THE PLANCHEREL FORMULA FOR THE HOROCYCLE SPACES AND GENERALIZATIONS, II

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Abstract. To be rewritten: This paper represents a continuation of the author's program to establish a new paradigm for concrete Plancherel analysis on homogeneous spaces wherein the distinction between finite and infinite multiplicity is de-emphasized. In this paper, the Plancherel formula is proven for arbitrary Strichartz homogeneous spaces $H\backslash G$, that is where both G and H are semidirect products of normal abelian subgroups by a reductive Lie group. In general, these spaces have infinite multiplicity and their generic irreducible unitary representations are induced from infinite-dimensional representations. Canonical nuclear operators for representations induced from infinite-dimensional representations are introduced in a very general context, and then utilized to derive the Plancherel Formula for Strichartz spaces.

1. Introduction.

This paper is a direct sequel to []. It is also heavily dependent on the ideas and results in [] [] [] and []. In this (and those) paper(s), the prime objective is to construct the Plancherel theory of the quasi-regular representation of a homogeneous space. More precisely, given a Lie group G and a closed subgroup H, let us assume that the quasi-regular representation $\tau = \tau_{G,H} = \mathrm{Ind}_H^G 1$ is type I. Then there is a unique direct integral decomposition

$$\tau = \int_{\widehat{G}(H)}^{\oplus} n_\pi \pi \, d\mu(\pi). \tag{1.1}$$

Here $\hat{G}(H)$ denotes the irreducible unitary representations of G that are weakly contained in τ, a closed subset of the unitary dual $\hat{G}$. The Plancherel theory that is derived in the previously cited papers includes a specific analytic formula that provides detailed information not only on the structure of $\hat{G}(H)$, the multiplicity function n_π and the Plancherel measure μ, but also on an intertwining operator that effects the direct integral decomposition. This is done in [] and [] in various cases that manifest finite multiplicity (*i.e.* $n_\pi < \infty$, μ-a.e.); it

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is done in [] and [] for certain infinite multiplicity situations. The fundamental philosophy of [] is that these two cases—usually thought of as very different—can be treated in a uniform manner. Moreover, the Penney-Fujiwara Plancherel formula (PFPF) and the Bonnet Plancherel formula (BPF), the analytic formulas that express the Plancherel theory in the two cases (resp.), are really the same gadget if interpreted properly (see [], []).

Our goal is to demonstrate very precisely that last assertion. We do that by considering a category of homogeneous space in which the multiplicity is either uniformly finite or uniformly infinite, depending on some geometric or measure-theoretic invariant. Then we show how to derive explicitly the Plancherel theory for both cases by parallel techniques. This program has been carried out for G nilpotent in Fujiwara's two papers [] []; for Strichartz spaces with trivial stabilizer in []; and for general Strichartz spaces in []. In [], I proposed pursuing this program for these categories of homogeneous spaces:

(1.2i) general Strichartz homogeneous spaces;
(1.2ii) semisimple homogeneous spaces with non-reductive stability group;
(1.2iii) nilpotent homogeneous spaces—a reformulation and simplification of the work of Fujiwara—and then exponential solvable homogeneous spaces;
(1.2iv) semidirect product homogeneous spaces $H\backslash G$, where G is the semidirect product of a subgroup H of the symplectic group with a normal Heisenberg group;
(1.2v) reductive homogeneous spaces.

We realized that goal for (1.2i) in []. In this paper we will realize it for (1.2ii).

The spectrum of the quasi-regular representations we consider will always consist of induced representations. Sometimes the inducing representations will be finite-dimensional, sometimes infinite-dimensional. The resulting representations will therefore be polynomial sometimes ([]), at other times exponential ([]). We will present our Plancherel formulae in the exact same fashion regardless—namely, as a so-called Bonnet Plancherel formula. This is usually done only in the case of infinite multiplicity; but as we explained in [], the finite-multiplicity formula is really the same thing once it's interpreted properly.

There is one more important point to be made before we proceed to the details. For many homogeneous spaces that we encounter—in particular, for those in this paper—the Plancherel formula is derived in a form different from formula (1.1). More seriously, it occurs in a different form from formula (2.2) in Theorem 2.2. Namely, the parametrization occurs in $\mathrm{Irr}(G)$, a concrete set of irreducible unitary representations of G, rather than in the dual $\hat{G}$:

$$\tau = \int_{\mathrm{Irr}(G)}^{\oplus} \pi \, d\nu(\pi). \tag{1.2}$$

To convert (1.2) into the form (1.1), one must factor by unitary equivalence. The Plancherel measure μ will then be a pseudo-image of ν. That is not so troublesome as the fact that the nuclear operators that appear in the BPF (see Thm. 2.2) can become considerably more complicated in the factored form. See §3a for an instance of that. We mention here only that the experience leaves one with the surprising conclusion that it is sometimes better to express a Plancherel formula in terms of a decomposition over $\mathrm{Irr}(G)$, where the multiplicity is not explicitly stated, than in terms of a decomposition over $\hat{G}$.

2. Nuclear Operators and the Plancherel Formula.

In this section we recall from [] the basic results on canonical nuclear operators associated to induced representations. We also reestablish the fundamental facts found in the BPF and the PFPF. The context is always that $\tau = \tau_{G,H}$ is a type I representation, resulting in a well-determined Plancherel decomposition (1.1). Finally, we always assume that G is unimodular.

We begin by fixing a choice of right Haar measures dg, dh on G and H respectively. We write Δ_H for the modular function of H (*i.e.* the derivative of right Haar measure with respect to left). By assumption, $\Delta_G \equiv 1$. We choose once and for all a smooth function $q = q_{H,G}$ satisfying $q(e) = 1$, $q(hg) = \Delta_H(h)q(g)$, $\forall h \in H$, $g \in G$. Now suppose π is a unitary representation of G acting on a Hilbert space $\mathcal{H}_\pi$. We write $\mathcal{H}_\pi^\infty$ to denote the Frechet space of C^∞ vectors and $\mathcal{H}_\pi^{-\infty}$ for its anti-dual space of distributions (*i.e.* conjugate linear functionals). As usual we set $\langle v, \alpha\rangle = \overline{\alpha(v)}$, $v \in \mathcal{H}_\pi^\infty$, $\alpha \in \mathcal{H}_\pi^{-\infty}$, a sesquilinear form. $\pi(G)$ acts on both $\mathcal{H}_\pi^\infty$ and $\mathcal{H}_\pi^{-\infty}$, and it's a well-known fact that $\pi(\mathcal{D}(G))\mathcal{H}_\pi^{-\infty} \subset \mathcal{H}_\pi^\infty$, if $\mathcal{D}(G)$ is the test space of compactly supported, infinitely differentiable functions on G. Given the direct integral decomposition (1.1), it is known from [5] and [14] that

$$\mathcal{H}_\tau^\infty = \int^\oplus n_\pi \mathcal{H}_\pi^\infty \, d\mu(\pi) \qquad \mathcal{H}_\tau^{-\infty} = \int^\oplus n_\pi \mathcal{H}_\pi^{-\infty} \, d\mu(\pi).$$

We refine the latter of these as follows. Set $(\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}} = \{\alpha \in \mathcal{H}_\pi^{-\infty} : \pi(h)\alpha = q^{-1/2}(h)\alpha, \forall h \in H\}$. It is established in [9] that the distribution $\alpha_\tau : f \to \overline{f(e)}$ lies in $(\mathcal{H}_\tau^{-\infty})^{H,q^{-1/2}}$, and its matrix coefficient $\langle \tau(\omega)\alpha_\tau, \alpha_\tau\rangle$, $\omega \in \mathcal{D}(G)$, is calculated there as follows:

$$\tau(\omega)\alpha_\tau(g) = \omega_H(g), \text{ where}$$

$$\omega_H(g) = q^{-1/2}(g)\int_H \omega(g^{-1}h^{-1})q^{-1/2}(h)\, dh, \; \omega \in \mathcal{D}(G); \tag{2.1}$$

and by a routine change of variable (see [9, Prop. 2.2])

$$\langle \tau(\omega)\alpha_\tau, \alpha_\tau\rangle = \omega_H(e) = \int_H \omega(h^{-1})q^{-1/2}(h)\, dh = \int_H \omega(h)q^{-1/2}(h)\, dh.$$

Now we state

Theorem 2.1 (BPF). [1] *For μ-a.a. π, there exist positive nuclear operators $U_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^{-\infty}$ such that*

$$\omega_H(e) = \int_{\widehat{G}(H)} \mathrm{Tr}[\pi(\omega)U_\pi]\, d\mu(\pi), \; \omega \in \mathcal{D}(G). \tag{2.2}$$

Furthermore, the pair $(\mu, \{U_\pi\}_{\pi\in\hat{G}})$ is unique up to positive scalars $c(\pi)$; i.e. any other pair $(\mu', \{U'_\pi\}_{\pi\in\hat{G}})$ satisfying formula (2.2) must be related to the original by $\frac{d\mu}{d\mu'} = c(\pi)$, $U_\pi = U'_\pi/c(\pi)$, for some positive Borel function $c(\pi)$.

We make use of the structure established in [13], §2. In particular, we have

$$\pi(h)U_\pi = q^{-1/2}(h)U_\pi, \quad U_\pi\pi(h) = U_\pi, \; h \in H. \tag{2.3}$$

The first of these is proven in [13, §2]. The second is not established there, but the method of proof is exactly the same as the first (using the left translate ω_h instead of the right translate ω^h of a test function $\omega \in \mathcal{D}(G)$). Now in case almost all of the operators U_π have finite rank, Theorem 2.1 specializes to

Theorem 2.2 (PFPF). [2] [14] *Suppose that*

$$n_\pi = \operatorname{rank} U_\pi < \infty, \mu - \text{a.e.}$$

Then of course, the multiplicity in (1.1) is finite μ-a.e., and for μ-a.a. π, there exist n_π linearly independent distributions $\beta_\pi^1, \dots, \beta_\pi^{n_\pi} \in \text{range}(U_\pi) \subset (\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}}$ such that

$$\langle \tau(\omega)\alpha_\tau, \alpha_\tau \rangle = \int_{\widehat{G}(H)} \sum_{j=1}^{n_\pi} \langle \pi(\omega)\beta_\pi^j, \beta_\pi^j \rangle \, d\mu(\pi), \ \omega \in \mathcal{D}(G). \tag{2.4}$$

To see how to construct the β_π out of U_π, consult [13], Remark 2.3.6. Now, as the phrase was coined in [13], by a *concrete* Plancherel formula for a category of homogeneous spaces $H\backslash G$, we mean explicit expressions for: μ, the nuclear operators U_π—or the distributions β_π^j, and the summable operators $\pi(\omega)U_\pi$—or smooth vectors $\pi(\omega)\beta_\pi^j$. These lead to a totally explicit intertwining operator in either case (see [9], [10], [13]).

Next, since we are concentrating on the situation wherein the spectrum of τ consists of induced representations, we focus on a single representation from the spectrum. Suppose that π is an induced representation $\pi = \operatorname{Ind}_B^G \sigma$. We shall say that π is *exponential* with respect to H if σ is infinite-dimensional and the following conditions are satisfied.

(2.5a) There exists a positive nuclear operator $U_\sigma : \mathcal{H}_\sigma^\infty \to \mathcal{H}_\sigma^{-\infty}$ that is left and right invariant under $H \cap B$.

(2.5b) BH is closed in G.

(2.5c) $q_{H\cap B,H}\, q_{H\cap B,B} \equiv 1$ on $H \cap B$.

(2.5d) For any $\omega \in C_c^\infty(B, H\cap B)$, the operator

$$\int_{H\cap B\backslash B} \omega(b)\sigma(b)U_\sigma \, d\dot{b}$$

is trace class.

See [] for a thorough discussion of the meaning and appropriateness of each of these conditions.

We take the usual realization and action for the induced representation π, namely its space $\mathcal{H}_\pi$ is the Hilbert space completion of

$$C_c^\infty(G,B,\sigma) = \{f : G \to \mathcal{H}_\sigma, f \text{ is } C^\infty, f(bg) = \sigma(b)f(g), b \in B, g \in G,$$
$$||f|| \text{ is compactly supported mod } B\}$$

and G acts by the formula

$$\pi(g)f(x) = f(xg)[q_{B,G}(xg)/q_{B,G}(x)]^{1/2}. \tag{2.6}$$

The norm is relative to the unique quasi-invariant measure on $B\backslash G$ determined by $q_{B,G}$. (For a review of q functions, see [7].)

Suppose that we are given a positive nuclear operator $U_\sigma : \mathcal{H}_\sigma^\infty \to \mathcal{H}_\sigma^{-\infty}$ that is left and right $H \cap B$-invariant. Given that, we now define the positive nuclear operator

$$U_\pi(\sigma) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^{-\infty}$$

by the formula

$$\begin{aligned}\langle f_1, U_\pi(\sigma) f_2 \rangle = &\int_{H\cap B\backslash H} \int_{H\cap B\backslash H} \langle f_1(h_1), U_\sigma f_2(h_2)\rangle \\ &\times q_{B,G}^{1/2}(h_1) q_{H,G}^{-1/2}(h_1) q_{H\cap B,H}^{-1}(h_1) q_{B,G}^{1/2}(h_2) q_{H,G}^{-1/2}(h_2) q_{H\cap B,H}^{-1}(h_2)\, d\dot{h}_1 d\dot{h}_2.\end{aligned} \tag{2.7}$$

Then we have

Theorem 2.3. []

(i) *$U_\pi(\sigma)$ is well-defined.*

(ii) *$U_\pi(\sigma)$ is relatively invariant under the action of H with modulus $q^{-1/2} = q_{H,G}^{-1/2}$, that is*

$$U_\pi(\sigma)\,(\mathcal{H}_\pi^\infty) \subseteq (\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}}. \tag{2.8}$$

(iii) *For $\omega \in \mathcal{D}(G)$, the operator-valued function $\pi(\omega)U_\pi(\sigma)$ is given by the formula*

$$\begin{aligned}\pi(\omega)U_\pi(\sigma)f(g) = \int_{H\cap B\backslash H}\int_{H\cap B\backslash B} &\omega_H(bg)\sigma(b)^{-1}U_\sigma f(h) \\ &\times q_{B,G}^{-1/2}(bg)q_{H,G}^{1/2}(bg)q_{H\cap B,B}^{-1}(b)q_{B,G}^{1/2}(h)q_{H,G}^{-1/2}(h)q_{H\cap B,H}^{-1}(h)\,d\dot{b}d\dot{h}\end{aligned} \tag{2.9}$$

where ω_H is defined in (2.1).

(iv) *For $\omega \in \mathcal{D}^+(G)$, the character $\mathrm{Tr}(\pi(\omega)U_\pi(\sigma))$ is*

$$\begin{aligned}&\mathrm{Tr}[\pi(\omega)U_\pi(\sigma)] \\ &= \int_{H\cap B\backslash H}\mathrm{Tr}\int_{H\cap B\backslash B}\omega_H(bh)\sigma(b)^{-1}U_\sigma q_{B,G}^{-1/2}(b)q_{H,G}^{1/2}(h^{-1}bh)q_{H\cap B,B}^{-1}(b)q_{H\cap B,H}^{-1}(h)\,d\dot{b}\,d\dot{h}.\end{aligned} \tag{2.10}$$

The trace is a non-negative number, possibly equal to $+\infty$, provided $\omega = \omega_1 \star \omega_1^*$, $\omega_1 \in \mathcal{D}(G)$.

In the next section we shall pass to the case of non-reductive semisimple homogeneous spaces. That is, we will assume G is a semisimple Lie group and H is a non-reductive subgroup. Actually, we shall be more demanding of H—we shall assume it is a canonical subgroup of a parabolic containing the nilradical. In that case, the representations that appear in the spectrum of the quasi-regular representation are always induced representations—in fact, induced from the parabolic itself. Speaking roughly, we are looking at semisimple homogeneous spaces whose spectrum avoids any discrete series. Said another way, we are concentrating on quasi-regular representations whose spectrum consists of representations whose associated Duflo-Kirillov functionals have non-totally complex polarizations.

3. Canonical Non-Reductive Semisimple Homogeneous Spaces.

Now let G be a semisimple Lie group. Let P be *any* parabolic subgroup. Denote its Langlands decomposition by $P = MAN$. If we fix Haar measures dm, da, dn on the unimodular groups M, A, N, then $dmdadn$ is left Haar measure on P, and $\int_N f(n)\,dn = e^{2\rho(\log a)}\int_N f(ana^{-1})\,dn$, where ρ is one-half the sum of the positive roots (on the Lie algebra $\mathfrak{a}$ with respect to $\mathfrak{n}$). In particular, $e^{2\rho(\log a)}dmdadn$ is right Haar measure on P and $\Delta_P(man) = e^{2\rho(\log a)}$ is the modular function.

Now we consider a subgroup H of G satisfying $N \subset H \subset P$. Moreover we assume that H is "canonical", that is, H is one of the groups associated naturally to the Langlands decomposition. In short, we will consider the four examples:

(a) $H = MN$;
(b) $H = MAN$, that is, $H = P$;
(c) $H = AN$;
(d) $H = N$.

In each case it is easy to derive the soft Plancherel formula, that is the abstract direct integral decomposition of the corresponding quasi-regular representation into irreducible unitary representations. It is our goal in this paper to give, in each case, using the theory described in the last section, the more explicit BPF. The soft decomposition will reveal the Plancherel measure. We have to compute the Bonnet nuclear operators in each case.

To give the soft Plancherel formulae, we only need to apply induction in stages, inducing through the parabolic to decompose the quasi-regular representation. We give the full computation for the first and third examples. The second is trivial, and the fourth is very similar to the third. Before doing so, we establish some notation for the principal series representations of G that one obtains by inducing from P. For any character $\lambda \in \hat{A}$, and any irreducible unitary representation $\sigma \in \hat{M}$, we write

$$\pi_{\lambda,\sigma} = \mathrm{Ind}_{P=MAN}^{G} \sigma \times \lambda \times 1. \tag{3.1}$$

It is well-known that these induced representations are generically irreducible. More precisely, if we restrict λ to the generic open set $\hat{A}'$ of regular characters (those not preserved by any non-trivial element of the Weyl group $W_P = \mathrm{Norm}_G(A)/\mathrm{Cent}_G(A)$), then $\pi_{\lambda,\sigma}$ is irreducible. It's also known that there is some duplication, namely $\pi_{\lambda,\sigma} \cong \pi_{\lambda',\sigma'}$ iff there is an element of the Weyl group that conjugates (λ, σ) into (λ', σ'). *Check this!!*

Now for the decompositions. Case (a) decomposes:

$$\begin{aligned}
\tau_{G,MN} &= \mathrm{Ind}_{MN}^{G} 1 \\
&= \mathrm{Ind}_{P}^{G} \mathrm{Ind}_{MN}^{P} 1 \\
&= \mathrm{Ind}_{P}^{G} \int_{\widehat{A}}^{\oplus} 1 \times \lambda \times 1 \, d\lambda \\
&= \int_{\widehat{A}}^{\oplus} \mathrm{Ind}_{P}^{G} 1 \times \lambda \times 1 \, d\lambda \\
&= \int_{\widehat{A}}^{\oplus} \pi_{\lambda,1} \, d\lambda && (3.2) \\
&= \int_{\widehat{A}'/W}^{\oplus} \#(W) \pi_{\lambda,1} \, d\tilde{\lambda}, && (3.3)
\end{aligned}$$

where $d\tilde{\lambda}$ is the image of Lebesgue measure $d\lambda$ under the canonical projection $\hat{A} \to \hat{A}/W$. As for case (c):

$$\begin{aligned}
\tau_{G,AN} &= \mathrm{Ind}_{AN}^{G} 1 \\
&= \mathrm{Ind}_{P}^{G} \mathrm{Ind}_{AN}^{P} 1 \\
&= \mathrm{Ind}_{P}^{G} \int_{\widehat{M}}^{\oplus} (\dim \sigma) \, \sigma \times 1 \times 1 \, d\mu_M(\sigma) \\
&= \int_{\widehat{M}}^{\oplus} (\dim \sigma) \, \mathrm{Ind}_{P}^{G} \sigma \times 1 \times 1 \, d\mu_M(\sigma) \\
&= \int_{\widehat{M}}^{\oplus} (\dim \sigma) \, \pi_{1,\sigma} \, d\mu_M(\sigma) && (3.4) \\
&= \int_{\widehat{M}/W}^{\oplus} \#(W) (\dim \sigma) \, \pi_{1,\sigma} \, d\tilde{\mu}_M(\sigma), && (3.5)
\end{aligned}$$

where μ_M is the Plancherel measure of the reductive group M, and $\tilde{\mu}_M(\sigma)$ is the image of Plancherel measure $\mu_M(\sigma)$ under the canonical projection $\hat{M} \to \hat{M}/W$.

The decomposiiton for (b) is trivial—the induced representation $\tau_{G,P}$ is irreducible. (*Check that!!*) Finally, we leave it to the reader to verify the decomposition for (d), namely:

$$\begin{aligned} \tau_{G,N} &= \mathrm{Ind}_N^G 1 \\ &= \int_{\widehat{MA}}^{\oplus} (\dim\sigma)\, \pi_{\lambda,\sigma}\, d\mu_{MA}(\sigma,\lambda) && (3.6) \\ &= \int_{(\widehat{M}\times\widehat{A}')/W}^{\oplus} \#(W)(\dim\sigma)\, \pi_{\lambda,\sigma}\, d\tilde{\mu}_{MA}(\sigma,\lambda), && (3.7) \end{aligned}$$

where $\mu_{MA} = \mu_M \times d\lambda$ is the Plancherel measure of the reductive group MA, and $\tilde{\mu}_{MA}(\sigma,\lambda)$ is its image under the canonical projection $\widehat{MA} \to (\widehat{MA})/W$.

3a. Generalized Horocycle Spaces. Now we deal with the concrete Plancherel formula for *generalized horocycle spaces*, that is the homogeneous space G/MN. We allow $P = MAN$ to be any parabolic subgroup of the semisimple group G. The soft Plancherel formula is prescribed in the decompositions (3.2), (3.3). In particular, we see that the multiplicity is finite and uniform. Thus one only needs the PFPF to describe the concrete Plancherel formula in this case. This is already done in []. Here we shall give a BPF (as if the multipliciy were infinite) and relate the description to the one in []. As we shall in every case, we utilize the machinery in §2.

First, we specify the q functions. In this case we have $H = MN$, a unimodular group. Hence, we choose $q_{H,G} = q_{MN,G} \equiv 1$ on all of G. Here, as in all the cases, the polarizing group B, from which the representations are induced, is P itself. Therefore, the intersection of the polarizing group and the stability group H will always be H (since in all cases $H \subset P$). Therefore we must have $q_{H\cap B,H} = q_{H,H} \equiv 1$ on H. Next, $q_{H\cap B,B} = q_{MN,MAN} = 1$ on MN. We extend it to P by setting $q_{MN,MAN}(man) = \Delta_{MAN}^{-1}(man) = e^{-2\rho(\log a)}$. Finally, we have $q_{B,G}(man) = q_{P,G}(man) = \Delta_P(man) = e^{2\rho(\log a)}$ on P. We extend it to G via the Iwasawa decomposition. Let K be a maximal compact subgroup of G. Then $G = PK = MANK$. The overlap is $M \cap K$. So we can choose unambiguously $q_{B,G}(mank) = q_{P,G}(mank) = e^{2\rho(\log a)}$.

The representations in the spectrum are $\pi_\lambda \equiv \pi_{\lambda,1}$. These representations are induced from the characters λ. So, since the inducing representations are not infinite-dimensional, the representations are not, strictly speaking, exponential. But the four conditions in (2.5) are satisfied. This is absolutely obvious for (b) and (c). As for (a) and (d), we make the only conceivable choice for the nuclear operator U_λ, namely the identity on the one-dimensional space of λ. We then invoke formula (2.7) and Theorem 2.3. The operator U_{π_λ} is defined on the space of $\mathcal{H}_{\pi_\lambda}$ by

$$\langle f_1, U_{\pi_\lambda} f_2\rangle = f_1(e)\bar{f}_2(e). \tag{3.8}$$

We see immediately from part (iv), formula (2.10), of Theorem 2.3, that

$$\mathrm{Tr}[\pi_\lambda(\omega)U_{\pi_\lambda}] = \int_A \omega_{MN}(a)\bar{\lambda}(a)e^{-\rho(\log a)}\, da d\lambda,$$

which is exactly the formula at the end of the computation on p. 48 of [horoI]. Concluding as in that case, we replicate the Plancherel formula in [horoI, Thm 2.2], namely

Theorem 3.1. *For τ_{MN}, we have the concrete Plancherel formula*

$$\omega_{MN}(e) = \int_{\widehat{A}} \mathrm{Tr}[\pi(\omega)U_{\pi_\lambda}]\, d\lambda, \quad \omega \in \mathcal{D}(G). \tag{3.9}$$

Formulas (3.8) and (3.9) represent the BPF corresponding to the soft Plancherel formula (3.2)—as well as corresponding to the PFPF in [horoI, Thm. 2.5]. We leave it to the reader to check that the Bonnet nuclear operators U_{π_λ} and the Penney distributions β_λ (see [,pp. 48-49]) are related as they should be (see [, Remark 2.3]). Instead, we observe that one may reasonably ask what is the BPF that corresponds to the formula (3.3). This question highlights the following general issue. Often one is able to derive a Plancherel formula (in either soft or hard fashion) in which the parameter space for the decomposition, say X, maps naturally to $\hat{G}$. That is, corresponding to each $x \in X$, we have an irreducible representation π_x, and the association of the unitary class $\{\pi_x\}$ of π_x to x is the asserted map. Of course the map $x \to \{\pi_x\}$, $X \to \hat{G}$ will rarely be surjective. The problem is that it also may not be injective. Thus a Plancherel formula, say in the BPF format, may be derrived in the form

$$\omega_H(e) = \int_X \mathrm{Tr}[\pi_x(\omega)U_{\pi_x}]\, du(x),\ \omega \in \mathcal{D}(G). \tag{3.10}$$

Strictly speaking, if $x \to \{\pi_x\}$ is not injective, formula (3.10) is not in accordance with the format of Theorem 2.1. To bring it into the proper format, one must pick a smooth cross-section, say $\tilde{X}$, for the equivalence relation $x \sim x' \Leftrightarrow \pi_x \cong \pi_{x'}$, pick an appropriate pseudo-image $\tilde{\mu}$ of μ on $\tilde{X} = X/\sim$, and, most difficult of all, recompute the nuclear operators U_x, $x \in X$, as nuclear operators $\tilde{U}_x$, $x \in \tilde{X}$, so that

$$\omega_H(e) = \int_{\tilde{X}} \mathrm{Tr}[\pi_x(\omega)U_{\pi_x}]\, d\tilde{\mu}(x),\ \omega \in \mathcal{D}(G). \tag{3.11}$$

The interesting point to note is that the new nuclear operators $\tilde{U}_x$ may be considerably more complicated than the original U_x. In fact, one of the key points of this paper is that the realization of the Plancherel formula in the "appropriate" form (3.11) may be much less natural than in the "inappropriate" form (3.10). This will be nicely illustrated by the current example. Let us now compute the BPF for the generalized horocycle spaces when we factor out unitary equivalence.

We select the usual cross-section $\hat{A}^+$ in $\hat{A}$ for $\hat{A}'/W$. Namely, since $\hat{A}$ is identified to the real vector space of all real linear functionals on $\mathfrak{a}$ via $\lambda(\exp(Y)) = e^{i\lambda(Y)}$, $Y \in \mathfrak{a}$, we can choose $\hat{A}^+$ to be the characters whose corresponding functional takes positive values on the positive Weyl chamber (in $\mathfrak{a}$, determined by $\mathfrak{n}$). Now let $s_1 = e, s_2, \ldots$ denote a (finite) set of representatives in $\mathrm{Norm}_G(A)$ for the elements of the Weyl group W_P. It is classical to specify the intertwining operator corresponding to the equivalent representations π_λ and $\pi_{\lambda\cdot s}$ by

$$T_s : f \to \int_{N_0\backslash N_s} f(n_s sg)\, d\dot{n}_s,$$

where $N_s = sNs^{-1}$, $N_0 = N \cap N_s$, and $d\dot{n}_s$ is the invariant measure on the quotient space. The integral plainly converges on the dense space of functions $f \in C_c^\infty(G, P, 1 \times \lambda \times 1) \subset$

$\mathcal{H}_{\pi_\lambda}$ and extends uniquely to a unitary intertwining operator of π_λ with $\pi_{\lambda\cdot s}$. Then a relatively simple computation gives us

$$\begin{aligned}\omega_{MN}(e) &= \int_{\widehat{A}} \mathrm{Tr}[\pi_\lambda(\omega)U_{\pi_\lambda}]\,d\lambda \\ &= \sum_{s\in W}\int_{\widehat{A}+} \mathrm{Tr}[\pi_{\lambda\cdot s}(\omega)U_{\pi_{\lambda\cdot s}}]\,d\lambda \\ &= \int_{\widehat{A}+} \mathrm{Tr}[\pi_\lambda(\omega)\tilde{U}_{\pi_\lambda}]\,d\lambda,\end{aligned}$$

where

$$\tilde{U}_{\pi_\lambda} = \sum_{s\in W} T_s^{-1}U_{\pi_{\lambda\cdot s}}T_s,\ \lambda\in\hat{A}^+.$$

It is a simple exercise to expand and obtain the formula for the nuclear operators

$$\langle f_1,\tilde{U}_{\pi_\lambda}f_2\rangle = \sum_{s\in W}\int_{(s^{-1}Ns\cap N)\backslash N}\int_{(s^{-1}Ns\cap N)\backslash N} f_1(sn)\bar{f}_2(sn')\,d\dot{n}d\dot{n}'. \tag{3.12}$$

Combined with Theorem 3.1, this yields

Theorem 3.2. *We have the second concrete Plancherel formula*

$$\omega_{MN}(e) = \int_{\widehat{A}+} \mathrm{Tr}[\pi_\lambda(\omega)\tilde{U}_{\pi_\lambda}]\,d\lambda, \quad \omega\in\mathcal{D}(G). \tag{3.13}$$

The reader may be the judge. Personally, I prefer the Plancherel formula of Theorem 3.1 (with formula (3.8)) to that of Theorem 3.2 (and formula (3.12)). The parameter space is very natural, even if not a subspace of $\hat{G}$, and the formulation of the nuclear operators is much clearer. This preference is even stronger when the multiplicity is infinite—as in the remaining three examples in the paper. Incidentally, this theme repeats itself in many other places in the literature. The BPF of Fujiwara [] can be simplified substantially if one does not insist on a parameter space in $\hat{G}$ for nilpotent homogeneous spaces. (The author plans to take that up in a future publication. *There may be a good ref to red homogeneous spaces in van-den Ban–Schlichtkrull? something on Baklouti-Ludwig. something on the fact that the intertwining operator is also rendered more complicated.*

3b. Generalized Flag Manifolds. We shall keep this section very short. Once again we shall have finite multiplicity and so the results here are based upon [,§2b] exactly as the results of the last subsection were related to those of [,§2a]. We let P be any parabolic subgroup of G. Then the quasi-regular representation $\tau_{G,P}$ is known to be irreducible. (*Do we know this??*) Let us be more general for a moment. Suppose we have a quasi-regular representation $\tau=\tau_{G,H}$ that is irreducible. Moreover, suppose π is an irreducible representation of G whose unitary equivalence class is the same as that of τ. Then by Theorem 2.1, there must exist a unique positive nuclear operator $U_\pi(H):\mathcal{H}_\pi^\infty\to\mathcal{H}_\pi^{-\infty}$ such that

$$\omega_H(e) = \mathrm{Tr}[\pi(\omega)U_\pi(H)], \quad \omega\in\mathcal{D}(G).$$

In some sense $U_\pi(H)$ measures the interaction between the two realizations of the point in $\hat{G}$ determined by τ and π. Indeed, it yields the intertwining operator between them (see []).

Now return to the case $\tau_{G,P}$. Let P_2 be another parabolic subgroup of G associate to P. Then it is also well known [] that $\tau_{G,P}$ and $\tau_2 = \tau_{G,P_2}$ are equivalent as well as irreducible. We can read off the Bonnet nuclear operator $U_{\tau_2}(P)$ from [] and [], namely it is

$$\langle f_1, U_{\tau_2}(P) f_2\rangle = \int_{N_0\backslash N} f_1(n)\, d\dot{n} \int_{N_0\backslash N} \bar{f}_2(n')\, d\dot{n}',$$

where N is the nilradical of P and N_0 is the intersection of N with the nilradical of P_2. The actual intertwining operator may be found in [, Theorem 2.6].

3c. Rossi-Vergne Spaces. In this subsection, we come to a situation in which multiplicities are (usually) infinite, and so much more germane to the fundamental premise of the paper. We consider $\tau_{G,AN}$ for an arbitrary parabolic $P = MAN$. Unless the parabolic is minimal or quasi-split, the numbers $\dim\sigma$ in the soft Plancherel formulae (3.4), (3.5) will be uniformly infinite. We shall make the tacit assumption that P is neither, although of course the BPF we shall derive is applicable even in those cases (for which the multiplicity is finite).

We start as in §3a by specifying the q functions. In this case we have $H = AN$, a non-unimodular group. In fact we have $q_{H,G} = q_{AN,G} = \Delta_{AN}$ on AN, so we extend it naturally to G by setting $q_{H,G}(g) = e^{2\rho(\log a)}$ if $g = mank$. Of course we still have $B = P$, so $q_{H\cap B,H} = q_{H,H} \equiv 1$ on H. Next, $q_{H\cap B,B} = q_{AN,MAN} = \Delta_{AN}/\Delta_{MAN} \equiv 1$ on AN. We extend it to P by setting it equal to 1 everywhere. Finally, we have $q_{B,G}(man) = q_{P,G}(man) = \Delta_P(man) = e^{2\rho(\log a)}$. We extend it to G by $q_{B,G}(mank) = q_{P,G}(mank) = e^{2\rho(\log a)}$.

The representations in the spectrum are $\pi^\sigma \equiv \pi_{1,\sigma}$. These representations are induced from the representations $\sigma\times 1\times 1$ on MAN. We are tacitly assuming the inducing representations σ are infinite-dimensional, although what follows holds even if they are not. The four conditions in (2.5) are satisfied for the following reasons. It is obvious for (b). Condition (c) holds because of the prior choices of q functions. To take care of (a) and (d), we again make the only conceivable choice for the nuclear operator U_σ, namely the identity operator. Condition (a) is clear, and (d) follows from the basic result of Harish-Chandra that irreducible unitary representations of reductive groups are traceable. It says that for a reductive Lie group M (in the Harish-Chandra class), the operators

$$\int_M \omega(m)\sigma(m)\, dm \quad \omega\in\mathcal{D}(M)$$

are trace class. We next invoke formula (2.7) and Theorem 2.3. The operator U_{π^σ} is defined on the space of $\mathcal{H}_{\pi^\sigma}$ by

$$\langle f_1, U_{\pi^\sigma} f_2\rangle = \langle f_1(e), f_2(e)\rangle. \tag{3.14}$$

We apply part (iv), formula (2.10), of Theorem 2.3, as well as the q function choices above to conclude that

$$\mathrm{Tr}[\pi^\sigma(\omega)U_{\pi^\sigma}] = \mathrm{Tr}\int_M \omega_{AN}(m)\sigma(m)^{-1}\, dm.$$

But then an application of the Plancherel formula on M yields immediately

Theorem 3.2. *For τ_{AN}, we have the concrete Plancherel formula*

$$\omega_{AN}(e) = \int_{\widehat{M}} \mathrm{Tr}[\pi^\sigma(\omega)U_{\pi^\sigma}]\, d\mu_M(\sigma), \quad \omega \in \mathcal{D}(G). \tag{3.15}$$

Formuals (3.14) and (3.15) represent the BPF corresponding to the soft Plancherel formula (3.4). As in §3a, we can derive the BPF corresponding to the alternate Plancherel formula (3.5) in which the parameter space is rendered into a subset of $\hat{G}$, but at the cost of making the description of the nuclear operators considerably more complicated. In fact the intertwining operators are exactly the same as in §3a, and we leave the virtually identical details to the reader.

3d. Whittaker Spaces. The results in this subsection are very similar to those in the last. We omit some details. We consider $\tau_{G,N}$ for an arbitrary parabolic $P = MAN$. As in §3c, unless the parabolic is minimal or quasi-split, the numbers $\dim\sigma$ in the soft Plancherel formulae (3.6), (3.7) will be uniformly infinite. The BPF we derive works in either instance, although we imagine we are in the infinite multiplicity case.

We specify the q functions. In this case $H = N$, is unimodular, so we set $q_{H,G} = q_{N,G} \equiv 1$ on G. As always $q_{H\cap B,H} = q_{H,H} \equiv 1$ on H. Next, $q_{H\cap B,B} = q_{N,MAN} = \Delta_{MAN}^{-1}$ on N. We extend it to P by setting $q_{H\cap B,B}(man) = e^{-2\rho(\log a)}$. Finally, we have as before $q_{B,G}(mank) = q_{P,G}(mank) = e^{2\rho(\log a)}$.

The representations in the spectrum are $\pi_{\lambda,\sigma}$ as defined in (3.1). The four conditions in (2.5) are satisfied for the following reasons. It is obvious for (b). Condition (c) holds because of the prior choices of q functions. (a) holds once we make the usual choice for the nuclear operator $U_{\lambda,\sigma}$, namely the identity operator. Finally, condition (d) follows from Harish-Chandra result, this time applied to the reductive group MA. We next invoke formula (2.7) and Theorem 2.3. The operator $U_{\pi_{\lambda,\sigma}}$ is defined on the space of $\mathcal{H}_{\pi_{\lambda,\sigma}}$ by

$$\langle f_1, U_{\pi_{\lambda,\sigma}} f_2\rangle = \langle f_1(e), f_2(e)\rangle. \tag{3.16}$$

We apply part (iv), formula (2.10), of Theorem 2.3, as well as the q function choices above to conclude that

$$\mathrm{Tr}[\pi_{\lambda,\sigma}(\omega)U_{\pi_{\lambda,\sigma}}] = \mathrm{Tr}\int_{MA} \omega_N(ma)\sigma(m)^{-1}\bar{\lambda}(a)\, dm da.$$

But then an application of the Plancherel formula on MA yields immediately

Theorem 3.3. *For τ_N, we have the concrete Plancherel formula*

$$\omega_N(e) = \int_{\widehat{MA}} \mathrm{Tr}[\pi_{\lambda,\sigma}(\omega)U_{\pi_{\lambda,\sigma}}]\, d\mu_M(\sigma)d\lambda, \quad \omega \in \mathcal{D}(G). \tag{3.17}$$

Formulas (3.16) and (3.17) represent the BPF corresponding to the soft Plancherel formula (3.6). As usual, we omit the details for the concrete Plancherel formula corresponding to (3.7).

4. Remarks

We will make several remarks to conclude the paper.

4.1. One difference we note between cases (3a) and (3b) on the one hand, and (3c) and (3d) on the other, is that the Bonnet nuclear operators are bounded in the former, not in the latter. It would be intersting to characterize when (almost) all the operators U_π that appear in a BPF are bounded. Just to lend more interest to the question, we supply another instance of a category of homogeneous spaces for which the Bonnet operators are bounded.

Proposition 4.1. *Let G be unimodular and type I, and let $K \subset G$ be a compact subgroup. Let $\tau = \tau_{G,K}$ be type I. Then the spectrum of τ is precisely $\hat{G}(K) = \{\pi \in \hat{G} : n_\pi = \dim \mathcal{H}_\pi^K > 0\}$, where $\mathcal{H}_\pi^K = \{\xi \in \mathcal{H}_\pi : \pi(k)\xi = \xi, \forall k \in K\}$. Let U_π be the projection of $\mathcal{H}_\pi$ onto $\mathcal{H}_\pi^K$. Then we have the Plancherel formula*

$$\omega_K(e) = \int_{\widehat{G}(K)}^{\oplus} \mathrm{Tr}[\pi(\omega)U_\pi]\, d\mu(\pi), \quad \omega \in \mathcal{D}(G),$$

where μ is the Plancherel measure of G restricted from $\hat{G}$ to the open subset $\hat{G}(K)$.

Proof. This is certainly not a new result, although an exact reference to the literature is difficult to locate. That $\hat{G}(K)$ is open in $\hat{G}$ follows from ?? (*Check!!*)The derivation of the BPF in this context follows from the Plancherel formula of G. It goes as follows. The Plancherel formula of G gives

$$\omega(e) = \int_{\widehat{G}} \mathrm{Tr}\,\pi(\omega)\, d\mu(\pi).$$

Apply the formula to the test function ω right translated by an element $k \in K$ to get

$$\omega(k) = \int_{\widehat{G}} \mathrm{Tr}[\pi(\omega)\pi(k)]\, d\mu(\pi).$$

Then integrate over K to obtain

$$\begin{aligned} \omega_K(e) &= \int_K \int_{\widehat{G}} \mathrm{Tr}[\pi(\omega)\pi(k)]\, d\mu(\pi)dk \\ &= \int_{\widehat{G}} \int_K \mathrm{Tr}[\pi(\omega)\pi(k)]\, d\mu(\pi)dk \\ &= \int_{\widehat{G}} \mathrm{Tr}[\pi(\omega)U_\pi]\, d\mu(\pi). \end{aligned}$$

This completes the proof.

Note of course that the case $K = \{e\}$ is included; as is the case of a semisimple Lie group G with a maximal compact subgroup K. These two cases seem to rule out any connection betweeen the boundedness of the nuclear operators and the multiplicity function in the direct integral decomposition. The case G compact is also included, so any connection with all representations being "intrinsically" induced representations is also excluded. In fact, I do not know what the proper characterization might be—but I think it is an interesting question.

4.2. In all the cases considered in §3, we observe the following structure. We have a unimodular group G and two subgroups $H \subset B \subset G$, where for almost every irreducible unitary representation σ that appears in the direct integral decomposition of $\tau_{B,H}$, the induced representation $\pi_\sigma = \mathrm{Ind}_B^G \sigma$ is irreducible. In such a situation we have the following soft representation-theoretic decomposition: suppose $\tau_{B,H}$ is type I, and that

$$\tau_{B,H} = \int_{\widehat{B}(H)}^{\oplus} n_\sigma \, \sigma \, d\mu(\sigma);$$

then

$$\tau_{G,H} = \int_{\widehat{B}(H)}^{\oplus} n_\sigma \, \pi_\sigma \, d\mu(\sigma).$$

Since all the irreducibles that appear in the decomposition are induced representations, it might be that we could use Theorem 2.3 to derive the BPF in this very general category, and so avoid the separate computations of §3. There is, as we shall see momentarily, a serious problem. But let us apply Theorem 2.3 and see where we get stuck. We reason heuristically by ignoring all modular functions and all q functions. The spaces of the representations π_σ are described by (2.6) and the corresponding nuclear operators by (2.7). Since $H \cap B = H$, we see the latter devolve to

$$\langle f_1, U_{\pi_\sigma} f_2 \rangle = \langle f_1(e), U_\sigma f_2(e) \rangle. \tag{4.1}$$

That is what happened in §3, although there U_σ was the identity. Applying Thereom 2.3, part (iv), and using the fact that U_σ is H-invariant, we compute

$$\begin{aligned}
&\int_{\widehat{B}(H)} \mathrm{Tr}[\pi(\omega) U_{\pi_\sigma}] \, d\mu(\sigma) \\
&\quad = \int_{\widehat{B}(H)} \mathrm{Tr} \int_{H \backslash B} \omega_H(b) \sigma(b)^{-1} U_\sigma \, d\dot{b} d\mu(\sigma) \\
&\quad = \int_{\widehat{B}(H)} \mathrm{Tr} \int_{H \backslash B} \int_H \omega(b^{-1} h^{-1}) \, dh \sigma(b)^{-1} U_\sigma \, d\dot{b} d\mu(\sigma) \\
&\quad = \int_{\widehat{B}(H)} \mathrm{Tr} \int_B \omega(b) \sigma(b) U_\sigma \, db d\mu(\sigma).
\end{aligned}$$

Now, presuming the operators U_σ arose in the BPF for $\tau_{B/H}$, we simply apply that result to conclude that the last expression is precisley $\omega_H(e)$. That would establish the BPF for $\tau_{G,H}$ with the nuclear operators U_{π_σ} arising from the U_σ by means of formula (4.1). What's the problem? The problem is the group B may not be unimodular. It certainly isn't in any of the examples in §3. Therefore the aplication of Theorem 2.1 is inappropriate, and we must treat each individual situation separately. Unless:

4.3. Can we develop a form of Theorem 2.1 for non-unimodular groups? That is, is there a Bonnet-type Plancherel Theorem for homogeneous spaces G/H in which no unimodularity assumption is placed on G. Well, we know that, with appropriate modifications, the classical Segal-Mautner-Godement Plancherel formula for unimodular groups was extended to non-unimodular groups [] []. I hope to take up a parallel extension for homogeneous spaces in a future paper.

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