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ATTRIBUTES AND APPLICATIONS OF THE CORWIN-GREENLEAF MULTIPLICITY FUNCTION

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1. Introduction. One of the most important ingredients in the Corwin-Greenleaf description of the direct integral decomposition of an induced representation for simply connected nilpotent Lie groups is the multiplicity function they obtain. It is given in terms of orbital geometry. Both its format and the values it can take on have proven to be very important in understanding the nature of induced representations for more general groups, as well as the harmonic analysis of various homogeneous spaces. It is my purpose in this paper to explore the many attributes of the Corwin-Greenleaf multiplicity function, and then to discuss several applications of it in harmonic analysis. The applications (in section four) will be both: direct — to maximal subgroups, and to irreducible induced representations; and indirect — to reciprocity, and to multiplicity of symmetric spaces. These results, and other considerations motivated by the Corwin-Greenleaf multiplicity function, lead to a general conjecture (in section five) on a characterization of bounded finite multiplicity for algebraic homogeneous spaces.

Corwin and Greenleaf have worked exclusively in the context of simply connected nilpotent Lie groups. As such, the governing principle in the description of the representation theory and harmonic analysis is the Kirillov method of coadjoint orbits. Here are the main ingredients. Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. The real linear dual of $\mathfrak{g}$ is denoted $\mathfrak{g}^*$. For any $\phi \in \mathfrak{g}^*$, we write B_ϕ for the alternating form $B_\phi(X,Y) = \phi[X,Y]$ on $\mathfrak{g}$. Then one can always choose a *real polarization* for ϕ, that is a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ which is maximally totally isotropic for the form B_ϕ. The character

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$\chi_\phi(\exp X) = e^{i\phi(X)}$ is then well-defined on the simply connected analytic subgroup A of G having Lie algebra $\mathfrak{a}$, and the induced representation $\pi_{\phi,\mathfrak{a}} = \mathrm{Ind}_A^G \chi_\phi$ is irreducible. Its class is independent of $\mathfrak{a}$, and so is unambiguously denoted π_ϕ. The map $\phi \to \pi_\phi$, $\mathfrak{g}^* \to \hat{G}$ is surjective and G-equivariant; and factors to a bijection $\mathfrak{g}^*/G \to \hat{G}$ of the coadjoint orbits onto the unitary dual. Now let H be any connected (and therefore simply connected) closed subgroup of G. Suppose that $\nu \in \hat{H}$ and $\mathcal{O}_\nu \subset \mathfrak{h}^*$ is the corresponding coadjoint orbit. Let $p : \mathfrak{g}^* \to \mathfrak{h}^*$ be the canonical projection. Then the *Corwin-Greenleaf orbital integral formula* for the description of the induced representation $\mathrm{Ind}_H^G \nu$ is as follows:

(CGoif) $$\mathrm{Ind}_H^G \nu = \int_{p^{-1}(\mathcal{O}_\nu)/H}^{\oplus} \pi_\phi \, d\dot{\phi} = \int_{G \cdot p^{-1}(\mathcal{O}_\nu)/G}^{\oplus} n_\phi^\nu \pi_\phi \, d\ddot{\phi},$$

where

(CGmf) $$n_\phi^\nu = \#(G \cdot \phi \cap p^{-1}(\mathcal{O}_\nu))/H.$$

n_ϕ^ν is called the *Corwin-Greenleaf multiplicity function.* The measures in (CGoif) are push-forwards of Lebesgue measure. The reader should consult [**7**] and [**19**] for these results. It is also worth noting that Corwin and Greenleaf had two powerful motivations for this result, namely the corresponding formulas derived by: Benoist [**3**] in the case that G/H is symmetric (and $\nu = 1$); and by Vergne [**35**] in the case G is exponential solvable, ν is a character, and $\mathfrak{h}$ is a real polarization for ν, not satisfying the Pukanszky condition.

For convenience we shall restate the Corwin-Greenleaf results in the special case that $\nu = 1$, that is the *quasi-regular representation.* In that case, the result becomes

(CGqr) $$\mathrm{Ind}_H^G 1 = \int_{\mathfrak{h}^\perp/H}^{\oplus} \pi_\phi \, d\dot{\phi} = \int_{G \cdot \mathfrak{h}^\perp/G}^{\oplus} n_\phi \pi_\phi \, d\ddot{\phi},$$

where if we write $\mathfrak{h}^\perp = \mathfrak{h}^\perp(\mathfrak{g}) = p^{-1}(\{0\}) = \{\phi \in \mathfrak{g}^* : \phi(\mathfrak{h}) = 0\}$, the formula for the multiplicity function $n_\phi = n_\phi^1$ is now

(CGmf1) $$n_\phi = \#(G \cdot \phi \cap \mathfrak{h}^\perp)/H.$$

It is the multiplicity functions (CGmf) and (CGmf1) that are the main focus of this paper. It is our goal to state and examine the various qualitative and quantitative properties of the function that have been discovered, and then to explore the possible preservation of those properties (or failure thereof) in various other structural contexts. In so doing we will uncover several applications, especially to algebraic homogeneous spaces and to symmetric spaces. The properties — or "attributes" as I prefer to call them — are discussed in section two. The more general structural contexts — "templates" is my terminology — are laid out in section three. As indicated earlier, the applications are in section four, and the general conjecture about criteria for specifying bounded finite multiplicity in algebraic homogeneous spaces is developed in section five.

This paper in an eclectic mixture of new results with proofs, exposition and elaboration of recent work, and a presentation of many new examples that have not appeared in the literature.

2. The Attributes. Now we examine the properties of the Corwin-Greenleaf multiplicity function ((CGmf) and (CGmf1)) that have been derived in [**7**], [**9**] and [**19**]. The reader should keep in mind that the function n_ϕ^ν is to be thought of as a function of ϕ for fixed ν. It is a remarkable fact (see [**8**] and [**19**]) that the same function serves as the multiplicity function for restricted representations

$$\pi_\phi|_H = \int_{p(G\cdot\phi)/H}^{\oplus} n_\phi^\nu \nu_\psi \, d\dot{\psi}.$$

Studying the multiplicity function for restricted representations amounts to fixing ϕ and investigating its properties as a function of ν (or ψ). We shall not do so in this paper — all of our attention will be focused on induced representations.

Here are the key properties enjoyed by n_ϕ^ν:

(1) n_ϕ^ν is either $+\infty$ *a.e.* or finite-valued *a.e.*;
(2) The multiplicity is finite iff

$$\text{(CGfmc)} \qquad \dim G\cdot\phi - 2\dim H\cdot\phi + \dim \mathcal{O}_\nu = 0, \text{ generically on } p^{-1}(\mathcal{O}_\nu);$$

(3) If the multiplicity is finite, it is bounded;
(4) If the multiplicity is finite, it has constant parity.

Here is some "philosophy" to explain the reasons behind the truth of these phenomena. First of all, the only way the multiplicity function

$$n_\phi^\nu = \#(G\cdot\phi \cap p^{-1}(\mathcal{O}_\nu))/H$$

can be finite is if the H-orbits on the varieties $G\cdot\phi\cap p^{-1}(\mathcal{O}_\nu)$ are open. The open H-orbits would then coincide with the components of the variety. But, since we are dealing with nilpotent groups, we are in an algebraic situation — in which case the components must be finite in number. Moreover, because the different varieties $G\cdot\phi\cap p^{-1}(\mathcal{O}_\nu), \phi \in p^{-1}(\mathcal{O}_\nu)$, will be described by the same polynomial equations — only the coefficients will change as ϕ varies in $p^{-1}(\mathcal{O}_\nu)$, relatively routine principles in algebraic geometry guarantee that there will be a single upper bound for the number of components as one varies over the manifold $p^{-1}(\mathcal{O}_\nu)$. On the other hand, the only way that infinite multiplicity could occur is if the generic (and hence all) H-orbits have lower dimension than the intersection varieties. These remarks explain properties (1) and (3). Property (4) is a straightforward consequence of the fact that for a single polynomial equation, as the coefficients vary, the number of roots is generically of the same parity. Finally, property (2) comes about because of the following computation (see [**19, p. 450**]): In fact, it is not hard to see that generically on $p^{-1}(\mathcal{O}_\nu)$, the variety $G\cdot\phi\cap p^{-1}(\mathcal{O}_\nu)$ is a manifold, and so if $\nu \in \hat{H}$ corresponds to a functional

$\psi \in \mathfrak{h}^*$, then

$$\begin{aligned}
\dim \mathfrak{h} \cdot \phi &= \dim H \cdot \phi \leq \dim G \cdot \phi \cap p^{-1}(\mathcal{O}_\nu) \leq \mathfrak{g} \cdot \phi \cap p^{-1}(\mathfrak{h} \cdot \psi) \\
&= \dim \mathfrak{g}_\phi^\perp(\mathfrak{g}) \cap p^{-1}(\mathfrak{h}_\psi^\perp(\mathfrak{h})) \\
&= \dim \mathfrak{g}_\phi^\perp(\mathfrak{g}) \cap \mathfrak{h}_\psi^\perp(\mathfrak{g}) \\
&= \dim(\mathfrak{g}_\phi \cap \mathfrak{h}_\psi)^\perp(\mathfrak{g}) \\
&= \dim \mathfrak{g} - (\dim \mathfrak{g}_\phi + \dim \mathfrak{h}_\psi - \dim \mathfrak{g}_\phi \cap \mathfrak{h}_\psi) \\
&= (\dim \mathfrak{g} - \dim \mathfrak{g}_\phi) - (\dim \mathfrak{h}_\psi - \dim(\mathfrak{h}_\psi \cap \mathfrak{g}_\phi) \\
&= (\dim \mathfrak{g} - \dim \mathfrak{g}_\phi) - (\dim \mathfrak{h} - \dim \mathfrak{h}_\phi) + (\dim \mathfrak{h} - \dim \mathfrak{h}_\psi) \\
&= \dim \mathfrak{g} \cdot \phi - \dim \mathfrak{h} \cdot \phi + \dim \mathcal{O}_\nu.
\end{aligned}$$

The second inequality is generically an equality. In any event, we see that on $p^{-1}(\mathcal{O}_\nu)$, we have

$$\dim G \cdot \phi - 2 \dim H \cdot \phi + \dim \mathcal{O}_\nu \geq 0,$$

and that generically the inequality is an equality precisely when generically H has open orbits on the intersection varieties $G \cdot \phi \cap p^{-1}(\mathcal{O}_\nu)$. This explains the *Corwin-Greenleaf finite multiplicity condition* (CGfmc). Once again for convenience we separate out the special case $\nu = 1$,

(CGfmc1) $$\dim G \cdot \phi = 2 \dim H \cdot \phi, \text{ generically on } \mathfrak{h}^\perp.$$

(We often refer to (CGfmc) or (CGfmc1) as the "CG condition."

Now the main question we address is the following. Consider the CG orbital integral formula ((CGoif), (CGqr)), the CG finite multiplicity condition (CGfmc), (CGfmc1), and especially the CG multiplicity function ((CGmf) or (CGmf1)) and its attributes. How particular are these formulas, conditions and attributes to simply connected nilpotent groups; how wide a domain of applicability might they enjoy for more general groups G and homogeneous spaces G/H? Experience has shown (see **[14]** **[22]** **[23]** **[26]** **[29]**) that the CG orbital integral formula for describing the direct integral decomposition of an induced representation — especially the quasi-regular representation — has a reasonably good track record as a candidate for generalization. If the orbit method can be used to describe the parameters of harmonic analysis on a class of groups, then the orbital formula for induced representations also has an excellent chance of being valid. In fact, I proved such formulas for completely solvable Lie groups **[22]**; then Fujiwara extended those results to exponential solvable groups **[14]**. For type I solvable Lie groups, there are additional parameters in the orbit data (aside from the coadjoint orbits themselves, the duals of the component groups of the stabilizers of linear functionals appear). But analogs of the formula are still valid **[26]**. It is even the case that, suitably interpreted, orbital formulas for direct integral decompositions hold in certain semidirect and semisimple group situations **[19]** **[24]**. Finally, in all of these instances, the expression of the CG multiplicity function in the form (CGmf) or (CGmf1) continues to be valid. On the other hand, the fate of the attributes (1) – (4) for the multiplicity function is very different. Let us elaborate.

First of all, property (4) is clearly special to nilpotent groups, where the multiplicity function is polynomial. It is very easy to write down completely solvable groups (and of course semisimple examples) where it is not true [**21**]. Next, while properties (1)–(3) do remain valid for completely solvable groups [**25**], property (2) fails for exponential solvable groups. The famous example, due to Grélaud, is the following: Let G be the simply connected 3-dimensional solvable Lie group whose Lie algebra $\mathfrak{g}$ has generators X, Y, Z satisfying bracket relations $[X,Y] = Y + Z$, $[X,Z] = -Y + Z$. If we parameterize $\mathfrak{g}^*$ by $\phi_{\xi,\eta,\zeta} = \xi X^* + \eta Y^* + \zeta Z^*$, then the coadjoint orbits consist of points $\{\phi_{\xi,0,0}\}, \xi \in \mathbb{R}$, and 2-dimensional spiral-shaped cylinders $\{\phi_{\xi, e^t(\eta\cos t - \zeta \sin t), e^t(\eta \sin t + \zeta \cos t)} : \xi, t \in \mathbb{R}\}, \eta^2 + \zeta^2 = 1$. Now take $\mathfrak{h} = \mathbb{R}Y, H$ the corresponding group, and $\nu = 1$. Then $\mathfrak{h}^\perp = \{\phi_{\xi,0,\zeta} : \xi, \zeta \in \mathbb{R}\}$, and the generic ($\zeta \neq 0$) orbit intersections are 1-dimensional varieties $\{\phi_{\xi,0,(-1)^n \zeta e^{n\pi}} : \xi \in \mathbb{R}, n \in \mathbb{Z}\}$, on which H has open orbits, but infinitely many on each variety. Thus the CG condition is satisfied, but the multiplicity is infinite. We note for future reference that this group is **not algebraic**.

Next to see how property (1) goes awry, all we have to do is drop the simply connected assumption. The easiest example is provided by a quotient of the Heisenberg group. In fact, if one takes the connected two-step three-dimensional nilpotent Lie group, with a circle group (instead of the real line) for the center, then the quasi-regular representation of that group contains (as a direct sum) the generic representations (a discrete family of infinite-dimensional representations parameterized by the non-trivial characters of the center) with infinite multiplicity, but also contains (as a direct integral) the two-parameter family of all characters with multiplicity 1. Once again I note for future reference that this is **not an algebraic** group. Finally, we observe that property (3) fails already for compact groups. Just take G compact and H trivial. The regular representation has unbounded finite multiplicity. We can summarize: The CG orbital formula and the CG multiplicity function have wide applicability beyond nilpotent groups, but the attributes listed in properties (1)-(4) do not survive very well outside the nilpotent habitat. How can we make sense of that? Indeed, property (4) is very special to nilpotent groups and we shall not be concerned with it further. But properties (1) – (3) are more universal. They reappear in many other situations. Can we categorize those situations, and can we characterize the representation-theoretic and/or analytic structure they are determining? We shall endeavor to do exactly that beginning in the next section by looking at several different classes of examples.

Before doing so, we make one more important and highly relevant point. In simply connected nilpotent groups, it was proven by Corwin and Greenleaf [**10**] that in the case of finite multiplicity, the algebra of G-invariant differential operators $D(G/H)$ is commutative. It is strongly believed that the converse is true, although that has not been verified yet. In our examples we shall test the commutativity of the algebra $D(G/H)$, and we shall see that the results are pertinent to the characterization problem we are investigating.

3. The Templates. We shall now examine some important and interesting homogeneous spaces G/H. With the exception of the first example below, the induced representation we consider shall be the quasi-regular representation. We always denote it by the symbol τ,

$$\tau = \mathrm{Ind}_H^G 1.$$

As we proceed, we keep in mind three questions:

(1) What is the multiplicity in the direct integral decomposition?
(2) To what extent is the Corwin-Greenleaf condition valid?
(3) Is the algebra of invariant differential operators $D(G/H)$ commutative?

Example 1. Exponential Solvable Lie Groups. If G is simply connected exponential solvable and H is a closed (simply) connected subgroup, then the Orbit Method applies. Everything asserted in section 1 for nilpotent groups is valid, except that the real polarizations must be taken to satisfy the Pukanszky condition — namely, $\phi + \mathfrak{a}^\perp = A \cdot \phi$. It is known [**14**] that the orbital integral formula (CGoif) is valid. A violation of the CG condition guarantees uniform infinite multiplicity, but as we have seen from the Grélaud example its observance does not insure finite multiplicity. However, if we assume that the group is also algebraic, then in fact it must be completely solvable. Therefore, by [**22**] and [**25**], we are once again in the same pleasant situation as for nilpotent groups, namely (CGoif), (CGmf) and properties (1) – (3) are true.

Regarding differential operators, we can give a partial answer to question (3) above.

Proposition 1. *Let G be algebraic exponential solvable with Levi decomposition $G = AN$. Suppose H is a closed connected subgroup with $\tau = \mathrm{Ind}_H^G 1$ having finite multiplicity. Suppose in addition that: (i) $H = BM, B \subset A, M \subset N$; (ii) $\tau_N = \mathrm{Ind}_M^N 1$ has finite multiplicity and (iii) $D(N/M)$ is isomorphic to $\mathbb{C}[\mathfrak{m}^\perp(\mathfrak{n})]^M$. Then $D(G/H)$ is commutative.*

Proof. Condition (i) amounts to assuming that H is algebraic. Now we observe that

$$\tau = \mathrm{Ind}_H^G 1 = \mathrm{Ind}_{BN}^{AN} \mathrm{Ind}_{BM}^{BN} 1.$$

It follows *a fortiori* that the quasi-regular representation $\mathrm{Ind}_{BM}^{BN} 1$ must have finite multiplicity. The results of [**20**] are now germane. It is pointed out there that the hypothesis (ii) may not be satisfied. However if it is, it follows that the algebra $D(BN/BM)$ — which injects into $D(N/M)$ — must be abelian. Next we can decompose the split abelian group A into a direct product $A = BC$. Writing $L = BN$, we can write $G = CL$ and observe that we are reduced to showing that $D(G/H) = D(CL/H)$ is commutative. But the latter also injects into $D(CN/M)$. In [**10**] strong evidence is presented for the assertion that the algebra $D(N/M)$ is naturally isomorphic to the M-invariants in the polynomial algebra $\mathbb{C}[\mathfrak{m}^\perp]$. If that is so, (assumption (iii)), then the algebra $D(CN/M)$ is clearly commutative.

It seems likely that the proposition — as well as its converse — is true without the three additional hypotheses.

Example 2. Symmetric Spaces. In this paper by a symmetric space we mean G/H where G is a Lie group and H is the set of fixed points of an involutive automorphism σ of G. (We abuse notation by also writing σ to denote the differentiated action on $\mathfrak{g}$ and the transpose action on $\mathfrak{g}^*$.) We will consider **five** different kinds of symmetric spaces. But first here is a general result.

PROPOSITION 2. *If G/H is symmetric, $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ is the ± 1 eigenspace decomposition with respect to σ, and $\phi \in \mathfrak{h}^\perp$, then $\mathfrak{g} \cdot \phi \cap \mathfrak{h}^\perp = \mathfrak{h} \cdot \phi$.*

PROOF. First we note that $\phi \in \mathfrak{h}^\perp$ iff $\sigma(\phi) = -\phi$. Then if $X \in \mathfrak{h}, Y \in \mathfrak{q}$ and both ϕ and $(X+Y)\cdot\phi$ belong to $\mathfrak{h}^\perp$, we can compute

$$-(X+Y)\cdot\phi = \sigma((X+Y)\cdot\phi) = \sigma(X+Y)\cdot\sigma(\phi) = (X-Y)\cdot -\phi.$$

Therefore $Y \cdot \phi = 0$.

COROLLARY. *The condition (CGfmc1) is valid for any symmetric space G/H.*

PROOF. This follows immediately from the proposition and the long computation in section two.

Thus for symmetric spaces, the answer to question (2) is "always." Now to the five examples.

(i) THE GROUP CASE. Here the ambient group is $G \times G$, with involution $\sigma(g_1, g_2) = (g_2, g_1)$. Then $H = \{(g,g) : g \in G\}, \tau = \lambda_G \times \rho_G$, the left and right regular representations of G respectively, and (provided G is type I) we have the multiplicity-free direct integral decomposition

$$\tau = \int_{\hat{G}}^{\oplus} \pi \times \bar{\pi}\, d\mu_G(\pi),$$

where μ_G is the usual Plancherel measure. The Corollary to Proposition 2 tells us that (CGfmc1) is satisfied and it is routine to check that $n_\phi = 1$ for $\phi \in \mathfrak{h}^\perp$. Moreover, the algebra of invariant differential operators is naturally isomorphic to the center of the enveloping algebra of $\mathfrak{g}$, certainly commutative.

(ii) REDUCTIVE SYMMETRIC SPACES. Suppose that G is a reductive Lie group in the Harish-Chandra class. Then it is a well-known result [**2**] that the quasi-regular representation of any symmetric space of G has finite multiplicity. The proof uses Penney's result [**32**]: in the direct integral decomposition

$$\tau = \int_{\hat{G}}^{\oplus} n_\pi\, \pi\, d\mu_{G,H}(\pi)$$

of the quasi-regular representation of (any, not just symmetric) type I homogeneous space, one has that

$$n_\pi \leq \dim(\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}}, \ \mu_{G,H} - a.e.;$$

where $\mathcal{H}_\pi$ denotes a Hilbert space on which π acts, $\mathcal{H}_\pi^\infty$ is the set of C^∞-vectors, $\mathcal{H}_\pi^{-\infty}$ is its dual of (anti-)distributions, and $(\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}}$ indicates the subspace of distributions which transform under the action of H by the character $q^{-1/2}$ where q is the quotient of the modular functions of H and G. For G reductive of the Harish-Chandra class, the latter spaces have finite dimension according to van den Ban's theorem [**2**]. Van den Ban's estimates incorporate dimensions of K-types and so no upper bound is evident. It is believed that the multiplicity function for a reductive symmetric space is bounded, but that is apparently not proven explicitly yet. It is also believed that the numbers $n_\phi = \#(G \cdot \phi \cap \mathfrak{h}^\perp)/H$, for $\phi \in \mathfrak{h}^\perp$ are identical to the indices of certain Weyl groups that should specify the multiplicity in τ — but that also has not been completely clarified yet. Finally, it is known that the algebra $D(G/H)$ is isomorphic to the corresponding algebra for the associated Riemannian symmetric space (see [**12**]), therefore it is abelian.

(iii) Nilpotent Symmetric Spaces. Now suppose G is simply connected nilpotent. Nilpotent symmetric spaces G/H are special cases of the homogeneous spaces discussed in section 1. As mentioned there, their analysis was derived by Benoist [**3**]. He lifts to the group the computation in Proposition 2. In fact in this case one knows [**3**] that $G = H \exp \mathfrak{q}$. Therefore, writing $g = h \exp X, h \in H, X \in \mathfrak{q}$, if ϕ and $g \cdot \phi$ are both in $\mathfrak{h}^\perp$, then

$$-h \exp X \cdot \phi = -g \cdot \phi = \sigma(g \cdot \phi) = \sigma(g) \cdot \sigma(\phi) = h \exp -X \cdot -\phi.$$

This implies $\exp 2X \cdot \phi = \phi$. But in the nilpotent case we have $\exp \mathfrak{g}_\phi = G_\phi$. Therefore $2X \cdot \phi = 0 \Rightarrow X \cdot \phi = 0 \Rightarrow g \cdot \phi = h \cdot \phi$, that is $G \cdot \phi \cap \mathfrak{h}^\perp = H \cdot \phi \Rightarrow n_\phi = 1$.

(iv) Abelian Symmetric Spaces. Let $G = H \ltimes V$ be a semidirect product of a Lie group H with a normal vector subgroup V. We make no assumption on H other than that $\hat{V}/H$ is countably separated. The automorphism σ which is the identity on H and minus the identity on V is an involution rendering G/H into what is called an abelian symmetric space. The theory of these spaces was worked out in [**18**]. The representations of G are described by the Mackey Machine: For $\chi \in \hat{V}$, we take any irreducible representation ν of H_χ and form the induced representation $\pi_{\chi,\nu} = \mathrm{Ind}_{H_\chi V}^G \nu \times \chi$. According to [**18**] we have

(a) $$\tau = \int_{\hat{V}/H}^{\oplus} \pi_{\chi,1} \, d\dot{\chi},$$

a multiplicity-free decomposition. Note that the space $\hat{V}/H$ is naturally identified to $\mathfrak{h}^\perp/H$, and that — by essentially the same proof as in part (iii) — for any $\phi \in \mathfrak{h}^\perp$ we have $G \cdot \phi \cap \mathfrak{h}^\perp = H \cdot \phi$. Furthermore the algebra $D(G/H)$ is obviously identified to the algebra of H-invariant constant coefficient differential operators on V — clearly abelian.

(v) Strichartz Symmetric Spaces. These form a natural generalization of abelian symmetric spaces. We assume the same setup as in (iv), but we suppose now that the conormal subgroup is reductive, $G = R \ltimes V$. We examine more complicated involutions — namely those which arise as a combination $\sigma(rv) = \sigma_R(r)\sigma_V(v)$, where σ_R, σ_V are involutions of R, V respectively. This requires that the two involutions σ_R, σ_V satisfy the equivariance relation

$$\sigma_R(r)\sigma_V(v)\sigma_R(r)^{-1} = \sigma_V(rvr^{-1}).$$

In fact, any involution of G is conjugate to one of this form (because of Theorem 7 below). If we set $S = R^\sigma$, $U = V^\sigma, H = G^\sigma = SU$, then G/H is a Strichartz symmetric space [**28**]. Here is a Mackey Machine decomposition of $\tau = \mathrm{Ind}_H^G 1$ into irreducibles:

$$\begin{aligned}
\tau = \mathrm{Ind}_{SU}^{RV} 1 &= \mathrm{Ind}_{SV}^{RV}\mathrm{Ind}_{SU}^{SV} 1 \\
&= \mathrm{Ind}_{SV}^{RV} \int_{U^\perp/S}^{\oplus} \mathrm{Ind}_{S_\lambda V}^{SV} 1 \times \lambda \, d\dot{\lambda} \qquad \text{(by (a))} \\
&= \int_{U^\perp/S}^{\oplus} \mathrm{Ind}_{S_\lambda V}^{RV} 1 \times \lambda \, d\dot{\lambda} \\
&= \int_{U^\perp/S}^{\oplus} \mathrm{Ind}_{R_\lambda V}^{RV} \mathrm{Ind}_{S_\lambda V}^{R_\lambda V} 1 \times \lambda \, d\dot{\lambda} \\
&= \int_{U^\perp/S}^{\oplus} \mathrm{Ind}_{R_\lambda V}^{RV} \int_{\hat{R}_\lambda(S_\lambda)}^{\oplus} n_\lambda(\nu)\nu \times \lambda \, d\mu_\lambda(\nu) d\dot{\lambda} \\
&= \int_{U^\perp/S}^{\oplus} \int_{\hat{R}_\lambda(S_\lambda)}^{\oplus} n_\lambda(\nu)\pi_{\lambda,\nu} \, d\mu_\lambda(\nu) d\dot{\lambda} \\
&= \int_{R\cdot U^\perp/R}^{\oplus} \int_{\hat{R}_\lambda(S_\lambda)}^{\oplus} n_\lambda n_\lambda(\nu)\pi_{\lambda,\nu} \, d\mu_\lambda(\nu) d\ddot{\lambda}
\end{aligned}$$

Here: $d\dot{\lambda}, d\ddot{\lambda}$ are the measure classes of pseudo-images of Lebesgue measure $d\lambda$; $(\hat{R}_\lambda(S_\lambda), n_\lambda(\cdot), d\mu_\lambda)$ are the Plancherel constituents of the "little" quasi-regular representation

$$\tau_\lambda = \mathrm{Ind}_{S_\lambda}^{R_\lambda} 1 = \int_{\hat{R}_\lambda(S_\lambda)}^{\oplus} n_\lambda(\nu)\nu \, d\mu_\lambda(\nu);$$

and $\hat{R}_\lambda(S_\lambda)$ is the spectrum of τ_λ, and $n_\lambda = \#(R \cdot \lambda \cap U^\perp)/S$ is its multiplicity. (See [**28**,§§**3-5**] for a discussion of how the symmetry of G/H simplifies the computation.) The spaces R_λ/S_λ are symmetric (since σ

preserves R_λ — see [**28,Thm.5.3**]), and they have lower dimension than that of G/H. This is exploited in [**30**] to prove that G/H has finite multiplicity. We will discuss the possible boundedness of the multiplicity and commutativity of $D(G/H)$ later.

Example 3. Generalized Oscillator Groups and Gelfand Pairs. Here is another category of homogeneous space that has received a great deal of scrutiny in recent times. Let $G = K \ltimes N$ be a semidirect product of a compact Lie group K by a normal simply connected nilpotent Lie group N. The homogeneous space to be considered is G/K. Such a space will be symmetric iff N is abelian (since that is the only way the inversion map on N can be an involution). By definition, G/K — or more accurately the couple (G, K) — is called a *Gelfand pair* if the convolution algebra $L^1(K\backslash\backslash G)$ of K-bi-invariant integrable functions on G is commutative. This is known [**4**] [**16, pp. 485 & 584**] to be equivalent to either of the conditions: τ is multiplicity-free, or $D(G/K)$ is commutative. Furthermore, according to [**4**], these can occur essentially only when N is a Heisenberg group (of dimension $2r+1$, say N_r, for some $r \geq 1$), in which case we may as well assume that K is a subgroup of the unitary group $U(r)$.

PROPOSITION 3. *The only possibilities for the multiplicity function in this case are: identically 1; finite and unbounded; or uniformly $+\infty$.*

PROOF. It is proven in [**5**] that if a representation occurs in τ with multiplicity larger than 1, then other representations occur with arbitrarily large multiplicity. What must be shown is that no example with a mixture of finite and infinite multiplicities is possible. But exactly as in [**5**], the multiplicities in τ are the same as the multiplicities of K in $P[V]$, where $V = N_r/\text{Cent } N_r$ and $P[V]$ is the polynomial algebra over the complex vector space V. The desired result follows from a standard fact in classical invariant theory which asserts that each K-isotypic component is finitely generated over the algebra of invariants. Thus a single K-module with infinite multiplicity forces the existence of a non-constant invariant polynomial p. But then if W is any irreducible K-module, the subspaces $p^j W$, $j \geq 0$, will yield infinitely many K-modules which are equivalent to W.

There is work underway to show that the Gelfand pair condition is equivalent to the CG condition. Partial results may be found in [**5**]. Moreover it appears that in the case of a Gelfand pair, the CG orbital formula is valid. (see [**5**]).

There is a reformulation of the CG condition which can be tied to Proposition 3. The formulation is in terms of the (non-normalized) *moment map*. Choose a K-invariant non-degenerate hermitian form $< \cdot, \cdot >$ on V and consider the K-equivariant map

$$\mu : V \to \mathfrak{k}^*, \quad \mu(v)(X) = < X \cdot v, v > .$$

Then [**5**] the equivalence of the CG and Gelfand pair conditions is the same as asserting that (G, K) is a Gelfand pair iff μ is 1-1 on K-orbits. The other two situations in Proposition 3 are conjectured to be equivalent (in terms of μ) to: $\mu^{-1}(\{0\})$ is a single K-orbit, and no condition, respectively. None of these issues has been settled yet.

Example 4. Compact Groups. It is instructive to see how the simple case where G is compact and H is trivial fits into the scheme of things. Of course, G is algebraic, its regular representation has finite multiplicity, but unbounded unless G is abelian. A similar fact pertains to $D(G)$ — namely, it will be commutative iff G is abelian. The CG condition will also hold exactly when G is abelian.

Note. It is beginning to look very much like the conditions of: bounded finite multiplicity, commutativity of $D(G/H)$, and validity of the CG condition (CGfmc) are all very much tied together. But before formulating a precise conjecture, we consider another class of examples, and also the four applications I have in mind.

Example 5. Classical Semisimple Homogeneous Spaces. Here we consider the familiar homogeneous spaces of a semisimple Lie group G. We suppose G has a faithful finite-dimensional representation and fix an Iwasawa decomposition $G = KAN$. Let M be the centralizer of A in K, so that MAN is a minimal parabolic subgroup.

(i) Riemannian Symmetric Space G/K. This is more special than Example 2(ii). One knows that

$$\tau = \operatorname{Ind}_K^G 1 = \int_{\hat{A}/W}^{\oplus} \pi_\alpha \, d\alpha,$$

a multiplicity-free decomposition into spherical principal series

$$\pi_\alpha = \operatorname{Ind}_{MAN}^G 1 \times \alpha \times 1.$$

W denotes the Weyl group $W = M'/M$, $M' = \operatorname{Norm}_K(A)$. Since $G = K \exp \mathfrak{p}$ ($\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition), Benoist's proof works to show that $G \cdot \phi \cap \mathfrak{k}^\perp = K \cdot \phi$, $\phi \in \mathfrak{k}^\perp$ — which agrees with the multiplicity one fact asserted. The CG condition holds (since (G/K is symmetric so Proposition 1 applies), and of course $D(G/K)$ is commutative.

(ii) Horocycle Space G/MN. In this case one knows that

$$\tau = \#(W) \operatorname{Ind}_K^G 1 = \int_{\hat{A}}^{\oplus} \pi_\alpha \, d\alpha,$$

that is the quasi-regular representation has the same spectrum as the Riemannian symmetric space — but with uniform multiplicity the order of the Weyl group W. The homogeneous space G/MN is the horocycle space. It is known [**15**] that $D(G/MN)$ is abelian. Moreover, the CG condition is valid. One can see that as follows. Use the Killing form to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Then $(\mathfrak{m} + \mathfrak{n})^\perp$ is identified to $\mathfrak{a} + \mathfrak{n}$. A generic element therein is purely hyperbolic and its stabilizer is conjugate to MA. Therefore, for generic $\phi \in (\mathfrak{m} + \mathfrak{n})^\perp$, we have $\dim \mathfrak{g} \cdot \phi = \dim \mathfrak{g} / \dim(\mathfrak{m} +$

$\mathfrak{a}) = 2\dim\mathfrak{n} = \dim(\mathfrak{m}+\mathfrak{n})\cdot\phi$. Moreover, the multiplicity is given correctly by (CGmf1). Indeed, it is clear that for a generic hyperbolic element $\phi \in (\mathfrak{m}+\mathfrak{n})^{\perp}$, the intersection $G\cdot\phi \cap (\mathfrak{m}+\mathfrak{n})^{\perp}$ will have as many MN orbits as there are representatives for the coset space $W = M'/M$.

(iii) Whittaker Space G/N. Now we have that the quasi-regular representation is quasi-equivalent to a direct integral of all minimal principal series

$$\pi_{\alpha,\nu} = \mathrm{Ind}_{MAN}^{G} \nu \times \alpha \times 1,\ \nu \in \hat{M},\ \alpha \in \hat{A}$$

each rep $\pi_{\alpha,\nu}$ occurring with multiplicity equal to $\#(W)\dim\nu$. Thus although the multiplicity is finite, it is bounded iff M is abelian. We also have that $D(G/N) \cong U(\mathfrak{m}+\mathfrak{a})$, the universal enveloping algebra. The latter is abelian iff M is abelian. What about the the CG condition and multiplicity formula? By analogy with the computation in example (ii), one sees that $\mathfrak{n}^{\perp}$ is identified to $\mathfrak{m}+\mathfrak{a}+\mathfrak{n}$, the generic elements are regular semisimple, and so have stabilizers of dimension that of a Cartan subalgebra. Letting T denote a maximal torus of M, we see that the generic dimension of G-orbits that meet $\mathfrak{n}^{\perp}$ is $\dim\mathfrak{g}/\dim(\mathfrak{t}+\mathfrak{a})$, while the generic dimension of the N-orbits is $\dim\mathfrak{n}$. The former will be twice the latter precisely when $\mathfrak{m} = \mathfrak{t}$. In that case, it is routine (as in (ii)) to check that the Corwin-Greenleaf multiplicity function (CGmf1) agrees with $\#(W)$ generically on $\mathfrak{n}^{\perp}$.

(iv) The Flag Manifold G/MAN. We assume here that G is complex so that MAN is a Borel subgroup of G. The quasi-regular representation is then irreducible. In fact

$$\tau = \mathrm{Ind}_{MAN}^{G} 1 = \pi_1,$$

the principal zonal spherical representation. Of course $D(G/MAN)$ is trivial. We have $(\mathfrak{m}+\mathfrak{a}+\mathfrak{n})^{\perp} = \mathfrak{n}$ and the results of [**34**] apply. In particular, there is a distinguished G-orbit which intersects $\mathfrak{n}$ in an open MAN-orbit. From [**34, Prop. 4**] we see that the dimension of that G-orbit is twice the dimension of the MAN-orbit.

(v) Moore's Example G/A. This homogeneous space was studied originally by Moore. In fact the multiplicity is uniform and infinite unless G is a direct product of r copies of $SL(2,\mathbb{R})$. In that case, we have a symmetric space (actually, not quite since according to our definition we would need G/MA, $M \equiv \mathbb{Z}_2^r$ here). But the analysis of G/A is nevertheless covered by the work in Example 2(ii). In particular, if $r = 1$, we have $D(G/A)$ is commutative. Also, generically on $\mathfrak{a}^{\perp}$, we have $\dim G\cdot\phi = 2\dim A\cdot\phi$ and

$$\begin{aligned} G\cdot\phi \cap \mathfrak{a}^{\perp} &= A\cdot\phi &&\text{if } \phi \text{ is elliptic}\\ \#(G\cdot\phi\cap\mathfrak{a}^{\perp})/A &= 2 &&\text{if } \phi \text{ is hyperbolic.}\end{aligned}$$

This agrees with the well-known fact that the quasi-regular representation is a direct sum of discrete series with multiplicity 1, together with a direct integral of principal series with multiplicity 2.

(vi) THE ROSSI-VERGNE SPACE G/AN. This is also an example that has been well-studied in the past. The quasi-regular representation is actually a direct sum of the irreducibles $\pi_{1,\nu}$, each occurring with multiplicity $\#(W)\dim\nu$. The situation is very similar to that in example (iii). Bounded finite multiplicity only occurs when M is abelian, and the usual comments apply to the algebra of invariant differential operators $D(G/AN)$, as well as the Corwin-Greenleaf conditions. I leave the details to the reader.

4. The Applications. The applications I intend to present are actually quite diverse. What they have in common is not so much that they are a direct consequence of the form or content of the CG conditions or formulas, but rather that they emerge from the strands woven from the attributes in section 2 and the template examples of section 3.

Application 1. Maximal Subgroups. It is plainly evident that the generic aspect of the CG condition is intrinsic. When we say "generically on $p^{-1}(\mathcal{O}_\nu)$" it is quite accurate; typically (CGmf) or (CGfmc) will fail on lower-dimensional varieties. On the other hand, the proof of the CG formula and conditions, especially in the case of nilpotent or exponential solvable groups, reveals a surprising phenomenon. The proof is by induction and is achieved by placing between H and G a maximal subgroup L, $H \subset L \subset G$. When G is nilpotent L can be selected normal of codimension 1; when G is only completely solvable L can still be found of codimension 1, but not necessarily normal; and for general exponential solvable groups, the best one can achieve is a subgroup of codimension 2. As an intermediate step in the proof, one must derive the CG formula and condition in these maximal situations directly. And the amazing fact is that in each of those maximal situations, all the generic restrictions disappear. In fact, one has the following [**29**]

THEOREM 4. *Let G be simply connected exponential solvable, $H \subset G$ a maximal connected subgroup of G. Then*

(i) *For any $\nu \in \hat{H}$, the induced representation $\pi_\nu = Ind_H^G \nu$ is of uniform multiplicity;*
(ii) *The numbers $\dim G\cdot\phi - 2\dim H\cdot\phi + \dim\mathcal{O}_\nu$ are constant on $p^{-1}(\mathcal{O}_\nu)$. π_ν has finite multiplicity iff that number is 0;*
(iii) *As a function of ϕ, the numbers $n_\phi^\nu = \#\left(G\cdot\phi\cap p^{-1}(\mathcal{O}_\nu)\right)/H$ are constant on $p^{-1}(\mathcal{O}_\nu)$. The common value is the multiplicity of π_ν.*

It is interesting to note that in this case all four attributes of the CG multiplicity function discussed in section 2 are valid. It is also important to observe that the phenomenon described in the theorem is not restricted to exponential solvable groups. It also applies to the quasi-regular representation of a Riemannian symmetric space. Namely, if G is semisimple and K is maximal compact, then for *every* $\phi \in \mathfrak{k}^\perp$, it is the case that $\dim G\cdot\phi = 2\dim K\cdot\phi$ and $G\cdot\phi\cap\mathfrak{k}^\perp = K\cdot\phi$. It would be interesting to know if that is also the case for any induced representation $\pi_\nu = \mathrm{Ind}_K^G \nu$, where $\nu \in \hat{K}$ is not the trivial representation. It also seems likely that analogs of Theorem 4 could hold in other maximal situations.

Application 2. Intertwining Operators and Reciprocity. In some instances (but especially in the nilpotent situation) the proof of the CG orbital integral formula is not constructive (by induction, *e.g.*). In particular, although one has a formula of the form

$$\tau = \mathrm{Ind}_H^G 1 = \int^\oplus n_\pi\, \pi\, d\mu_{G,H}(\pi), \tag{b}$$

one does not have an explicit expression for the intertwining operator that effects the decomposition. One usually wants it, not only for its intrinsic interest, but also because one needs it — for example in work on solvability properties of invariant differential operators. Here is a scheme for obtaining the intertwining operator that works very well when the spectrum is monomial — that is, when almost all the irreducibles that appear in the decomposition are induced from characters. We know from Penney's result that

$$n_\pi \le \dim(\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}}. \tag{c}$$

Now interpret the formula (b) from a distribution-theoretic point of view. Let $\delta_\tau : f \to \overline{f(1)}$ be the canonical cyclic distribution in $(\mathcal{H}_\tau^{-\infty})^{H,q^{-1/2}}$. Then the distribution-theoretic companion to (b) is

$$\delta_\tau = \int \sum_{j=1}^{n_\pi} \delta_\pi^j \, d\mu_{G,H}(\pi),\ \delta_\pi^j \in (\mathcal{H}_\tau^{-\infty})^{H,q^{-1/2}}.$$

For example, if we are in the case that (CGqr) holds and the q functions are trivial, then

$$\delta_\tau = \int_{\mathfrak{h}^\perp/H} \delta_\phi \, d\dot{\phi},$$

where since the spectrum is monomial each π_ϕ is induced by a character $\pi_\phi = \mathrm{Ind}_B^G \chi_\phi$ of some "polarizing" subgroup B of ϕ. One of the main results of [**24**] is that the corresponding Penney distributions δ_ϕ are given by

$$\delta_\phi : f \to \int_{H\cap B\backslash H} \bar{f}(h)\, d\dot{h}.$$

Moreover, the intertwining operator is precisely

$$\tau(\omega)\delta_\tau \to \{\pi_\phi(\omega)\delta_\phi\}_{\phi\in\mathfrak{h}^\perp/H}, \quad \omega \in C_c^\infty(G).$$

These quantities are easily expanded to yield the intertwining operator

$$f \in \mathcal{D}(G,H) \to f_\phi(g) = \int_{H\cap B\backslash B} f(bg)\overline{\chi_\phi(b)}\, d\dot{b}.$$

(Here $\mathcal{D}(G, H)$ denotes the smooth, H-invariant functions on G which are compactly supported mod H.) This formula applies of course to exponential solvable groups. But it also applies to abelian symmetric spaces, Strichartz symmetric spaces and semisimple homogeneous spaces like G/MN and G/K. The convergence of the integrals specified by the distributions is sometimes problematic, and depending upon which kind of groups one is considering, one may have to do some approximation or analytic continuation (see [**24**] *e.g.*).

The reader is urged to consult [**36**] where the Penney-Fujiwara Plancherel formula, in the format I have presented it, was originally developed. In particular, a very nice treatment of how the presentation is "independent of polarization" can be found there.

There is a separate issue (whose discussion was also initiated by Fujiwara) that is extremely interesting and largely unresolved — namely, is it possible that there are more distributions than are accounted for by the multiplicity. That is, could the inequality in formula (c) be strict. The answer is yes in a trivial way. The left side is clearly only defined *a.e.* while the right side is defined pointwise. Thus one would expect at least a discrepancy on a set of measure zero. Here is an illustration. Let G be the three-dimensional Heisenberg group, with Lie algebra having generators X, Y, Z satisfying the bracket relation $[X, Y] = Z$. Let $H = \exp \mathbb{R}X$. There are two types of representations of G: the characters $\chi_{\xi,\eta} : \exp xX \exp yY \exp zZ \to \exp[i(\xi x + \eta y)]$, ξ, $\eta \in \mathbb{R}$; and the one-parameter family of infinite-dimensional representations π_ζ, $\zeta \neq 0$, uniquely specified by $\pi_\zeta(\exp zZ) = \exp(i\zeta z)\,\mathrm{Id}$. The latter act on $L^2(\mathbb{R})$, the space of C^∞-vectors consists precisely of the Schwartz functions, and in fact the unique (up to scalar) H-invariant distribution is

$$\delta_\zeta : f \to \int_{\mathbb{R}} \bar{f}.$$

Moreover, the quasi-regular representation is decomposed

$$\tau = \mathrm{Ind}_H^G 1 = \int_{\mathbb{R}}^{\oplus} \pi_\zeta \, d\zeta.$$

But a character $\chi_{\xi,\eta}$ is H-invariant iff $\xi = 0$.

The truly interesting question is: Can there be a quasi-regular representation in which the inequality (c) is strict on a set of positive measure? It is suspected that that is the case for reductive symmetric spaces, but I have not seen a specific example. On the other hand, it is believed that for exponential solvable homogeneous spaces, it cannot happen. Yet only very partial results in that direction are known [**13**]. One positive result is found in [**30**] — namely, if H is compact and τ is multiplicity-free, then $n_\pi = \dim(\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}} = 1$, *a.e.* (Richard Penney has just informed me that in his paper [**33**] he proves equality *a.e.* in (c) without the multiplicity-free assumption but with H still compact.) I will give another positive result in the next subsection.

Application 3. Orbital Symmetric Spaces. Consider an abelian symmetric space $G/H, G = H \ltimes V$ and suppose that G is algebraic. It is possible that H could have a Zariski-open orbit on $\hat{V}$. In such an eventuality, the quotient space $\hat{V}/H$ would, from a measure-theoretic point of view, be a singleton. Therefore, by Example 2(iv), the quasi-regular representation would be irreducible. This is surprising at first glance — a symmetric space whose quasi-regular representation is irreducible. In [**30**], it is shown that this happens with surprising regularity. (These symmetric spaces are called *orbital* because of Theorem 5 below.) Clearly the algebra of invariant differential operators is trivial, and so abelian. The orbital characterization of this situation is in [**30**].

THEOREM 5. *Let G be algebraic, σ an involution of G, $H = G^\sigma$. Then G/H has an irreducible quasi-regular representation iff there is a Zariski-open H-orbit on $\mathfrak{h}^\perp$, and for any ϕ in that orbit we have $G_\phi \subset H$.*

It is trivially clear from the theorem that for the Zariski-open orbit we have $G \cdot \phi \cap \mathfrak{h}^\perp = H \cdot \phi$. It also follows from [**30**] that the G-orbit has twice the dimension of the H-orbit (because $\mathfrak{h}$ is a real polarization for ϕ and $\mathfrak{g}_\phi = \mathfrak{h}_\phi$). We also have [**30**]

PROPOSITION 6. *If the quasi-regular representation of a homogeneous space G/H is an irreducible representation π_0, then we have*

$$\dim(\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}} = \begin{cases} 1 & \pi = \pi_0 \\ 0 & \pi \ncong \pi_0 \end{cases}.$$

Application 4. Finite Multiplicity for Symmetric Spaces. Here is the main application. First take G a connected Lie group, σ an involution of G, and $H = G^\sigma$. By Proposition 2, generically on $\mathfrak{h}^\perp$, there are open H-orbits on the orbit intersections $G \cdot \phi \cap \mathfrak{h}^\perp$. Now if in addition we assume G is algebraic, then the components on $G \cdot \phi \cap \mathfrak{h}^\perp$ can be at most finite in number. Moreover on $\mathfrak{h}^\perp$, those finite numbers will be bounded. Therefore the numbers $n_\phi = \#\left(G \cdot \phi \cap \mathfrak{h}^\perp\right)/H$ are finite and bounded, and we are strongly motivated to conjecture that (algebraic) symmetric spaces have finite bounded multiplicity. This result was originally conjectured in [**27**]. The finiteness part was finally proven in [**30**] as a special case of the following results.

THEOREM 7. *Let G be algebraic, σ an involution of G. Then there is a Levi subgroup of G which is preserved by σ.*

THEOREM 8. *Let G be algebraic, σ an involution of G, $H = G^\sigma$. Suppose Z is a central, connected, unipotent, σ-invariant subgroup of G, and let χ be a character of HZ which satisfies $\sigma\chi = \bar{\chi}$. Then the induced representation $\tau_\chi = Ind_{HZ}^G \chi$ has finite multiplicity.*

The proof in [**30**] amounts to a generalization of the computation in section 3, Example 2(v). It involves dropping the assumption of abelian unipotent radical and requires the integration of the orbital parameters into the Mackey Machine that was effected by Duflo in [**11**].

Now we observe that in virtually all symmetric spaces G/H that have been investigated it turns out that $D(G/H)$ is abelian. Moreover, by Proposition 2, the CG condition is valid. These facts and the discussion to this point suggest strongly that, at least for the algebraic symmetric spaces under consideration, the multiplicity should also be bounded. The complete resolution of that issue awaits the outcome of the reductive situation. It is commonly asserted that reductive symmetric spaces have bounded multiplicity, but once again to my knowledge, that has not been definitively established. Here is a proof of a companion result.

THEOREM 9. *Let G be algebraic and solvable. Suppose that σ is an involution of G and $H = G^\sigma$. Then the symmetric space G/H has bounded multiplicity.*

PROOF. We adopt the structure presented in [**30**]. The solvability assumption simplifies matters somewhat. We can choose a Levi decomposition $G = AN$ with A σ-invariant (by Theorem 7). Then $H = BM$, with $B = A^\sigma, M = N^\sigma$. We can parameterize the representations of G à la Duflo [**11**]. Let $\theta \in \mathfrak{n}^*, \gamma = \gamma_\theta \in \hat{N}$, the corresponding representation. Then, because of [**1**], [**11**] and the solvability, there is a canonical lifting $\tilde{\gamma}_\theta$ to G_θ satisfying

$$\tilde{\gamma}_\theta(g_\theta)\gamma(n)\tilde{\gamma}_\theta(g_\theta)^{-1} = \gamma(g_\theta n g_\theta^{-1}), \quad g_\theta \in G_\theta,\, n \in N$$

$$\tilde{\gamma}_\theta(n_\theta) = \chi_\theta(n_\theta)^{-1}\gamma(n_\theta), \quad n_\theta \in N_\theta,$$

where $\chi_\theta(\exp X) = e^{i\theta(X)}$, $X \in \mathfrak{n}_\theta$. (Here G_θ, N_θ, and $\mathfrak{n}_\theta$ denote the stability subgroups and subalgebra of θ.) The representations of G lying over $\gamma = \gamma_\theta$ are obtained by forming

$$\pi_{\gamma,\nu} = \mathrm{Ind}_{G_\theta N}^{G} \nu \otimes \tilde{\gamma}_\theta \times \gamma_\theta,$$

where ν is any irreducible representation of G_θ satisfying $\nu(n_\theta) = \chi(n_\theta)\mathrm{Id}$. Thus the little group spectrum is parameterized by the spectrum of $\mathrm{Ind}_{N_\theta}^{G_\theta}\chi_\theta$ — which we write as $G_\theta(N_\theta, \chi_\theta)^\wedge$.

Now we bring in the symmetric space structure. Indeed, we do not need the entire set $G_\theta(N_\theta, \chi_\theta)^\wedge$. We only need the part that contributes to the quasi-regular representation τ. To see exactly what that is we invoke the results of [**30**]. First write

$$\tau_N = \mathrm{Ind}_M^N 1 = \int_{\hat{N}(M)}^{\oplus} \gamma \, d\mu_M(\gamma).$$

Then set $K = BN$. By the Alignment Lemma [**17, Lemma 3.3**], we may assume (conjugating θ if necessary) that $G_\theta = A_\theta N_\theta$. Thus $K_\theta = B_\theta N_\theta$ and χ_θ extends naturally to K_θ by setting it $= 1$ on B_θ. Next express $\mathrm{Ind}_{K_\theta}^{G_\theta}\chi_\theta$ as a direct integral

$$\mathrm{Ind}_{K_\theta}^{G_\theta}\chi_\theta = \int_{G_\theta(K_\theta,\chi_\theta)^\wedge}^{\oplus} n_\theta(\nu)\nu \, d\mu_\theta(\nu).$$

Then

$$\mathrm{Ind}_{K_\theta N}^{G_\theta N}\chi_\theta\gamma_\theta \times \gamma_\theta = \int_{G_\theta(K_\theta,\chi_\theta)^\wedge}^{\oplus} n_\theta(\nu)\nu \otimes \gamma_\theta \times \gamma_\theta \, d\mu_\theta(\nu);$$

and so

$$\begin{aligned}\tau = \mathrm{Ind}_H^G 1 &= \int_{\hat{N}(M)/B}^{\oplus} \int_{G_\theta(K_\theta,\chi_\theta)^\wedge}^{\oplus} n_\theta(\nu)\pi_{\theta,\nu}\, d\mu_\theta(\nu) d\dot{\mu}_M(\gamma_\theta) \\ &= \int_{G\cdot\mathfrak{m}^\perp/G}^{\oplus} \int_{G_\theta(K_\theta,\chi_\theta)^\wedge}^{\oplus} n_\theta n_\theta(\nu)\pi_{\theta,\nu}\, d\mu_\theta(\nu) d\ddot{\mu}_M(\gamma_\theta),\end{aligned}$$

where $n_\theta = \#(G\cdot\theta \cap \mathfrak{m}^\perp)/H$.

Now we employ a trick due to Auslander-Kostant [**1**]. Set $\mathfrak{q}_\theta = \ker\theta|_{\mathfrak{n}_\theta}$, Q_θ the corresponding analytic subgroup of N_θ. It is clear that $[\mathfrak{g}_\theta, \mathfrak{q}_\theta] \subset \mathfrak{n}_\theta$, and obviously $\theta[\mathfrak{g}_\theta, \mathfrak{q}_\theta] = 0$. That is, $\mathfrak{q}_\theta$ is an ideal in $\mathfrak{g}_\theta$. Furthermore $\mathfrak{n}_\theta/\mathfrak{q}_\theta$ is central in $\mathfrak{g}_\theta/\mathfrak{q}_\theta$. This is because $[\mathfrak{g}_\theta, \mathfrak{n}_\theta] \subset \mathfrak{n}_\theta$ and $\theta[\mathfrak{g}_\theta, \mathfrak{n}_\theta] = 0$, so that $[\mathfrak{g}_\theta, \mathfrak{n}_\theta] \subset \mathfrak{q}_\theta$. In fact these properties lift to the groups. The reasoning follows [**6,p. 207**]. For $g \in G_\theta$ and $X \in \mathfrak{n}_\theta$, it is clear that $g\cdot X - X \in \mathfrak{q}_\theta$. That is, the adjoint action of G_θ on $\mathfrak{n}_\theta/\mathfrak{q}_\theta$ is trivial. It follows that $[G_\theta, N_\theta] \subset Q_\theta$. In fact in this case we actually have $[G_\theta, G_\theta] \subset Q_\theta$. This is because $G_\theta = A_\theta N_\theta$. Hence G_θ/Q_θ is abelian, and thus the representation $\tau_\theta = \mathrm{Ind}_{K_\theta/Q_\theta}^{G_\theta/Q_\theta}\chi_\theta$ must be multiplicity-free. We are reduced to showing that the numbers n_θ are uniformly bounded. But this follows from the same principal we have invoked several times. The reader may consult [**31**] — but basically the algebraic situation says we have components of a family of algebraic varieties determined as the coefficients in a fixed set of polynomials varies. The number of components in such a situation is bounded above.

Remark. In [**30**] it is proven that $n_\phi = n_\theta n_\theta(\nu)$, whence in this scenario we actually have $n_\phi = n_\theta$. Thus the CG multiplicity function describes the bounded multiplicity for algebraic solvable symmetric spaces. If one traces through the usual orbital parameters, one sees that this is saying in addition, that in the two parameters of the orbital data, the multiplicity only depends on the orbital base, not at all on the orbital fiber. That is a principal that probably has much wider applicability, but I shall say no more about it now.

5. The Conjecture. The thrust of the preceding discussions leads inexorably to the following

CONJECTURE. *Let G/H be an algebraic homogeneous space. Then the following are equivalent*

(i) *The quasi-regular representation $\tau = Ind_H^G 1$ has bounded finite multiplicity.*
(ii) *The algebra $D(G/H)$ of G-invariant differential operators is commutative.*
(iii) *Generically on $\mathfrak{h}^\perp$, the G-orbits have dimension twice the H-orbits.*
(iv) *Generically on $\mathfrak{h}^\perp$, H has open orbits on $G\cdot\phi\cap\mathfrak{h}^\perp$.*

We conclude the paper with several

Remarks.

(1) Unitary representations only correspond to integral or admissible orbits. Thus it is tempting to include an integrality condition in parts (iii) or (iv) of the conjecture. It is one of my current philosophies that in matters pertaining to the conjecture, it is the fulfillment of the conditions on the full (generic) portion of the varieties that governs the situation. In this regard see [**5**].

(2) Implicit in the conjecture is the belief that for algebraic homogeneous spaces there are only three possibilities for the multiplicity function: bounded finite multiplicity, unbounded finite multiplicity, or uniform infinite multiplicity. That is, no mixed finite and infinite multiplicity examples can occur. This is also discussed in [**5**], where it is related to three distinct properties of the moment map (see Example 3).

(3) If there is an orbital parameterization in control of $\hat{G}$, we expect that the CG formula will apply. But even without a completely refined orbital parameterization, the orbit method appears in the form of conditions (iii) and (iv) of the conjecture. (A similar situation obtains in Theorem 5.)

(4) We expect that the bound on the multiplicity for an algebraic symmetric space should be exactly ess $\sup_{\{\phi\in\mathfrak{h}^\perp\}} n_\phi =$ ess $\sup_{\{\phi\in\mathfrak{h}^\perp\}} \#(G\cdot\phi\cap\mathfrak{h}^\perp)/H$. The argument in [**30**] reduces that prospect to the reductive case.

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