(iv). By part (vi) of Lemma 3.7, there is a unique such E_ϕ of minimal rank.

Finally, the required filtration in Proposition 3.1 is constructed by setting $F_i = \pi^{-1}(Q_i)$, where

$$0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_n = E/E_\phi$$

is the Seshadri filtration for E/E_ϕ, and $\pi : E \mapsto E/E_\phi$ is the projection map.

Definition 3.9. Given an $(n+2)$-tuple $\vec{\rho} \in I_\tau$ (see Definition 2.19), let

$$\mathscr{W}_\tau^\pm(\vec{\rho}) = \{[E,\phi] \in \hat{\mathscr{B}}: (E,\phi) \text{ is stable and } \mathrm{gr}^\pm(E,\phi) \in \mathscr{Z}_\tau(\vec{\rho})\} \bigcup \mathscr{Z}_\tau(\vec{\rho})$$

$$\mathscr{W}_\tau^\pm = \bigcup_{\vec{\rho}\in I_\tau} \mathscr{W}_\tau^\pm(\vec{\rho}).$$

It is not hard to see that $\mathscr{W}_\tau^\pm(\vec{\rho})$ form a stratification of $\hat{\mathscr{B}}$ in the sense of Kirwan ([12, Definition 2.11]). This will also follow directly from the results of Section 4.1 and the connection with Morse theory.

We end this section with the following estimate which will be used in the next section.

Proposition 3.10. *For critical values $\tau \in (d/R, d/(R-1))$ the complex codimension of $\mathscr{W}_\tau^\pm$ in $\hat{\mathscr{B}}$ is $\geqslant 2$, with the exception of the case $R = 2, \tau = d - 1$.*

Lemma 3.11. *Suppose (E_ϕ, ϕ) is a τ-stable pair and E_s is a semistable bundle with slope τ. Then $H^0(E_\phi \otimes E_s^*) = 0$.*

Proof. Suppose $\alpha \in H^0(E_\phi \otimes E_s^*)$ and $\alpha \not\equiv 0$. Then α defines a map of sheaves $E_s \to E_\phi$, and since $\alpha \not\equiv 0$, $\mathrm{rank}(\ker\alpha) < \mathrm{rank}(E_s)$. But then the semistability of E_s and the τ-stability of E_ϕ imply $\tau = \mu(E_s) \leqslant \mu(E_s/\ker\alpha) = \mu(\mathrm{im}\,\alpha) < \tau$, which is a contradiction. This proves the lemma.

Proof of Proposition 3.10. The case $R = 2$ is straightforward, so assume $R > 2$. It suffices to compute the codimension of the largest stratum $\mathscr{W}_\tau^\pm(\vec{\rho})$, where $\vec{\rho} = (d_\phi, R_\phi, R_s) \in I_\tau$. We first consider $\mathscr{W}_\tau^-(\vec{\rho})$. Let (E,ϕ) be a stable pair such that $gr^+(E,\phi) \in \mathscr{Z}_\tau(\vec{\rho})$. Then we have an exact sequence $0 \to E_\phi \to E \to E_s \to 0$. The tangent space to $\mathscr{W}_\tau^-(\vec{\rho})$ at (E,ϕ) naturally splits

$$T_{(E,\phi)}\mathscr{W}_\tau^-(\vec{\rho}) \simeq T_{(E_\phi,\phi;E_s)}\mathscr{Z}_\tau(\vec{\rho}) \oplus \mathrm{Ext}^1(E_s, E_\phi).$$

The dimension of $\mathscr{Z}_\tau(\vec{\rho})$ is computed as in Section 3 of [5]:

$$\dim \mathscr{Z}_\tau(\vec{\rho}) = d - \tau R_s + 1 + (R - R_s - 1)(R - R_s)(g-1) + R_s^2(g-1),$$

and $\mathrm{Ext}^1(E_s, E_\phi) \simeq H^1(E_\phi \otimes E_s^*)$. By Lemma 3.11 and Riemann–Roch we have that

$$\begin{aligned} \dim T_{(E,\phi)}\mathscr{W}_\tau^-(\vec{\rho}) &= \dim \mathscr{Z}_\tau(\vec{\rho}) + (\tau R - d + (R - R_s)(g-1))R_s \\ &= d + 1 + (R^2 - R)(g-1) + R_s(R_s - R + 1)(g-1) \\ &\quad + (\tau(R-1) - d)R_s. \end{aligned}$$

Hence, by Theorem 2.12,

$$p_-(R_s, \tau) \equiv \mathrm{codim}\,\mathscr{W}_\tau^-(\vec{\rho}) = (d - \tau(R-1))R_s + R_s(R - R_s - 1)(g-1),$$

and since $\tau(R-1) < d$ one sees that $p_-(R_s,\tau) \geqslant 2$ for all possible R_s except when $R = 2$ and $\tau = d - 1$.

Now consider $\mathscr{W}_\tau^+(\vec{\rho})$. Let (E,ϕ) be a stable pair such that $\mathrm{gr}^+(E,\phi) \in \mathscr{Z}_\tau(\vec{\rho})$. Then E may be written (see Proposition 3.5): $0 \to F \to E \xrightarrow{\pi} Q \to 0$. As in the case of $\mathscr{W}_\tau^-(\vec{\rho})$ the tangent space $T_{(E,\phi)}\mathscr{W}_\tau^+(\vec{\rho})$ contains $T_{(Q,\pi(\phi);F)}\mathscr{X}_\tau(\vec{\rho})$ as a summand. The complement is naturally isomorphic to the space of extensions of Q by F direct sum with the equivalence classes of liftings of $\pi(\phi)$. The liftings are parametrized by $H^0(F)$, and two liftings are equivalent if and only if they differ by an element of $H^0(Q^* \otimes F)$. Therefore,

$$\dim T_{(E,\phi)}\mathscr{W}_\tau^+(\vec{\rho}) = \dim T_{(Q,\pi(\phi);F)}\mathscr{X}_\tau(\vec{\rho}) + \dim H^1(Q^* \otimes F) + \dim H^0(F) - \dim H^0(Q^* \otimes F).$$

Since F is stable with slope $\tau > d/R > 2g-2$, $H^1(F) = 0$. Therefore, by Riemann–Roch

$$\begin{aligned}\dim T_{(E,\phi)}\mathscr{W}_\tau^+(\vec{\rho}) &= d - \tau R_s + 1 + (R - R_s - 1)(R - R_s)(g-1) + R_s^2(g-1)\\ &\quad + (d - \tau(R-1) + (R - R_s - 1)(g-1))R_s\\ &= d + 1 + (R^2 - R)(g-1) + (d - R\tau)R_s + R_s(R_s - R)(g-1).\end{aligned}$$

By Theorem 2.12, this implies

$$p_+(R_s,\tau) \equiv \operatorname{codim} \mathscr{W}_\tau^+(\vec{\rho}) = (R\tau - d)R_s + R_s(R - R_s)(g-1),$$

and since $R\tau > d$, it is clear that $p_+(R_s,\tau) > 2$. This completes the proof of Proposition 3.10.

4. BIRATIONAL EQUIVALENCE OF STABLE PAIRS

In this section we describe how the moduli of vortices $\mathscr{B}_\tau$ change with respect to τ. The analogous situation has been studied in the symplectic category by Guillemin and Sternberg [11] and in the algebraic category by Goresky and MacPherson [9]. However, since $\hat{\mathscr{B}}$ has singularities and no obvious embedding in projective space compatible with the $U(1)$ action, the results of [11, 9] are not directly applicable to the case at hand.

4.1. The Morse theory of f

We first consider the Morse theory of the function f. We shall write down solutions to the gradient flow of f and describe the stable and unstable manifold stratifications of $\hat{\mathscr{B}}$. Furthermore, we show that the Morse theoretical stratification of $\hat{\mathscr{B}}$ coincides with the algebraic stratification of Definition 3.9. The results of this section are similar in spirit to the results of [7]. However, the situation here is technically simpler because we are dealing with a finite-dimensional problem and an abelian group action (see also [12]).

PROPOSITION 4.1. *Let $\Phi: \hat{\mathscr{B}} \times [0,\infty) \to \hat{\mathscr{B}}$ be the flow $\Phi_t[\bar{\partial}_E, \phi] = [\bar{\partial}_E, e^{-t/2\pi R}\phi]$. Then Φ is continuous. Moreover, Φ preserves $\hat{\mathscr{B}}_0$ and coincides with the gradient flow of f on $\hat{\mathscr{B}}_0$.*

Proof. We must verify that $d\Phi_t/dt = -\nabla_{\Phi_t} f$. First recall that after identifying $T_{[\bar{\partial}_E,\phi]}\hat{\mathscr{B}}$ with $H^1(C^{\bar{\partial}_E}_{\phi,0})$, the infinitesimal vector field of the $U(1)$ action on $\hat{\mathscr{B}}$ is given by $\xi^\#[\bar{\partial}_E, \phi] = (\sqrt{-1}/R)(0,\phi)$. Moreover,

$$\begin{aligned}\nabla_{\Phi_t[\bar{\partial}_E,\phi]} f &= \frac{-1}{2\pi i}\nabla_{\Phi_t[\bar{\partial}_E,\phi]}\Psi = \frac{1}{2\pi i}\xi^\#(\Phi_t[\bar{\partial}_E,\phi])\\ &= \frac{1}{2\pi R}(0, e^{-t/2\pi R}\phi) = -\frac{d}{dt}(0, e^{-t/2\pi R}\phi) = -\frac{d\Phi_t[\bar{\partial}_E,\phi]}{dt}\end{aligned}$$

which is what was to be shown.

Definition 4.2. Given a critical τ and $\vec{\rho} \in I_\tau$ as in Definition 2.19, let

$$\mathscr{W}_\tau^{\mathrm{s}}(\vec{\rho}) = \left\{ [\bar{\partial}_E, \phi] \in \hat{\mathscr{B}}: \lim_{t\to\infty} \Phi_t[\bar{\partial}_E, \phi] \in \mathscr{Z}_\tau(\vec{\rho}) \right\},$$

and let $\mathscr{W}_\tau^{\mathrm{u}}(\vec{\rho})$ be defined similarly as $t \to -\infty$. Also, for a critical value of τ, we set $\mathscr{W}_\tau^{\mathrm{s}} = \bigcup_{\vec{\rho}\in I_\tau} \mathscr{W}_\tau^{\mathrm{u}}(\vec{\rho})$ and $\mathscr{W}_\tau^{\mathrm{u}} = \bigcup_{\vec{\rho}\in I_\tau} \mathscr{W}_\tau^{\mathrm{u}}(\vec{\rho})$. We call $\{\mathscr{W}_\tau^{\mathrm{s}}\}$ and $\{\mathscr{W}_\tau^{\mathrm{u}}\}$ the stable and unstable Morse stratifications of $\hat{\mathscr{B}}$.

THEOREM 4.3. *For each critical value τ, $\mathscr{W}_\tau^{\mathrm{s}} = \mathscr{W}_\tau^{+}$, and $\mathscr{W}_\tau^{\mathrm{u}} = \mathscr{W}_\tau^{-}$. Consequently, the Morse stratification of $\hat{\mathscr{B}}$ coincides with the algebraic stratification of Section* 3.

Proof. We shall show that $\mathscr{W}_\tau^{\mathrm{u}} = \mathscr{W}_\tau^{-}$. Indeed, since both $\{\mathscr{W}_\tau^{\mathrm{u}}\}$ and $\{\mathscr{W}_\tau^{-}\}$ are stratifications of $\hat{\mathscr{B}}$, it suffices to prove the inclusion $\mathscr{W}_\tau^{-} \subset \mathscr{W}_\tau^{\mathrm{u}}$ for all τ. In fact, we are going to show that $\mathscr{W}_\tau^{-}(\vec{\rho}) \subset \mathscr{W}_\tau^{\mathrm{u}}(\vec{\rho})$ for all $\vec{\rho} \in I_\tau$. Fix $[\bar{\partial}_E, \phi] \in \mathscr{W}_\tau^{-}(\vec{\rho})$. Let $0 = E_\phi = F_0 \subset F_1 \subset \cdots \subset F_n = E$ denote the μ_- filtration of the pair (E, ϕ). Fix real numbers $0 < \mu_1 < \mu_2 < \cdots < \mu_n$ such that $\sum_{i=1}^n R_i\mu_i = R_\phi$, and consider the 1-parameter sub-group of gauge transformations in $\mathfrak{G}_0^{\mathbb{C}}$, $g_t = \mathrm{diag}(e^{t/2\pi}, e^{-t\mu_1/2\pi R}, \ldots, e^{-t\mu_n/2\pi R})$ written diagonally with respect to the filtration above. Then

$$\lim_{t\to-\infty} \Phi_t[\bar{\partial}_E, \phi] = \lim_{t\to-\infty} [\bar{\partial}_E, g_t^{-1}\phi] = \lim_{t\to-\infty} [g_t(\bar{\partial}_E), \phi] = [\mathrm{gr}^-(E,\phi), \phi],$$

where the last equality follows the same way as in [7, p. 716]. Hence, $[\bar{\partial}_E, \phi] \in \mathscr{W}_\tau^{\mathrm{u}}$. The case of the stable manifolds is similar. To prove that $\mathscr{W}_\tau^{+}(\vec{\rho}) \subset \mathscr{W}_\tau^{\mathrm{s}}(\vec{\rho})$ for all $\vec{\rho} \in I_\tau$, one must show that for all $[\bar{\partial}_E, \phi] \in \mathscr{W}_\tau^{+}(\vec{\rho}), \lim_{t\to\infty} \Phi_t[\bar{\partial}_E, \phi] = [\mathrm{gr}^+(E,\phi)]$, where $[\mathrm{gr}^+(E,\phi)]$ is the μ_+-grading (Definition 3.4). The above method can be used, but now the complex gauge transformations $g_t \in \mathfrak{G}_0^{\mathbb{C}}$ must be defined as follows. Let $0 = F_0 \subset F_1 \subset \cdots \subset F_{n+1} = E$ denote the μ_+-filtration of the pair (E, ϕ). If E is a semistable bundle, fix real numbers $1 > \mu_1 > \mu_2 > \cdots > \mu_{n+1}$ such that $\sum_{i=1}^n R_i\mu_i = 0$, and let $g_t = \mathrm{diag}(e^{t\mu_1/2\pi}, e^{t\mu_2/2\pi R}, \ldots, e^{t\mu_{n+1}/2\pi R})$ written diagonally with respect to the filtration above. If E is not semistable, then one must take $\mu_{n+1} = 1$ and impose the constraint $R_{n+1} + \sum_{i=1}^n R_i\mu_i = 0$. The rest of the argument proceeds as before.

4.2. Modifications

We are now in a position to prove

THEOREM 4.4. (i) *Suppose the interval $[\tau, \tau+\varepsilon]$ contains no critical value of the function f. Then the Morse flow induces a biholomorphism between $\mathscr{B}_{\tau+\varepsilon}$ and $\mathscr{B}_\tau$.* (ii) *Suppose that τ is the only critical value of f in the interval $[\tau, \tau+\varepsilon]$. Then the Morse flow defines a continuous map from $\mathscr{B}_{\tau+\varepsilon}$ onto $\mathscr{B}_\tau$ which restricts to a biholomorphism between $\mathscr{B}_{\tau+\varepsilon}\backslash\mathbb{P}_\varepsilon(\mathscr{W}_\tau^{+})$ and $\mathscr{B}_\tau\backslash\mathscr{Z}_\tau$, where $\mathbb{P}_\varepsilon(\mathscr{W}_\tau^{+}) = \mathscr{W}_\tau^{+} \cap f^{-1}(\tau+\varepsilon)/U(1)$.* (iii) *In the case $R = 2$, the restriction of the Morse flow to $\mathbb{P}_\varepsilon(\mathscr{W}_\tau^{+})$ induces a map $\mathbb{P}_\varepsilon(\mathscr{W}_\tau^{+}) \xrightarrow{\pi} \mathscr{Z}_\tau$ which is a holomorphic projective bundle (unless d is even and $\tau = d/2$). In particular, in rank two $\mathbb{P}_\varepsilon(\mathscr{W}_\tau^{+})$ is a smooth subvariety of $\mathscr{B}_{\tau+\varepsilon}$.*

Proof. (i) For the sake of notational simplicity, we denote the equivalence class of the pair $[\bar{\partial}_E, \phi] \in \hat{\mathscr{B}}$ by x. Let $F: f^{-1}(\tau+\varepsilon) \times [0,\infty) \to \mathbb{R}$ denote the map $F(x,t) = f(\Phi_t(x))$. By our assumption on $[\tau, \tau+\varepsilon]$, F is smooth. Moreover, given $(x,t) \in F^{-1}(\tau)$ we have

$$\left.\frac{\partial F}{\partial t}\right|_{(x,t)} = df_{\Phi_t(x)}\left(\left.\frac{\partial \Phi_t}{\partial x}\right|_x\right) = \|\nabla_{\Phi_t(x)} f\|^2 \neq 0$$

since $\tau = f(\Phi_t(x))$ is not a critical value of f. Then by the implicit function theorem we can solve $F(x,t) = \tau$ as $t = t(x)$, where t is a smooth function of x. We define

$\hat{\sigma}_+ : f^{-1}(\tau + \varepsilon) \to f^{-1}(\tau)$ by $\hat{\sigma}_+(x) = f(x, t(x))$. It follows that $\hat{\sigma}_+$ is a diffeomorphism between $f^{-1}(\tau + \varepsilon)$ and $f^{-1}(\tau)$.

Next we show that $\hat{\sigma}_+$ is a CR-map with respect to the induced CR-structure on the level sets $f^{-1}(\tau + \varepsilon)$ and $f^{-1}(\tau)$. Indeed, let $X \in T^{1,0}\hat{\mathscr{B}} \cap Tf^{-1}(\tau + \varepsilon) \otimes \mathbb{C}$, and let $\bar{X}$ denote the complex conjugate. Then

$$d\hat{\sigma}_+(\bar{X}) = \frac{\partial \Phi_t}{\partial t}\left(\frac{\partial t}{\partial x}(\bar{X})\right) + \frac{\partial \Phi_t}{\partial x}(\bar{X}) = \nabla f\left(\frac{\partial t}{\partial x}(\bar{X})\right) + \frac{\partial \Phi_t}{\partial x}(\bar{X}).$$

Since Φ_t is holomorphic in x, $\partial\Phi_t/\partial x(\bar{X}) = 0$. Hence, $d\hat{\sigma}_+(\bar{X}) = \nabla f(\partial t/\partial x(\bar{X}))$ is both tangential and normal to $f^{-1}(\tau)$, and therefore $d\hat{\sigma}_+(\bar{X}) = 0$. Thus, $\hat{\sigma}_+$ is a CR-map. Since $\hat{\sigma}_+$ and the CR-structures on $f^{-1}(\tau + \varepsilon)$ and $f^{-1}(\tau)$ are $U(1)$-invariant, $\hat{\sigma}_+$ induces a biholomorphism $\sigma_+ : \mathscr{B}_{\tau+\varepsilon} = f^{-1}(\tau + \varepsilon)/U(1) \to f^{-1}(\tau)/U(1) = \mathscr{B}_\tau$. (ii) The same argument as in (i) gives a smooth map $\hat{\sigma}_+ : f^{-1}(\tau + \varepsilon)\backslash \mathscr{W}_\tau^+ \to f^{-1}(\tau)\backslash \mathscr{Z}_\tau$. We extend $\hat{\sigma}_+$ across $\mathscr{W}_\tau^+$ by setting $\hat{\sigma}_+(x) = \lim_{t\to\infty} \Phi_t(x)$ for $x \in \mathscr{W}_\tau^+$. We are going to show that $\hat{\sigma}_+$ is continuous. It is easily seen (e.g. from the uniqueness of the filtration) that the restrictions of $\hat{\sigma}_+$ to $f^{-1}(\tau_\varepsilon)\backslash \mathscr{W}_\tau^+$ and $\mathscr{W}_\tau^+$ are continuous. Therefore, it suffices to prove that if $\{x_l\}$ is a sequence in $f^{-1}(\tau + \varepsilon)$, $x \in \mathscr{W}_\tau^+$ and $x_l \to x$, then $\hat{\sigma}_+(x_l) \to \hat{\sigma}_+(x)$. Let dist be a metric compatible with the topology of $\hat{\mathscr{B}}$. Set $t_l = t(x_l)$, where $\hat{\sigma}_+(x_l) = f(\Phi_{t(x_l)}(x_l))$. Then

$$\operatorname{dist}(\hat{\sigma}_+(x_l), \hat{\sigma}_+(x)) \leqslant \operatorname{dist}(\Phi_{t_l}(x_l), \Phi_{t_l}(x)) + \operatorname{dist}(\Phi_{t_l}(x), \hat{\sigma}_+(x)).$$

Since clearly $t_l \to \infty$ and Φ_t is uniformly continuous, both terms on the right-hand side of the above inequality go to zero, and this proves the continuity of $\hat{\sigma}_+$. Since $\hat{\sigma}_+$ is also $U(1)$-invariant, it induces a continuous map $\sigma_+ : \mathscr{B}_{\tau+\varepsilon} = f^{-1}(\tau + \varepsilon)/U(1) \to f^{-1}(\tau)/U(1) = \mathscr{B}_\tau$. On the other hand, by the same argument as in (i), σ_+ defines a biholomorphism onto its image away from $\mathbb{P}_\varepsilon(\mathscr{W}^+) = \mathscr{W}^+ \cap f^{-1}(\tau + \varepsilon)/U(1)$. (iii) First, we suppose that $\tau > d/2$ since otherwise $\mathbb{P}_\varepsilon(\mathscr{W}_\tau^+) = \mathscr{B}_{\tau+\varepsilon}$. Then the fixed point sets $\mathscr{Z}_\tau$ in $\hat{\mathscr{B}}$ are smooth, and hence $\mathscr{W}^+ \cap f^{-1}(\tau + \varepsilon)$ is a smooth submanifold of $\hat{\mathscr{B}}\backslash f^{-1}(d/2)$. The Morse flow clearly induces a continuous map $\mathscr{W}^+ \cap f^{-1}(\tau + \varepsilon) \xrightarrow{\pi} \mathscr{Z}_\tau$, which is an odd-dimensional sphere bundle (say with fiber S^{2n+1}) over $\mathscr{Z}_\tau$. Since $\mathscr{W}^+$ is an analytic subvariety, the CR-structure on $f^{-1}(\tau + \varepsilon)$ induces a CR-structure on the intersection with $\mathscr{W}^+$, and as in the proof of part (ii) above, the map π is a CR-map. Since π is also $U(1)$-invariant and the $U(1)$ action is CR, π descends to a holomorphic map $\mathscr{W}^+ \cap f^{-1}(\tau + \varepsilon)/U(1) \xrightarrow{\pi} \mathscr{Z}_\tau$, with fiber $S^{2n+1}/U(1) \simeq \mathbb{P}^n$ (it is easily checked that the action is standard). This completes the proof of Theorem 4.5.

By reversing the orientation of the flow lines, we obtain a similar result relating $\mathscr{B}_\tau$ and $\mathscr{B}_{\tau-\varepsilon}$. Combining the two results immediately proves the

COROLLARY 4.5. *If τ is the only critical value of Ψ in $[\tau - \varepsilon, \tau + \varepsilon]$, then $\mathscr{B}_{\tau-\varepsilon}, \mathscr{B}_{\tau+\varepsilon}$ are related by the diagram*

$$\begin{array}{ccc} \mathscr{B}_{\tau-\varepsilon} & & \mathscr{B}_{\tau+\varepsilon} \\ & {\scriptstyle \sigma_-}\searrow \quad \swarrow{\scriptstyle \sigma_+} & \\ & \mathscr{B}_\tau & \end{array}$$

where $\sigma_\pm$ are continuous maps. Moreover, $\sigma_\pm : \mathscr{B}_{\tau\pm\varepsilon}\backslash\sigma_\pm^{-1}(\mathscr{Z}_\tau) \to \mathscr{B}_\tau\backslash\mathscr{Z}_\tau$ are biholomorphisms.

Before continuing, we digress to prove

THEOREM 4.6 (cf. [6]). *For all noncritical values of τ, $\mathscr{B}_\tau$ is a nonsingular projective variety.*

Proof. According to [15], since $\mathscr{B}_\tau$ is a Kähler manifold we need only prove that $\mathscr{B}_\tau$ is Moishezon. First, let us assume τ is close to d/R. By Siu's proof of the Grauert–Riemenschneider conjecture (see [17]) it suffices to prove that $\mathscr{B}_\tau$ admits a Hermitian, holomorphic line bundle whose curvature is positive on a dense set. According to [5], Theorem 6.4, there is a holomorphic map $\mathscr{B}_\tau \xrightarrow{\pi} \mathscr{M}(d,R)$, where $\mathscr{M}(d,R)$ denotes the Seshadri compactification of stable bundles. Moreover, the restriction of π to the preimage of the open set $\mathscr{M}^s(d,R)$ consisting of stable bundles is a fibration with fiber $\mathbb{P}^N$. From [6] there is a Hermitian, holomorphic line bundle γ on $\mathscr{B}_\tau$ whose restriction to a fiber over $\mathscr{M}^s(d,R)$ is $\mathscr{O}_{\mathbb{P}^N}(R)$ (an $\mathscr{O}(1)$ is not always possible, due to the Brauer obstruction on $\mathscr{M}(d,R)$). By pulling back a sufficiently high power k of an ample bundle $H \to \mathscr{M}(d,R)$ we can arrange that $L = \gamma \otimes \pi^* H^k$ be positive on $\pi^{-1}(\mathscr{M}(d,R))$, and Siu's theorem then implies the projectivity of $\mathscr{B}_\tau$.

Now for the general case: Note that according to Corollary 4.5, for any critical $d/R < \tau < d/(R-1)$ the complex manifolds $\mathscr{B}_{\tau\pm\varepsilon} \backslash \sigma_\pm^{-1}(\mathscr{Z}_\tau)$ are biholomorphic. On the other hand, by Proposition 3.10, $\sigma_\pm^{-1}(\mathscr{Z}_\tau)$ has codimension at least 2 in $\mathscr{B}_{\tau\pm\varepsilon}$ (except when $R=2$ and $\tau = d-1$, but in this case the result is obvious) and $\mathscr{B}_{\tau\pm\varepsilon}$ are smooth. It then follows from the Levi extension theorem (cf. [10, p. 396]) that $\mathscr{B}_{\tau+\varepsilon}$ is bimeromorphic to $\mathscr{B}_{\tau-\varepsilon}$, and hence for all noncritical τ, $\mathscr{B}_\tau$ is Moishezon. This completes the proof. Notice that by GAGA we have also proven the following:

COROLLARY 4.7. *For all noncritical τ in $(d/R, d/(R-1))$, the spaces $\mathscr{B}_\tau$ are mutually birational.*

It is interesting to apply Corollary 4.7 to the case of $\mathscr{B}_{d/R+\varepsilon}$ and $\mathscr{B}_{(d/(R-1))-\varepsilon}$. Assume that d is coprime to both R and $R-1$ and $d > R(2g-2)$. Let $U(d,R) \to \Sigma \times \mathscr{M}(d,R)$ denote the universal bundle over the moduli space of vector bundles of rank R and degree d, and let $\pi: \Sigma \times \mathscr{M}(d,R) \to \mathscr{M}(d,R)$ denote the projection onto the second factor. It follows that $\mathscr{B}_{d/R+\varepsilon}$ is biholomorphic to the projectivization of the vector bundle $\pi_* U(d,R)$ (cf. [3]). Similarly, it follows that $\mathscr{B}_{(d/(R-1))-\varepsilon}$ is biholomorphic to the projectivization of the first direct image sheaf of $U(d,R-1)^*$ on $\mathscr{M}(d,R-1)$, which we denote by $\mathbb{P}\,(\mathrm{Ext}^1(U(d,R-1),\mathscr{O}))$. By combining with Corollary 4.7, we obtain

COROLLARY 4.8. *Assume that $d > R(2g-2)$ is coprime to both R and $R-1$. Then $\mathbb{P}(\pi_* U(d,R))$ over $\mathscr{M}(d,R)$ is birational to $\mathbb{P}\,(\mathrm{Ext}^1(U(d,R-1),\mathscr{O}))$ over $\mathscr{M}(d,R-1)$.*

Presumably, Corollary 4.8 may also be obtained by carrying out a GIT construction of these spaces as in [2, 18].

In the case of rank two our theorem combined with the result of [11] implies the following theorem of Thaddeus [18].

THEOREM 4.9. *Let $R=2$ and $d > 4(g-1)$. Suppose that $\tau \in (d/2, d)$ is the only critical value of f in the interval $[\tau-\varepsilon, \tau+\varepsilon]$. Then there is a projective variety $\tilde{\mathscr{B}}_\tau$ and holomorphic maps*

$$\begin{array}{ccc} & \tilde{\mathscr{B}}_\tau & \\ {}^{\beta_-}\swarrow & & \searrow^{\beta_+} \\ \mathscr{B}_{\tau-\varepsilon} & & \mathscr{B}_{\tau+\varepsilon} \end{array}$$

Moreover, for $\tau < d-1$, $\beta_\pm$ are blow-down maps onto the smooth subvarieties $\mathbb{P}_\varepsilon(\mathscr{W}_\tau^\pm)$. For $\tau = d-1$, β_+ is the blow-down map onto $\mathbb{P}_\varepsilon(\mathscr{W}_\tau^+)$ and β_- is the identity.

Proof. By Theorem 4.4(iii), $\mathscr{W}_\tau^\pm$ have codimension $p_+(\tau) = 2\tau - d + g - 1$ and $p_-(\tau) = d - \tau$ (see the proof of Proposition 3.10). Assume $\tau \in (d/2, d-1)$ so that $p_\pm(\tau) \geqslant 2$.

Let $\tilde{\mathscr{B}}$ denote the blow-up of $\hat{\mathscr{B}}$ along $\mathscr{Z}_\tau$. Since the Kähler structure on $\hat{\mathscr{B}}$ is defined by the curvature of a holomorphic, Hermitian line bundle (the obvious line bundle on $\mathscr{H}^*$ defining the symplectic structure descends via holomorphic and symplectic reduction to $\hat{\mathscr{B}}$) the relative version of the argument in [10, pp. 186–187] puts an explicit Kähler structure on $\tilde{\mathscr{B}}$ which agrees with the one on $\hat{\mathscr{B}}$ away from a neighborhood of the exceptional set. Extend the $U(1)$ action to $\tilde{\mathscr{B}}$. The fixed point set in a neighborhood of the exceptional set splits into $\mathbb{P}(\mathscr{W}_\tau^+)$ and $\mathbb{P}(\mathscr{W}_\tau^-)$. By dividing by the finite stabilizer as in [11, Section 11], we can make this new action quasifree. This procedure reduces the problem to the case where the signature of the Hessian in the normal directions to the fixed point set is of the form $(2, 2p)$, and the following claim, essentially due to Guillemin and Sternberg, implies the existence of the blow-down maps $\beta_\pm$:

CLAIM. *Let $\tilde{\mathscr{B}}$ be a Kähler manifold with a quasifree, holomorphic, symplectic $U(1)$ action and proper moment map Ψ. If $\mathscr{Z}_\tau$ is a critical manifold of signature $(2, 2p)$, then the gradient flow of Ψ induces a holomorphic map of the reduced spaces $\beta: \mathscr{B}_{\tau+\varepsilon} \to \mathscr{B}_{\tau-\varepsilon}$ which is a blow-down map onto $\mathbb{P}(\mathscr{W}_\tau^-) \simeq \mathscr{Z}_\tau$.*

Although the proof of the claim can be extracted from the details in [11], we shall give a brief sketch.

We first verify the claim for the "local model" consisting of $\mathbb{C}^{n+1}$ with the $U(1)$ action $e^{i\theta}(u_0, u_1, \ldots, u_n) = (e^{-i\theta}u_0, e^{i\theta}u_1, \ldots, e^{i\theta}u_n)$. In this case the gradient flow of the associated moment map $\Psi(u_0, u_1, \ldots, u_n) = -|u_0|^2 + |u_1|^2 + \cdots + |u_n|^2$ is $(e^{-t}u_0, e^t u_1, \ldots, e^t u_n)$. The reduced spaces $\Psi^{-1}(-\varepsilon)/U(1)$ and $\Psi^{-1}(\varepsilon)/U(1)$ can be identified with $\mathbb{C}^n$ and the tautological line bundle over $\mathbb{P}^n$, respectively. Under these identifications it is straightforward to see that the map

$$L = \Psi^{-1}(\varepsilon)/U(1) \to \Psi^{-1}(0)/U(1) \simeq \Psi^{-1}(-\varepsilon)/U(1) = \mathbb{C}^n$$

defined by the gradient flow coincides with the blow-up at the origin $0 \in \mathbb{C}^n$. This completes the proof in this case since the stable manifold in L is a divisor.

For the general case, choose a self-diffeomorphism φ of $\tilde{\mathscr{B}}$ which is the identity outside of a neighborhood of $\mathscr{Z}_\tau$ and is such that the pull-back symplectic structure agrees with the local model in a neighborhood of $\mathscr{Z}_\tau$, as in [11, Section 8]. We pull back the complex structure by φ so that it remains compatible with the pull-back symplectic form. By the result above, the induced map $\beta' : (\psi')^{-1}(\tau + \varepsilon)/U(1) \to (\Psi')^{-1}(\tau - \varepsilon)/U(1)$ defined by the new flow is topologically a blow-up. On the other hand, β' corresponds to β via the diffeomorphism φ. This proves the claim.

Returning to the proof of Theorem 4.9; for $\tau = d - 1$, we have $p_-(d-1) = 1$, and we are in the case of signature $(2p, 2)$. Thus, $\tilde{\mathscr{B}}_{\tau-\varepsilon} = \mathscr{B}_{\tau-\varepsilon}$, and $\tilde{\mathscr{B}}_{\tau+\xi}$ is the blow-up of $\mathscr{B}_{\tau+\varepsilon}$ along $\mathbb{P}_\varepsilon(\mathscr{W}_\tau^+)$, which may be identified with $\Sigma \times \mathscr{J}_{d-1}$ by Example 2.21.

We close with a few remarks. First, let $\mathscr{J}_d$ denote component of the Jacobian of Σ corresponding to degree d line bundles and $\det: \hat{\mathscr{B}} \to \mathscr{J}_d$ the determinant map. For $L \in \mathscr{J}_d$ let $\hat{\mathscr{B}}(L) = \det^{-1}(L)$ and $\hat{\mathscr{B}}_0(L) = \hat{\mathscr{B}}(L) \cap \hat{\mathscr{B}}_0$. Clearly, $\hat{\mathscr{B}}(L)$ and $\hat{\mathscr{B}}_0(L)$ are preserved by the $U(1)$ action, and for any noncritical value of τ, $f^{-1}(\tau) \cap \hat{\mathscr{B}}(L)/U(1)$ is biholomorphic to the moduli space $\mathscr{B}_\tau(L)$ of stable pairs of fixed determinant (see also [6, 18]). It is easily seen that all the constructions performed in the previous sections commute with the map det and thus one has the analogous theorems for $\mathscr{B}_\tau(L)$.

Perhaps the most important question is how to resolve the birational maps of Corollary 4.7. The problem is that the master space $\hat{\mathscr{B}}$ is singular along some of the critical sets. This means that $\mathbb{P}_\varepsilon(\mathscr{W}_\tau^\pm)$, the centers along which we wish to blow-up, are singular in general.

One way to proceed might be to desingularize the master space $\hat{\mathscr{B}}$ as in Kirwan [13] and extend the circle action. However, one would still have to deal with finite quotient singularities. Such a description is desirable because by Corollary 4.8 one would then have a relationship between the moduli spaces of rank R bundles which is inductive on the rank. This could be used to compute, for example, Verlinde dimensions as in [18] or perhaps even the cohomology ring structure of these spaces in a manner similar to [3].

Acknowledgement—The authors are pleased to acknowledge the warm hospitality of the Mathematics Institute at the University of Warwick where part of this work was completed.

REFERENCES

1. M. F. ATIYAH, and R. BOTT: The Yang–Mills equations over Riemann surfaces, *Phil. Trans. R. Soc. Lond. A* **308** (1982), 523–615.
2. A. BERTRAM: Stable pairs and stable parabolic pairs, preprint, 1992.
3. A, BERTRAM, G. DASKALOPOULOS and R. WENTWORTH: Gromov invariants for holomorphic maps from Riemann surfaces to Grassmannians, to appear in *J. AMS.*
4. S. B. BRADLOW: Special metrics and stability for holomorphic bundles with global sections, *J. Differential Geom.* **33** (1991), 169–214.
5. S. B. BRADLOW and G. D. DASKALOPOULOS: Moduli of stable pairs for holomorphic bundles over Riemann surfaces, *Internat. J. Math.* **2** (1991), 477–513.
6. S. B. BRADLOW and G. D. DASKALOPOULOS: Moduli of stable pairs for holomorphic bundles over Riemann surfaces II, *Internat. J. Math.* **4** (1993), 903–925.
7. G. D. DASKALOPOULOS: The topology of the space of stable bundles over a compact Riemann surface, *J. Differential Geom.* **36** (1992), 699–746.
8. O. GARCIA-PRADA: Dimensional reduction of stable bundles, vortices and stable pairs, preprint.
9. M. GORESKY and R. MACPHERSON: On the topology of torus actions, SLNM 1271, Springer Verlag Berlin, Heidelberg (1987).
10. P. GRIFFITHS and J. HARRIS: *Principles of algebraic geometry*, Wiley, New York (1978).
11. V. GUILLEMIN and S. STERNBERG: Birational equivalence in the symplectic category, *Invent. Math.* **97** (1989), 485–522.
12. F. KIRWAN: Cohomology of quotients in symplectic and algebraic geometry, Princeton University Press, Princeton (1984).
13. F. KIRWAN: On the homology of compactifications of moduli spaces of vector bundles over a Riemann surface, *Proc. London Math. Soc.* (3) **53** (1986), 237–266.
14. S. KOBAYASHI: *Differential geometry of complex vector bundles*, Princeton University Press, Princeton (1987).
15. B. G. MOISHEZON: A criterion for projectivity of complete algebraic abstract varieties, *AMS Transl.* **63** (1967), 1–50.
16. P. E. NEWSTEAD: *Introduction to moduli problems and orbit spaces*, Tata Inst. Lectures **51**, Springer, Heidelberg (1978).
17. Y.-T. SIU: A vanishing theorem for semipositive line bundles over non-Kähler manifolds, *J. Differential Geom.* **19** (1984), 431–452.
18. M. THADDEUS: Stable pairs, linear systems, and the Verlinde formula, preprint (1992).

Department of Mathematics
University of Illinois
Urbana, IL 61801, U.S.A.

Department of Mathematics
Brown University
Providence, RI 02912, U.S.A.

Department of Mathematics
University of California, Irvine
Irvine, CA 92717, U.S.A.