DISTRIBUTIONAL RECIPROCITY AND GENERALIZED GELFAND PAIRS

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Abstract. There has been a great deal of study recently of so-called *generalized Gelfand pairs*; that is, couples of groups $H \subset G$ in which (for the case that G/H has an invariant measure) every space of H-invariant distributions over an irreducible unitary representation of G has dimension at most one. It is very well-known that, for any such couple, the quasi-regular representation $\mathrm{Ind}_H^G 1$ is multiplicity-free. In this paper I investigate the validity of the converse implication; that is: when is it the case that $\mathrm{Ind}_H^G 1$ multiplicity-free implies that (G, H) is a generalized Gelfand pair?

0. Acknowledgements

Like most individuals who came of mathematical age in the 1960s working in Lie group representations, I was greatly influenced by Harish-Chandra. More than thirty years later, his work on the Plancherel theory of semisimple Lie groups and Riemannian symmetric spaces continues to influence my research, this paper being a good illustration. I would like to thank Bob Doran for organizing the conference in Harish-Chandra's memory, for arranging this memorial volume, and for tolerating my tardiness in submitting this work.

1. Introduction

If G and H are both unimodular Lie groups and the quasi-regular representation $\tau = \tau_{G,H} = \mathrm{Ind}_H^G 1$ is type I, then it is a classical result of Penney [17] that the multiplicity function in the direct integral decomposition of τ is dominated by the dimension of the space of H-invariant distributions. More precisely, if

$$\tau_{G,H} = \int_{\widehat{G}}^{\oplus} n(\pi)\pi \, d\mu(\pi), \tag{1.1}$$

then we have

$$n(\pi) \leq \dim(\mathcal{H}_\pi^{-\infty})^H, \qquad \text{for } \mu\text{-almost all } \pi \in \hat{G}. \tag{1.2}$$

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In (1.1), the objects π denote irreducible unitary representations of G, $n(\pi)$ is the multiplicity function, and μ is the Plancherel measure. In (1.2), the symbol $\mathcal{H}_\pi$ represents the Hilbert space on which π acts, $\mathcal{H}_\pi^\infty$ denotes the subspace of C^∞-vectors, $\mathcal{H}_\pi^{-\infty}$ is the dual space of distributions, and $(\mathcal{H}_\pi^{-\infty})^H$ stands for the H-invariants therein. We can drop unimodularity, but then we must replace $(\mathcal{H}_\pi^{-\infty})^H$ by $(\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}}$, $q = q_{H,G} = \Delta_H/\Delta_G$; that is, we replace the space of H-invariant distributions by the space of semi-invariant distributions that transform under the action of H according to a modulus equal to the reciprocal of the square root of the quotient of the modular functions of H and G. The assertion that *distributional reciprocity* holds is the claim that strict inequality can occur in (1.2) on at most a set of measure zero.

The Plancherel decomposition (1.1) is "soft" in the sense that it does not specify the intertwining operator that effects the direct integral decomposition. It is rendered "hard" by expressing it in the form of the Penney-Fujiwara Plancherel formula **PFPF** or the Bonnet Plancherel formula **BPF**. The former normally obtains in the case that the multiplicity function is finite, the latter in the case of infinite multiplicity. In [15] I have argued that the distinction is somewhat artificial. Indeed, if we set

$$\omega_H(g) = q^{-1/2}(g) \int_H \omega(g^{-1}h^{-1}) q^{-1/2}(h)\, dh,\ \omega \in \mathcal{D}(G), \tag{1.3}$$

where $\mathcal{D}(G)$ is the test space of compactly supported, infinitely differentiable functions on G, then we have the following

Theorem 1.1 (BPF). [3] *For μ-a.a. π, there exist positive nuclear operators $U_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^{-\infty}$ such that*

$$\omega_H(e) = \int_{\widehat{G}} \mathrm{Tr}[\pi(\omega) U_\pi]\, d\mu(\pi),\ \omega \in \mathcal{D}(G). \tag{1.4}$$

The pair $(\mu, \{U_\pi\}_{\pi\in\hat{G}})$ is unique up to positive scalars $c(\pi)$; i.e. any other pair $(\mu', \{U'_\pi\}_{\pi\in\hat{G}})$ satisfying formula (1.4) must be related to the original by $\frac{d\mu}{d\mu'} = c(\pi)$, $U_\pi = U'_\pi/c(\pi)$, for some positive Borel function $c(\pi)$.

Furthermore, if we consult [15, Remark 2.3], we see that we can make the following

Observations 1.2.

(1) The nuclear operators U_π actually yield, in a concrete way, the explicit intertwining operator for the direct integral decomposition (1.1).

(2) We have $n(\pi) = \mathrm{rank}(U_\pi)$. In the case of finite multiplicity, one can derive the formulas for the Penney distributions in the PFPF (see *e.g.* [10, formula (1.2) and Theorem 2.1]) directly from the nuclear operators U_π; and *vice versa*, the knowledge of the Penney distributions determines the nuclear operators.

(3) Finally, we have

$$U_\pi(\mathcal{H}_\pi^\infty) \subseteq (\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}}. \tag{1.5}$$

The assertion of distributional reciprocity is precisely the claim that the inclusion in (1.5) is an equality μ-*a.e.*.

There seem to be two schools of thought on the question of distributional reciprocity. Some believe it is false in general, and have sought counterexamples among reductive homogeneous spaces. Others have operated under the assumption that it is likely to be true, and have concentrated their attention on nilpotent homogeneous spaces. Neither camp has obtained a definitive answer. I am inclined toward the latter camp, but will pursue a slightly different question in this paper. In the case that the commuting algebra of $\tau_{G,H}(G)$ is abelian, that is the multiplicity-free case, distributional reciprocity would assert:

$$n_\pi \leq 1,\ \mu - a.e. \Rightarrow \dim(\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}} \leq 1,\ \mu - a.e. \tag{1.6}$$

Let us consider the stronger assertion:

$$n_\pi \leq 1,\ \mu - a.e. \Rightarrow \dim(\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}} \leq 1,\ \forall \pi \in \hat{G}. \tag{1.7}$$

The right side of implication (1.7) is precisely the defining criterion for a pair of groups (G, H) to be a *generalized Gelfand pair* ([18] [19]). So, in this paper, we will investigate the question: In what situations is it the case that a multiplicity-free quasi-regular representation $\tau_{G,H}$ forces (G, H) to be a generalized Gelfand pair? Note that, by formula (1.2) or by [19, Thms. 2.3 and 2.4], the converse is always true, that is, if (G, H) is a generalized Gelfand pair, then $\tau_{G,H}$ must be multiplicity-free.

2. More observations and statements of results

We continue to make some important observations and, as we do so, we indicate the main results that are found in the paper.

Observations 2.1.

(1) It is most natural to consider first the implication (1.7) in the case of compact groups. Of course, if G is compact, then $\tau_{G,H}$ decomposes discretely, the implications (1.6) and (1.7) amount to the same thing, and they are both true by ordinary Frobenius reciprocity. If we only restrict H to be compact, then we are in less familiar territory. In fact, I will show in section 3 that if $H = K$ is compact and G is unimodular, then $(\mathcal{H}_\pi^{-\infty})^K = \mathcal{H}_\pi^K$, and implication (1.7) is valid. I shall also show that distributional reciprocity holds (Theorem 3.1). If one brings in the CMC conjecture (see [14, section 1]), it suggests that bounded finite multiplicity is enough to warrant these conclusions. I conclude section 3 with the statement (and outline of a proof) of a new result: if the multiplicity of $\tau_{G,K}$ is bounded ($a.e.$), then in fact it is multiplicity-free and the couple (G, K) is an ordinary Gelfand pair (Theorem 3.2).

(2) The implication (1.7) is *not* true in general. I will supply counterexamples in section 4. I will even give counterexamples that apply when the homogeneous space G/H is a symmetric space.

(3) On the other hand, I will show in section 5 that whenever $\tau_{G,H}$ decomposes discretely, then in fact the implication (1.7) is true (Theorem 5.1).

(4) In light of these results it is reasonable to organize our thoughts as follows. Given a specific category of homogeneous space, then exactly one of the following situations obtains:

- (i) Every object in the category is a generalized Gelfand pair, and so also is multiplicity-free. Examples are provided by Riemannian symmetric spaces [7, section III.12] and nilpotent symmetric spaces [1].

(ii) Within the category, sometimes an object that is multiplicity-free is also a generalized Gelfand pair, and sometimes it is not. The category of general nilpotent homogeneous spaces provides an example.

(iii) The most interesting categories are those wherein not every object is a generalized Gelfand pair, but the objects that are generalized Gelfand pairs constitute precisely those that are multiplicity-free. An example occurs when H is compact.

Note that in the above (and below) we sometimes abuse terminology and refer to the homogeneous space G/H, rather than to the couple (G, H), as a generalized Gelfand pair; and we sometimes refer to the homogeneous space G/H, rather than $\tau_{G,H}$, as multiplicity-free. Now in the remainder of the paper, we examine a few interesting categories of homogeneous spaces and place them into one of these three categories.

(5) We examine two types of Strichartz homogeneous spaces. In section 6, we show that partially Riemannian Strichartz spaces are in category (i); and in section 7, we prove that metabelian Strichartz spaces are in category (iii). Finally, in section 8 we examine the generalized horocycle spaces of [13] [16].

(6) How does one prove that a homogeneous space is a generalized Gelfand pair? There are only two known strategies. Either one verifies directly the inequality

$$\dim\left(\mathcal{H}_\pi^{-\infty}\right)^{H,q^{-1/2}} \leq 1, \forall \pi \in \hat{G};$$

or one must proceed by the technique pioneered originally by Gelfand. Namely, one must find an involution of the group G that preserves H and is also an anti-automorphism in the sense of [19, Criterion 2.5]. The construction of such a gadget always follows from a structural result of the form $G = HQ$, where the "cross-section" Q for G/H needs to have certain properties (see [18] and [19] for good examples). This is reminiscent of a symmetric space, but many non-symmetric space examples are known. Unfortunately, there seems to be no general paradigm for finding such decompositions. We will find one in section 6.

(7) Regardless of the status of a category of homogeneous space in regard to the implication (1.7), nothing precludes the possibility that implication (1.6) could be true for that category. In fact, every counterexample I provide to (1.7) in this paper does not provide a counterexample to (1.6).

3. Compact stability group

In this section we assume G is unimodular and H is compact. We denote the compact stability group by the letter K in this section. We adopt the usual terminology:

$$\mathcal{H}_\pi^K = \{\xi \in \mathcal{H}_\pi : \pi(k)\xi = \xi, \forall k \in K\}.$$

Then by [11, Prop. 2.1], the multiplicity function in (1.1) satisfies $n_\pi = \dim \mathcal{H}_\pi^K$. Furthermore, we have

Theorem 3.1. *If $\tau_{G,K}$ is type I, then $\forall \pi \in \hat{G}$,*

$$\dim(\mathcal{H}_\pi^{-\infty})^K = \dim \mathcal{H}_\pi^K.$$

Proof. This theorem is proven in [11, Thm. 3.1] under the assumption that the quasi-regular representation is multiplicity-free. We adapt that proof. Consider the conjugate-linear map $\xi_0 \to \beta_{\xi_0}$, $\mathcal{H}_\pi^K \to (\mathcal{H}_\pi^{-\infty})^K$ defined by $\langle \beta_{\xi_0}, \xi\rangle = \langle \xi_0, \xi\rangle$. The map is clearly an injection. We only need to show that it is surjective under the presumption that $\dim \mathcal{H}_\pi^K < \infty$. As in [11], define the orthogonal projection of $\mathcal{H}_\pi$ onto $\mathcal{H}_\pi^K$ by

$$P = \int_K \pi(k)\, dk.$$

Then exactly as in [11, p. 661] the space $\mathcal{V}_\pi = \left(\mathcal{H}_\pi^K\right)^\perp$ agrees with $\ker P$, and it respects the C^∞ decomposition—namely,

$$\mathcal{H}_\pi^\infty = \mathcal{H}_\pi^K \oplus \mathcal{V}_\pi^\infty, \tag{3.1}$$

where $\mathcal{V}_\pi^\infty = \mathcal{H}_\pi^\infty \cap \mathcal{V}_\pi$. To conclude the argument, let $\beta \in (\mathcal{H}_\pi^{-\infty})^K$. Then β is determined by its restriction to $\mathcal{H}_\pi^K$. To see that, it's enough to show $\beta = 0$ on $\mathcal{V}_\pi^\infty$. If not, then there is some $\eta \in \mathcal{V}_\pi^\infty$ with $\langle \beta, \eta \rangle \neq 0$. But then

$$0 \neq \langle \beta, \eta \rangle = \langle P\beta, \eta \rangle = \langle \beta, P\eta \rangle = 0.$$

The surjection is therefore clear.

Theorem 3.1 establishes the fact that for compact stability group, distributional reciprocity is true. Now what if $\tau_{G,H}$ is multiplicity-free? It is still conceivable that on a set of Plancherel measure zero, the inequality $\dim(\mathcal{H}_\pi^{-\infty})^K = \dim \mathcal{H}_\pi^K \leq 1$ could be violated. But exactly as in [11, Remark 2 in section 3], this cannot happen because of [6, IV.3]. So implication (1.7) is true here: any couple (G, K) with K compact and $\tau_{G,K}$ multiplicity-free is a generalized Gelfand pair. This is also established in [19, Thm 2.1] or in [18, section 4].

Let's finish this section by invoking the CMC [14]. It asserts, in particular, that if G and H are algebraic, and we have bounded finite multiplicity, then it should be the case that the algebra of invariant differential operators $\mathbf{D}(G/H)$ is abelian. But, if $H = K$ is compact, that can only happen if (G, K) is an ordinary Gelfand pair. In fact we have

Theorem 3.2. *Suppose that G is connected, unimodular, K is compact, both are algebraic, and the multiplicity in $\tau_{G,K}$ is finite and bounded μ-a.e. Then (G, K) is an ordinary Gelfand pair.*

This result is somewhat peripheral to our main concern, so we shall leave the details of the proof for another occasion. I only remark that it proceeds by induction on dimension, uses the results from [2], and eventually comes down to the case that G is reductive—from which, given the bounded multiplicity hypothesis, one deduces that K must be maximal compact in G, and so the homogeneous space is Riemannian symmetric and the multiplicity is free.

4. Counterexamples

In fact, multiplicity-free homogeneous spaces need not be generalized Gelfand pairs. Here is the basic way to get a counterexample to implication (1.7). Consider the simply connected nilpotent Lie group whose Lie algebra has generators W, X, Y, Z, U satisfying the bracket relations:

$$[W, X] = Y,\ [W, Y] = Z,\ [X, Y] = U.$$

Take for H the group $H = \exp \mathbb{R}X$. Then $\tau_{G,H}$ is multiplicity-free. One can see this by computing co-adjoint orbits, or by the following simple exercise in the Mackey Machine. The normal subgroup L generated by X, Y, Z, U is the direct product of a Heisenberg group and a line. It's irreducibles are parameterized by two real variables μ and ζ, where μ parametrizes the central character (on $\exp \mathbb{R}U$) of the irreducible of the Heisenberg portion, $\mu \neq 0$, and ζ parametrizes a character of $\exp \mathbb{R}Z$. The representation $\mathrm{Ind}_H^L 1$ decomposes as a multiplicity-free direct integral over all $\mu \neq 0$ and ζ. The action of $\exp \mathbb{R}W$ on every one of those representations is trivial, and so (by induction in stages), $\tau_{G,H}$ is also multiplicity-free. But the group $C = \exp \mathbb{R}U$ is central in G. If we divide it out, what remains is the unique simply connected nilpotent Lie goup of dimension four that has no non-trivial abelian central subgroup. It is well-known that $\tau_{G/C,H}$ has uniform multiplicity two [4, Example 2]. Therefore, the H-invariant distributions on the representation spaces corresponding to those representations—which of course have null Plancherel measure in the direct integral decomposition of $\tau_{G,H}$—have dimension at least two. So (G, H) is definitely not a generalized Gelfand pair.

One cannot duplicate this feat for nilpotent homogeneous spaces that are symmetric—because of Benoist's results [1]. However, there are non-nilpotent examples of symmetric spaces for which implication (1.7) fails. It's not likely that we can find an example in which G is reductive. But here are two semidirect product examples. In both instances, although $\tau_{G,H}$ is multiplicity-free, there is a normal subgroup $L \subset G$ for which $\tau_{G/L,HL/L}$ has multiplicity two. Then the H-invariant distribution spaces corresponding to the irreducibles of G (lifted from G/L) that appear in the spectrum of $\tau_{G/L,HL/L}$ have dimension at least two. Interestingly, both of the examples we present are *symmetric* spaces.

The reader might consult [9, p.471] in reference to the next example. Let G be the *oscillator* group. That is, take G to be the semidirect product $G = KN$, where N is a Heisenberg group whose Lie algebra has generators X, Y, Z satisfying $[X, Y] = Z$, $K = \exp \mathbb{R}T$, and the generator T acts on the Lie algebra of N by the bracket relations $[T, X] = Y, [T, Y] = -X, [T, Z] = 0$. Select $H = \exp \mathbb{R}X$. It is easy to check that G/H is a symmetric space; the appropriate involution on G is generated by the Lie algebra mapping that moves the generators as follows: $X \to X$, $Y \to -Y$, $Z \to -Z, T \to -T$. The direct integral decomposition $\mathrm{Ind}_H^G 1$ is computed in [9]—it is multiplicity-free. However, if we divide out by the central subgroup $C = \exp \mathbb{R}Z$, then again from [9], we see that the direct integral decomposition $\mathrm{Ind}_H^{G/C} 1$ has uniform multiplicity two. The irreducibles in the latter decomposition constitute a set of Plancherel measure zero in $\mathrm{Ind}_H^G 1$, but the corresponding H-invariant distribution spaces have dimension two, and so (G, H) is not a generalized Gelfand pair.

The final example is similar to the last, except that the compact toral group is replaced by a non-compact semisimple Lie group. This time consider the matrix group

$$G = \left\{ g = \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix} : \det(g) = 1 \right\},$$

where the coefficients are real. Then G is a semidirect product $G = SN$, where $S = SL(2, \mathbb{R})$ and $N = \mathbb{R}^2$. Consider the involutive automorphism on G determined by conjugation by the diagonal marix whose entries are $1, -1, 1$. If H is the stability group of the involution, then

$$H = \left\{ \begin{pmatrix} a & 0 & x \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} : a \neq 0, x \in \mathbb{R} \right\}$$

is a semidirect product of $\mathbb{R}^\times$ with $\mathbb{R}$. Following the scheme laid out in [10, sections 3 and 4], one computes easily that $\tau_{G,H}$ is multiplicity-free. Take for the normal subgroup $L = N$. Then the corresponding homogeneous space $(G/L)/(HL/L)$ is nothing more than $SL(2,\mathbb{R})$ over its diagonal subgroup. The spectrum of this reductive homogeneous space is well-known: the discrete series of $SL(2,\mathbb{R})$ appears with uniform multiplicity one, but the principal series representations appear with uniform multiplicity two. There is a slight twist here though that one must take into account. Namely, although G is unimodular, the group H is not. However, both of the groups G/L and HL/L are unimodular. So the two-dimensional distributions spaces that appear contain H-invariant distributions. It is a simple matter to modify the distributions so that they are H-semiinvariant with the correct modulus. I omit all the details—suffice it to mention that the representations in question act in $L^2(\mathbb{R})$, and the corresponding distributions, both invariant and semiinvariant, are of the form

$$f \to \int_0^{\pm\infty} f(v)|v|^{-\alpha}\, dv,$$

for appropriate choices of the positive scalar α.

5. The direct sum case

In this section we assume that the direct integral decomposition of the homogeneous space G/H is actually a direct sum. We show in this case that the implication (1.7) is indeed true—that is, any couple (G,H) whose corresponding quasi-regular representation decomposes as a multiplicity-free direct sum is a generalized Gelfand pair. Actually, we shall derive this as a virtual direct consequence of some classical results of Richard Penney ([17]). Of course, Penney's argument does not pass to the direct integral case. However, I shall examine one instance of a type of homogeneous space for which every multiplicity-free direct integral situation guarantees that (G,H) forms a generalized Gelfand pair.

In fact, we have

Theorem 5.1. *If both G and H are unimodular, and*

$$\tau_{G,H} = Ind_H^G 1 = \sum_i^{\oplus} \pi_i,$$

is a multiplicity-free direct sum of irreducibles, then it is the case that

$$\dim \left(\mathcal{H}_\pi^{-\infty}\right)^H = \begin{cases} 1, & \text{if } \pi \text{ occurs in } \tau_{G,H} \\ 0, & \text{if } \pi \text{ does not occur in } \tau_{G,H}. \end{cases}$$

Proof. In fact, if one examines [17, IV Theorem], and observes that in the notation of [17, p. 188] we have $\mathcal{R}(U) = (U^{-\infty})^H$, then the result follows immediately.

We have seen both positive and negative resolutions of the implication (1.7) when the direct sum is replaced by a direct integral. Here is another positive case relating to a category of homogeneous space that I have studied in [2] and [14] among other places. Consider $G = HN$, where N is a Heisenberg group of dimension $2r+1$, and H is a unimodular subgroup of the symplectic group $Sp(r,\mathbb{R})$. Then I have computed the direct integral decomposition of $\tau_{G,H}$ explicitly in [14, Prop. 2.2 and Cor. 2.3]. Summarizing the notation prior to the statement of these results (in [14]) very quickly, we let γ_χ denote a generic irreducible representation of N, where χ parameterizes the non-trivial real characters of the central subgroup of N, and let $\tilde{\gamma}_\chi$ denote the corresponding Weil representation of H (I assume for simplicity that there is no obstruction). The latter only depends on the sign of χ.

Proposition 5.2. *Let the Weil representation decompose*

$$\tilde{\gamma}^{\pm} = \int_{\widehat{H}}^{\oplus} n^{\pm}(\nu)\nu \, d\mu^{\pm}(\nu). \tag{5.1}$$

Then

$$Ind_H^G 1 = \int_{\chi>0}^{\oplus} \int_{\widehat{H}}^{\oplus} n^{+}(\nu)\pi_{\bar{\nu},\chi} \, d\mu^{+}(\nu)d\chi + \int_{\chi<0}^{\oplus} \int_{\widehat{H}}^{\oplus} n^{-}(\nu)\pi_{\bar{\nu},\chi} \, d\mu^{-}(\nu)d\chi. \tag{5.2}$$

In particular, the multiplicity of the quasi-regular representation $\tau_{G,H}$ is exactly the multiplicity of the Weil representation $\tilde{\gamma}$, meaning that $n_{\pi_{\nu,\chi}} = n^{\mathrm{sgn}(\chi)}(\bar{\nu})$.

Now suppose that the Weil representation decomposes in (5.1) as a direct sum. Then the inner direct integrals in formula (5.2) become direct sums, but the outer decompositions remain direct integrals. So Theorem 5.1 does not apply. However, we can prove

Theorem 5.3. *If the Weil representation is a multiplicity-free direct sum, then (G,H) is a generalized Gelfand pair.*

Proof. This proof will supply one of the very few instances I know of whereby a homogeneous space is proven to be a generalized Gelfand pair without the use of a structural decomposition as described in Observation 2.1.6. Namely, I will show directly that the dimension of every H-invariant distribution space is at most one. Now there are three types of irreducibles of the semidirect product G. First, there are the totally degenerate representations that are trivial on the normal subgroup N. These are just the irreducibles of H itself, and there is clearly no nontrivial distribution on any such representation space that is H-invariant. Next, there are the partially degenerate representations, namely those that are not trivial on N, but are trivial on its center C. But then we are basically dealing with the abelian homogeneous space $(G/C)/H = H(N/C)/H = (H \ltimes \mathbb{R}^{2r})/H$. The theory of abelian homogeneous spaces is very well-known ([8] [18] [19]); they are symmetric spaces and always form generalized Gelfand pairs. In particular, the space of H-invariant distributions corresponding to any of these representations will have dimension either zero or one. Finally, we come to the generic representations of G:

$$\pi_{\nu,\chi} = \nu \otimes \tilde{\gamma}_\chi \times \gamma_\chi,$$

where $\chi \neq 1, \gamma_\chi \in \hat{N}$, and $\nu \in \hat{H}$. But it is virtually a direct consequence of generalized Gelfand pair theory, applied to the symmetric space $(H \times H)/\mathrm{diag}(H)$, that the representation space of $\pi_{\nu,\chi}$, namely $\bar{\mathcal{H}}_\nu \otimes \mathcal{H}_\gamma$ supports an H-invariant distribution iff $\mathcal{H}_\nu$ is one of the Hilbert spaces that appears in the multiplicity-free direct sum

$$\mathcal{H}_\gamma = \sum_{\sigma \in \hat{H}}^{\oplus} n_\gamma(\sigma)\mathcal{H}_\gamma(\sigma),\ n_\gamma(\sigma) = 0 \text{ or } 1.$$

Moreover, the only invariant distribution on that space is (up to scalar) the distribution given as follows:

On $\bar{\mathcal{H}}_\gamma(\nu) \otimes \mathcal{H}_\gamma = \sum_\sigma \bar{\mathcal{H}}_\gamma(\nu) \otimes \mathcal{H}_\gamma(\sigma)$ the distribution is concentrated in $\bar{\mathcal{H}}_\gamma(\nu) \otimes \mathcal{H}_\gamma(\nu)$, and is given there by $T \to \mathrm{Tr}\bar{T}$.

There are of course many interesting subgroups of the symplectic group (including the symplectic group itself) on which the Weil representation breaks up as a multiplicity-free direct sum, but I do not believe any kind of classification is known.

6. Partially Riemannian symmetric spaces

In this and the next section we present two more categories in which there is a positive answer to the implication (1.7). Both of these are Strichartz symmetric spaces, as was the second example we looked at in section 4. We refer to [10] for background on Strichartz symmetric spaces. They are homogeneous spaces of semidirect product groups $G = RV$ where V is a normal vector subgroup, and R is a (usually) reductive subgroup. We assume the existence of an involutive automorphism that preserves both V and R, so that if the corresponding stability groups are U and S, then $H = SU$ is the stability subgroup for the automorphism in G and G/H is a symmetric space. Such a space is called *partially Riemannian* if R is reductive and S is compact. It is proven in [10, Thm. 5.5] that if G/H is partially Riemannian and R is non-compact, then G/H is multiplicity-free. We have

Proposition 6.1. *Any partially Riemannian Strichartz symmetric space in which the reductive subgroup is non-compact is a generalized Gelfand pair.*

Proof. This is an instance in which there are proofs by both methods—directly, by computation of the dimensions of the spaces of invariant distributions for all irreducibles; or indirectly, using the symmetric space structure to arrive at a structural decomposition that enables the Gelfand-Thomas-van Dijk method. The former is easier since it is in fact already done. One need only consult Theorem 8.1 in [10]. The proof there of distributional Frobenius reciprocity assumes that R is compact. But if one examines the proof, one sees that only the compactness of S is used. The alternative method, one that is perhaps more susceptible to generalization, is a proof using structure theory. We supply one. Let σ denote the symmetric space involution. Set $Q = \{g \in G : \sigma(g) = g^{-1}\}$ and, using σ to denote the involution on the Lie algebra level as well, we also let $\mathfrak{q}$ denote the -1 eigenspace for σ. Then we use the well-known fact that [19, Criterion 2.5] is satisfied if

$$\exp : \mathfrak{q} \to Q$$

is a diffeomorphism. (See *e.g.* [1] for a demonstration of how this works.) The diffeomorphism can be established as follows. First, we observe that S is a maximal compact subgroup of the reductive group R, and that σ on the Lie algebra of R is a Cartan involution. Let $\mathfrak{p}$ be the -1 eigenspace, and let $\mathfrak{a}$ be maximal abelian in $\mathfrak{p}$. Then we diagonalize the action of $\mathfrak{a}$ on V

$$V = V^0 + \sum_{\beta>0}(V^\beta + V^{-\beta}).$$

Of course $Y \in V^\beta \Leftrightarrow \sigma(Y) \in V^{-\beta}$. Also, the elements in V^β decompose according to the involution by

$$Y \to Y + \sigma(Y),\ V^\beta \to \mathfrak{u}; \quad Y \to Y - \sigma(Y),\ V^\beta \to \mathfrak{w},$$

where $\mathfrak{w}$ (resp. $\mathfrak{u}$) is the -1 (resp. $+1$) eigenspace of σ on V. It is easy to see that $V^0 = (V^0 \cap \mathfrak{u}) + (V^0 \cap \mathfrak{w})$.

Now, utilizing conjugation by the maximal compact subgroup S of R, we see that it is enough to show that the map $(\mathfrak{a}+\mathfrak{w}) \to \exp(\mathfrak{a}+\mathfrak{w})$ is a diffeomorphism. We need to be more specific about the set $\exp(\mathfrak{a}+\mathfrak{w})$. In fact $Q = \{rv \in RV : \sigma(r) = r^{-1} \text{ and } \sigma(v) = rv^{-1}r^{-1}\}$. Now restrict the R component to $A = \exp \mathfrak{a}$. It is easy to see, restricting to one β component, that the element $a \exp(Y_1 + Y_2)$, $a = \exp X$, $X \in \mathfrak{a}, Y_1 \in V^\beta$, $Y_2 \in V^{-\beta}$, is in Q iff $Y_2 = -e^{-\beta(X)}\sigma(Y_1)$. (We have identified $Y \in V$ with $\exp Y$.) In particular, the contributing elements of Q may be written

$$\exp X(e^{\beta(X)/2}Y - e^{-\beta(X)/2}\sigma(Y)),\ X \in \mathfrak{a},\ Y \in V^\beta.$$

But then it's a simple computation to show that: for $X \in \mathfrak{a}, Y_0 \in V^0 \cap \mathfrak{w}$, $Y_\beta \in V^\beta$,

$$\begin{aligned}
&\exp(X + Y_0 + \sum_{\beta>0}(Y_\beta - \sigma(Y_\beta))) = \\
&\exp(X)\exp(Y_0)\Pi_{\beta>0}\exp\cosh(\beta(X)/2)(Y_\beta - \sigma(Y_\beta))\exp\sinh(\beta(X)/2)(Y_\beta + \sigma(Y_\beta))) = \\
&\qquad \exp(X)\exp(Y_0)\Pi_{\beta>0}(e^{\beta(X)/2}Y_\beta - e^{-\beta(X)/2}\sigma(Y_\beta)).
\end{aligned}$$

This establishes the diffeomorphism.

7. Metabelian symmetric spaces

Here we consider another category, namely metabelian Strichartz spaces. In this case there is no structural proof.

Proposition 7.1. *Let $G = RV, H = SU$, and suppose $\tau_{G,H}$ is multiplicity-free and that R is abelian. Then (G, H) is a generalized Gelfand pair.*

Proof. We have to show that [10, Thm. 8.1] still holds when we replace the assumption that R is compact by the assumption that R is abelian. The initial portion of the proof of part (i) goes through unchanged. The second part—in which the compactness of S was used to obtain an S-invariant function from an H-invariant distribution no longer applies. But we can use the commutativity of R as follows. Suppose a distribution ξ is H-invariant. Then

$$\langle \xi, f \rangle = \langle \xi, f^s \rangle$$

for all $s \in S$. But then for $s \in S_\lambda$, we have

$$\langle \xi, f \rangle = \langle \xi, f^s \rangle = \langle \xi, f_{s^{-1}} \rangle = \sigma(s)^{-1}\langle \xi, f \rangle.$$

Therefore $\sigma(s) = 1$ and so $\sigma \in \hat{R}_\lambda(S_\lambda)$. This completes the argument for part (i). As for part (ii), the multiplicity-free assumption can only happen if generically on $\lambda \in U^\perp$, the equality

$$R \cdot \lambda \cap U^\perp = S \cdot \lambda$$

is valid. If it is, then it is clear, by the uniqueness of Haar measure and the same reasoning as in [10, p. 377], that the only possible invariant distribution is

$$f \to \int_{S/S_\lambda} f(s)\, d\dot{s}.$$

This completes the proof.

It is worth noting that the proof in [10] really only requires S to be compact, and the proof here only requires S to be abelian.

8. Horocycle spaces

In this paper I have touched upon all of the homogeneous spaces, but one, that have played a prominent role in my recent work on Plancherel theory—namely, reductive homogeneous spaces, nilpotent homogeneous spaces, Strichartz spaces, and homogeneous spaces of the form G/H where $G = HN$, N is normal and Heisenberg, and H is a subgroup of the symplectic group. The missing category is generalized horocyclic spaces; that is, G/H where G is a semisimple Lie group and H is a subgroup of a parabolic group that contains its nilradical. What can we say about the issues of this paper in the context of those groups? Considering [16], we see that virtually all of those homogeneous spaces have uniform infinite multiplicity and so are not germane to the discussion in this paper. With one exception, we get a multiplicity-free decomposition only in some extremely special cases that do not merit general attention. The one exceptional case is when H is the full parabolic—in which instance the quasi-regular representation $\tau_{G,H}$ is often irreducible, and at worst a finite, multiplicity-free direct sum. In the irreducible case we invoke [12, Thm. 3.3], which actually holds whether G/H is symmetric or not, to see that G/P is indeed a generalized Gelfand pair. In particular, if P is a minimal parabolic subgroup of a semisimple Lie group G, then $\tau_{G,P}$ is an irreducible principal series representation and (G, P) is a generalized Gelfand pair.

Finally, I will look at the related question of distributional reciprocity. And I shall only look at the usual horocycle space associated to a minimal parabolic. So suppose G is a semisimple Lie group, MAN is a minimal parabolic; then the homogeneous space in which we are interested is G/MN. It is well-known ([13, section 2a] that the representation $\tau_{G,MN}$ breaks up as a direct integral over the spherical principal series with uniform multiplicity the order of the Weyl group. The issue of distributional reciprocity is dealt with in [5] or [7, esp. Thm. 5.16, p. 176]. Namely, for sufficiently regular elements in the series, we have $(\mathcal{H}_\pi^{-\infty})^{MN} = \#(W)$. The question of the dimension of the space of MN-invariant distributions on other representation spaces is an interesting one that is not completely settled. In a private communication, Helgason has outlined an argument for showing that the dimension is zero for a non-conical representation, and at least $\#(W)$ otherwise.

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