ORBITAL SYMMETRIC SPACES AND FINITE MULTIPLICITY

Ronald L. Lipsman

Department of Mathematics
University of Maryland
College Park, Maryland 20742 USA

Abstract. It is proven that *any* algebraic symmetric space has *finite* multiplicity. The multiplicity is then shown to be *bounded* provided that is the case for reductive symmetric spaces (a widely-believed, but as yet unproven fact). The method of proof involves an amalgamation of the Mackey Machine and the Orbit Method. A necessary and sufficient orbital condition is also derived for the quasi-regular representation of a symmetric space to be *irreducible*.

1. Introduction. On several instances previously I have examined the harmonic analysis of non-semisimple symmetric spaces (e.g. in [11] [13, §6] [14] [16,§4] and [19]). One interesting category is that of *abelian* symmetric spaces — namely G/H, where $G = H \ltimes V$ is a semidirect product of a normal real vector subgroup V by any Lie group H acting on V satisfying the condition that $\hat{V}/H$ is countably separated. Then the spectrum of the quasi-regular representation $\tau = \mathrm{Ind}_H^G 1$ is multiplicity-free and parameterized by $\hat{V}/H$. However, it is not at all difficult to find examples in which $\hat{V}/H$ is measure-theoretically a singleton — e.g. when H has a Zariski-open orbit on $\hat{V}$. In that case the representation τ is actually irreducible. At first glance that would seem almost pathological — a symmetric space whose associated quasi-regular representation is irreducible! One unavoidable consequence (we shall discover) is that the algebra of G-invariant differential operators $D(G/H)$ is trivial. Hence, on a first encounter, it is entirely reasonable to view such spaces as unusual or atypical, and therefore not to concentrate attention too closely on them.

But surprisingly, as one broadens the category of non-semisimple symmetric spaces one investigates, this phenomenon keeps recurring. For example, in a natural generalization of abelian symmetric spaces — the so-called Strichartz symmetric spaces — we see it happening with great regularity (see sections 2 and 8 for examples). And although it does not happen for semisimple or nilpotent symmetric spaces, it does occur quite frequently for solvable symmetric spaces. If we only ask that τ be a *finite* sum of irreducibles — a situation which also appears unusual at first glance — then the occurrence of examples is even greater. Finally, if we write $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ for the ± 1-eigenspace decomposition of the Lie algebra (with respect to the differentiated involution), then typically (even though $D(G/H)$ is trivial) the algebra of H-invariant differential operators on $\mathfrak{q}$ will not be trivial, if H is not reductive. For all these reasons, one is obliged to look more closely at these spaces.

1991 *Mathematics Subject Classification.* 22E45.

Supported by NSF # DMS-90-02642 and # DMS-92-25-063

In fact, it is possible to give an orbital characterization of these "minimal" symmetric spaces. I shall do so in this paper. But interestingly, I shall show how the tools developed for arriving at that characterization provide me with a method for settling an important open conjecture concerning general symmetric spaces. What might be an orbital characterization of symmetric spaces with irreducible τ? If one consults the formulas in [12] or [15], it is clear that the phenomenon should demand that the space $\mathfrak{h}^\perp/H$ of H-orbits on $\mathfrak{h}^\perp$ be measure-theoretically a singleton. As in the abelian symmetric space example above, that suggests a Zariski-open orbit, and in fact I shall work in the context of real algebraic groups. Such a context is already strongly suggested by the structure of Strichartz symmetric spaces. But we must also deal with the fact that (outside exponential solvable groups) there is a second parameter — in addition to geometric (coadjoint) orbits — that appears in the orbital description of the irreducibles — namely, the dual of the component group of the stabilizer of functionals in the orbits. Taking that into consideration, I shall prove in this paper

Theorem 1.1. *Let G be real algebraic, σ an involution of G, $H = G^\sigma$. Then the quasi-regular representation $\tau = \mathrm{Ind}_H^G 1$ is*

(i) *irreducible $\Leftrightarrow$ H has a Zariski-open orbit on $\mathfrak{h}^\perp$ and for any ϕ in that orbit, $G_\phi \subset H$;*
(ii) *a finite sum of irreducibles $\Leftrightarrow$ there is a finite collection of open H-orbits in $\mathfrak{h}^\perp$ whose union is conull.*

(Note that either condition in (i) requires that Cent $G \subset H$.) How can such a theorem be proven? Based on [8], it seems clear that a proof must involve a scheme which incorporates an amalgamation of the Mackey Machine and the Orbit Method. In attempting to set up such a scheme I noticed that the "little" homogeneous spaces that arise nearly reflect the nature of the original symmetric space G/H, but of course have smaller dimension. This suggests the possibility of an inductive procedure. In fact I can implement such a procedure (see §5). Moreover, in addition to the proof of Theorem 1.1, it enables me to settle the conjecture in [18] — namely I can prove

Theorem 1.2. *Let G be real algebraic, σ an involution of G, $H = G^\sigma$. Then the quasi-regular representation $\tau = \mathrm{Ind}_H^G 1$ has finite multiplicity.*

This theorem generalizes van den Ban's theorem in the reductive case [2] and Benoist's theorem in the exponential solvable case [4]. In the latter, the multiplicity is free. While this is not so in the reductive case, it is expected (although not proven yet apparently) that the multiplicity is *bounded*. I shall show that the multiplicity of any algebraic symmetric space is bounded if it is true in the reductive case (see Theorem 7.3 below).

One detail in the proof of Theorem 1.2 is worth mentioning at this point. One knows that typically one can expect cocycle obstructions to be associated with the little groups (or homogeneous spaces) that arise in the Mackey Machine. However, their description has been essentially integrated into the Orbit Method in [7]. Nevertheless, this appearance has the potential of disrupting the inductive scheme since the little homogeneous spaces may have a twist that does not appear in the original. One can overcome that problem by installing the twist into the original structure — the little homogeneous space twist will be of exactly the same type. We shall do this, and in fact we shall deduce Theorem 1.2 from the following more general theorem, which we shall prove by the enhanced inductive scheme.

Theorem 1.3. *Let G be real algebraic, σ an involution of G, $H = G^\sigma$. Let $Z \subset \operatorname{Cent} G$ be a unipotent connected σ-invariant subgroup. Suppose χ is a unitary character of HZ satisfying $\sigma\chi = \bar{\chi}$. Then the induced representation*

$$\tau_\chi = \mathrm{Ind}_{HZ}^G \chi$$

has finite multiplicity.

The organization of the paper is as follows. In section 2 we present some of the most elementary examples of symmetric spaces with τ irreducible or a finite sum of irreducibles. Motivated by Theorem 1.1, we call these symmetric spaces (or their quasi-regular representations τ) *orbital* or *quasi-orbital* respectively. Section 3 contains two auxiliary results: (i) the fact that any involution preserves some Levi component; and (ii) the computation of the Penney distribution spaces for an orbital symmetric space. The latter establishes that the distribution-theoretic form of Frobenius reciprocity holds for orbital symmetric spaces. In section 4 we make the orbital terminology precise. Then in section 5 we develop in detail the inductive scheme. It is at this point that we effect the basic integration of the Mackey Machine and Orbit Method that activates the scheme. We also prove here that the little homogeneous spaces are symmetric. In section 6 we prove Theorem 1.1, and in section 7 we prove Theorems 1.2 and 1.3. We present in section 8 another suggestive example and two conjectures. Finally, section 9 is an appendix containing an argument supplied by the referee.

2. Elementary examples and observations. In this section we initiate our study by exhibiting a few of the simplest examples of orbital and quasi-orbital symmetric spaces. We also examine the effect of passing to a component stability group. Finally, we investigate the likelihood of the phenomena occurring for several standard types of groups.

Examples 2.1. (i) *Abelian symmetric spaces.* $G = H \ltimes V$, V a simply connected abelian normal Lie group (i.e. a vector subgroup), H arbitrary except $\hat{V}/H$ is countably separated. Then by [11,Thm.2.1], we have

$$\tau = \int_{\hat{V}/H}^{\oplus} \pi_\chi \, d\dot{\chi},$$

where $d\dot{\chi}$ is the pseudo-image measure class of Lebesgue measure $d\chi$ and $\pi_\chi = \pi_{1,\chi} = \mathrm{Ind}_{H_\chi V}^G 1 \times \chi$. If the measure $d\dot{\chi}$ is concentrated in a finite number of points, then G/H is quasi-orbital; and it is orbital if $d\dot{\chi}$ is concentrated in a point. The latter occurs of course if there is a Zariski-open orbit; and the former if there is a finite number of open orbits with conull union. We illustrate the simplest examples of both situations:

(ia) $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0,\, b \in \mathbb{R} \right\}$, $\quad \sigma =$ conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a \neq 0 \right\}$. τ is irreducible.

(ib) Let G be as in (ia) and take the component group G^0. Then $(G^0)^\sigma = H^0$ and $\tau = \pi_+ \oplus \pi_-$, where $\pi_\pm = \mathrm{Ind}_V^G \chi_{\pm 1}$, $\chi_t \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = e^{itb}$.

(ii) *Strichartz symmetric spaces.* Let $G = R \ltimes V$, where V is a normal vector subgroup and R is reductive. Suppose σ is an involution of G leaving both R and V invariant. Set

$S = R^\sigma$, $U = V^\sigma$, $H = G^\sigma = S \ltimes U$. Then from [19,Thm.3.1], we obtain:

$$\tau = \int_{U^\perp/S}^{\oplus} \int_{\hat{R}_\lambda(S_\lambda)}^{\oplus} n_\lambda(\nu)\pi_{\nu,\lambda}\, d\mu_\lambda(\nu) d\dot{\lambda};$$

where $d\dot{\lambda}$ is the pseudo-image class of Lebesgue measure $d\lambda$ (on $U^\perp$); $(\hat{R}_\lambda(S_\lambda), n_\lambda, \mu_\lambda)$ is the Plancherel data in the "little homogeneous space" decomposition

$$\tau_\lambda = \mathrm{Ind}_{S_\lambda}^{R_\lambda} 1 = \int_{\hat{R}_\lambda(S_\lambda)}^{\oplus} n_\lambda(\nu)\nu\, d\mu_\lambda(\nu);$$

and the irreducibles prescribed by the Mackey Machine are

$$\pi_{\nu,\lambda} = \mathrm{Ind}_{R_\lambda V}^{G} \nu \times \lambda,\ \nu \in \hat{R}_\lambda.$$

($\hat{R}_\lambda(S_\lambda)$ refers to the ν's weakly contained in τ_λ.) Thus G/H will be orbital exactly when $U^\perp/S$ is a singleton (measure-theoretically) and R_λ/S_λ is orbital. We will prove (below in Lemma 5.1 — see also [19,Thm.5.3]) that R_λ/S_λ is symmetric — and so the outline of our inductive scheme begins to emerge. For example,

(iia) $G = \left\{ \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix} : \det g \neq 0 \right\} \cong GL(2,\mathbb{R}) \ltimes \mathbb{R}^2$. Let σ be conjugation by the diagonal matrix with entries $+1, -1, +1$ respectively. Then $H = S \ltimes U$, with $S = \begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $U^\perp/S$ is a point (measure-theoretically) and the little homogeneous space is exactly (ia).

(iib) If we replace $GL(2,\mathbb{R})$ in the last example by $GL^+(2,\mathbb{R})$, i.e. positive determinant, then $U^\perp/S$ is still one point, but R_λ/S_λ is now quasi-orbital.

(iic) The reverse of (iib) can happen. For example, take $R = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $a \neq 0$, $d > 0$. Then $U^\perp/S$ becomes two points while $R_\lambda = S_\lambda$.

Finally, here is an example wherein the unipotent radical is really nilpotent, not abelian.

(iii) Let $\mathfrak{g}$ be the 4-dimensional solvable Lie algebra with generators T, X, Y, Z satisfying bracket relations

$$[X,Y] = Z \quad [T,X] = X \quad [T,Z] = Z.$$

Define σ by $\sigma : T \to T$, $X \to X$, $Y \to -Y$, $Z \to -Z$. Let $\mathfrak{h} = \mathfrak{g}^\sigma$ and let G, H be the corresponding simply connected groups. Using the Orbit Method [15], it is easy to check that G/H is quasi-orbital. We leave it to the reader to render the example orbital by enlarging G to a disconnected group with the same Lie algebra.

We conclude this preliminary section with two results suggested in part by the above (and other) examples.

Proposition 2.2. *Let σ be a non-trivial involution of a Lie group G, $H = G^\sigma$. If G is connected exponential solvable, then τ is not orbital. If G is nilpotent or reductive, and non-compact, then τ cannot be quasi-orbital.*

Proof. First suppose that G is simply connected exponential solvable (in which case H is also). In this case one knows the direct integral decomposition of τ is given by

$$\tau = \int_{\mathfrak{h}^\perp/H}^{\oplus} \pi_\phi \, d\dot{\phi},$$

(see [10]). But it follows easily from [5,Ch.I] that the exponential action of H on the vector space $\mathfrak{h}^\perp$ cannot result in a single conull orbit. Now suppose G is only connected, but not simply connected, and $\tau = \mathrm{Ind}_H^G 1$ is irreducible. Consider the simply connected covering group $\tilde{G}$. Then $G = \tilde{G}/D$, where D is a discrete subgroup of $\tilde{Z} = \mathrm{Cent}\ \tilde{G}$. Any involution σ differentiates to $\mathfrak{g}$ and then lifts to $\tilde{G}$. If we denote by $p : \tilde{G} \to G$ the canonical projection, then we have a commutative diagram

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\sigma} & \tilde{G} \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ G & \xrightarrow{\sigma} & G. \end{array}$$

In particular σ preserves D. Write $\tilde{H} = \tilde{G}^\sigma$ and consider $p^{-1}(H)$, which is easily seen to satisfy

$$p^{-1}(H) = \left\{\tilde{g} : \tilde{g}^{-1}\sigma(\tilde{g}) \in D\right\}.$$

We use the fact that in the exponential solvable case one knows that $\tilde{G} = \tilde{H} \exp \mathfrak{q}$ (see [4]). Therefore if $\tilde{g} = \tilde{h} \exp X \in p^{-1}(H)$, then $\tilde{g}^{-1}\sigma(\tilde{g})$ is in D and also equals $\exp -2X$. Hence $2X \in \mathfrak{z} \Rightarrow X \in \mathfrak{z} \Rightarrow p^{-1}(H) \subset \tilde{H}\tilde{Z}$. Suppose the inequality is strict. We have $(\mathrm{Ind}_H^G 1) \circ p = \mathrm{Ind}_{p^{-1}(H)}^{\tilde{G}} 1$. But the latter is not irreducible, since $\mathrm{Ind}_{p^{-1}(H)}^{\tilde{H}\tilde{Z}} 1$ is not irreducible. This is a contradiction. On the other hand, if $p^{-1}(H) = \tilde{H}\tilde{Z}$, then since H and $p^{-1}(H)$ have the same Lie algebra, this would force $\tilde{Z} \subset \tilde{H}$, i.e. $p^{-1}(H) = \tilde{H}$. Then $(\mathrm{Ind}_H^G 1) \circ p = \mathrm{Ind}_{\tilde{H}}^{\tilde{G}} 1$, and we are reduced back to the simply connected case.

If G is nilpotent, the H-orbits in $\mathfrak{h}^\perp$ are closed, and therefore lower-dimensional. Thus if G is simply connected, τ cannot be quasi-orbital. Basically the same argument as above shows that this fact extends to the non-simply connected case — the only change would be that quasi-orbital forces $p^{-1}(H)$ to be of finite index in $\tilde{H}\tilde{Z}$, therefore open, and therefore everything since the latter is connected.

Finally, if G is reductive the result is clear from well-known results on semisimple symmetric spaces (see [3]).

The second result is the following

Proposition 2.3. *Let G be algebraic and suppose that H is a proper open subgroup of G^σ. Then $\mathrm{Ind}_H^G 1$ cannot be orbital.*

Proof. Because of the algebraic assumption, the component group $G^\sigma/(G^\sigma)^o$ is finite. Then $\mathrm{Ind}_H^{G^\sigma} 1$ cannot be irreducible. It follows that $\mathrm{Ind}_H^G 1 = \mathrm{Ind}_{G^\sigma}^G \mathrm{Ind}_H^{G^\sigma} 1$ cannot be irreducible.

Remarks. It would be interesting to know if the algebraic assumption could be dropped in Prop. 2.3. Without it, the component group could be really nasty, and a quasi-regular representation of a "nasty" discrete group could be irreducible. We also remark that, as Example 2.1(ib) shows, the proposition cannot be strengthened to quasi-orbital.

3. Two auxiliary results. In this section we prove two (unrelated) results. The first will be extremely useful in the sequel. Both may be useful in other contexts as well. The first result says that *any* involution of an algebraic group is made up of a Levi and unipotent part. By an algebraic group G we shall always mean in this paper that G is the set of real points $\mathbf{G}(\mathbb{R})$ of a complex algebraic group $\mathbf{G}$ def/$\mathbb{R}$. If $\mathbf{N}$ is the unipotent radical of $\mathbf{G}$, we call $N = \mathbf{N}(\mathbb{R})$ the unipotent radical of G. By a Levi component R of G we mean $R = \mathbf{R}(\mathbb{R})$ for any Levi component $\mathbf{R}$ of $\mathbf{G}$ that is def/$\mathbb{R}$. Now let G be algebraic, N its unipotent radical. Since N is characteristic any involution of G preserves N. But we also have

Theorem 3.1. *Let σ be an involution of the algebraic group G. Then there is a Levi component of G that is preserved by σ.*

Proof. We first prove the result when N is abelian. Select a Levi component R. Then $\sigma(R)$ is another Levi component. But any two Levi components are conjugate by an element of N. So $\sigma(R) = nRn^{-1}$ for some $n \in N$.

Lemma 3.2. *Suppose $\sigma(R) = nRn^{-1}$ and $\sigma(n) = n$. Then $nRn^{-1} = R$, i.e. R is σ-invariant.*

Proof. We have $R^2 = \sigma^2(R) = \sigma(\sigma(R)) = \sigma(nRn^{-1}) = \sigma(n)\sigma(R)\sigma(n)^{-1} = n^2Rn^{-2}$. That is, $m = n^2$ normalizes R. So for any $r \in R$, we have

$$r^{-1}mrm^{-1} \in R \cap N \Rightarrow r^{-1}mrm^{-1} = 1.$$

That is, $m \in Z_G(R)$. Let $m = \exp X$, $X \in \mathfrak{n}$. It must be that $X \in \mathfrak{Z}_{\mathfrak{g}}(R)$. But the set of vectors in $\mathfrak{n}$ which are fixed by R is a vector space. Hence $\frac{1}{2}X \in \mathfrak{Z}_{\mathfrak{g}}(R) \Rightarrow n \in Z_G(R)$. $\square$

Now continuing with the proof of Theorem 3.1, we have $\sigma(R) = nRn^{-1}$. Write

$$n = pq,\ p, q \in N, \sigma(p) = p, \sigma(q) = q^{-1}.$$

Choose $\ell \in \exp(\mathfrak{q} \cap \mathfrak{n})$ satisfying $\ell^2 = q$. Then we can compute

$$\begin{aligned}\sigma(\ell R\ell^{-1}) &= \sigma(\ell)\sigma(R)\sigma(\ell)^{-1}\\ &= \ell^{-1}nRn^{-1}\ell\\ &= n\ell^{-1}R\ell n^{-1} \qquad (N \text{ abelian})\\ &= p\ell R\ell^{-1}p^{-1}.\end{aligned}$$

Applying Lemma 3.2 to $\ell R\ell^{-1}$ and p, we obtain $\sigma(\ell R\ell^{-1}) = \ell R\ell^{-1}$.

We turn at last to the general case of unipotent N. The proof is by induction on $\dim N$. If $\dim N = 1$, then N is abelian and we are done. Otherwise, let $Z = \operatorname{Cent} N$. Z is both R and σ-invariant. Consider $G_1 = G/Z$. The unipotent radical of G_1 is N/Z which has lower dimension than that of N. Moreover, σ passes to G_1, call the resulting involution σ_1. If $p : G \to G_1$ is the canonical projection, we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\sigma} & G \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ G_1 & \xrightarrow{\sigma_1} & G_1. \end{array}$$

Let R be any Levi component of G. Then $R_1 = p(R)$ is a Levi component of G_1 and $p : R \to R_1$ is an isomorphism. By induction, we can choose R_1' conjugate to R_1, a Levi subgroup of G_1, which is σ_1-invariant. Suppose $R_1' = n_1 R_1 n_1^{-1}, n_1 \in N/Z$. Select any $n \in N$ satisfying $p(n) = n_1$. Then

$$p(nRn^{-1}) = p(n)p(R)p(n)^{-1} = n_1 R_1 n_1^{-1} = R_1'.$$

Therefore

$$p^{-1}(R_1') = nRn^{-1}Z,$$

and since R_1' is σ_1-invariant, it follows easily that the group $G_2 = nRn^{-1}Z$ is σ-invariant. Then we apply the case of abelian unipotent radical to the group G_2. The resulting σ-invariant Levi subgroup of G_2 will be a σ-invariant Levi subgroup of G.

The second result is a strong form of Frobenius reciprocity for orbital symmetric spaces. In order to present it we need further notation which will also be important in the rest of the paper. For $H \subset G$ we choose right Haar measures dh, dg on H and G, and we also choose a smooth function q satisfying

$$q(hg) = \frac{\Delta_H(h)}{\Delta_G(h)} q(g),\ h \in H,\ g \in G,$$

Δ denoting Haar measures. Then a quasi-invariant measure $d\dot{g}$ on $H\backslash G$ is determined by

$$\int_G f(g)q(g)\,dg = \int_{H\backslash G} \int_H f(hg)\,dh d\dot{g}.$$

If χ is a character of H, the induced representation $\tau_\chi = \mathrm{Ind}_H^G \chi$ acts in the space

$$L^2(G; H; \chi) = \{f : G \to \mathbb{C}, f(hg) = \chi(h)f(g), \int_{H\backslash G} |f|^2\,d\dot{g} < \infty\}$$

by the action

$$\tau_\chi(g)f(x) = f(xg)[q(xg)/q(x)]^{1/2}.$$

If q is a character on all of G, then $d\dot{g}$ is relatively invariant with modulus q. More generally, if α is an automorphism of G preserving H, then we shall write $q_{H,G}$ for the modulus of the action of α on $H\backslash G$,

$$q_{H,G}(\alpha) \int_{H\backslash G} f(\alpha(g))\,d\dot{g} = \int_{H\backslash G} f(g)\,d\dot{g}.$$

Now we take our motivation from the basic result of Penney [20]. If $\tau = \mathrm{Ind}_H^G 1$ is type I and has a direct integral decomposition

$$\tau = \int_{\hat{G}}^{\oplus} n(\pi)\pi\,d\mu_\tau(\pi)$$

with respect to a Plancherel measure class μ_τ, then the multiplicity function $n(\pi)$ satisfies

$$n(\pi) \leq \dim\left(\mathcal{H}_\pi^{-\infty}\right)^{H,q^{-1/2}}, \quad \mu_\tau - a.e. \tag{3.1}$$

Here $\mathcal{H}_\pi$ denotes a space on which π acts: $\mathcal{H}_\pi^\infty$ is the dense subspace of C^∞ vectors; $\mathcal{H}_\pi^{-\infty}$ is the anti-dual space of distributions; and $(\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}}$ denotes the subspace of distributions which transform under H according to $q^{-1/2}$, where q is the quotient of the modular functions of H and G as above. The strong (distribution-theoretic) form of Frobenius reciprocity is equality in formula (3.1). Now if τ is irreducible, then the ideal statement of reciprocity would be that the right side of (3.1) is 1 precisely for the class τ and 0 otherwise. Indeed we have

Theorem 3.3. *Let G/H be an orbital symmetric space. Let $\pi \in \hat{G}$. Then*

$$\dim \left(\mathcal{H}_\pi^{-\infty}\right)^{H,q^{-1/2}} = \begin{cases} 1 & \pi \cong \tau \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We first observe that the canonical cyclic distribution

$$\alpha : f \to \bar{f}(1)$$

is a member of $(\mathcal{H}_\tau^{-\infty})^{H,q^{-1/2}}$ [16,Prop.2.2]. We show that any other distribution is a multiple of it. Let $\beta \in (\mathcal{H}_\tau^{-\infty})^{H,q^{-1/2}}$. We associate to β a distribution $\tilde{\beta}$ on G defined by

$$\langle \tilde{\beta}, \phi \rangle = \langle \bar{\beta}, \tilde{\phi} \rangle$$

$$\tilde{\phi}(g) = \int_H \phi(hg) q^{-1/2}(hg)\, dh,\ \phi \in \mathcal{D}(G).$$

Here, $\mathcal{D}(G)$ denotes the space of smooth, compactly supported (i.e. test) functions on G. Of course $\tilde{\phi}$ is a C^∞ vector for τ, so $\tilde{\beta}$ is well-defined. We observe that $\tilde{\beta}$ transforms under the action of H according to

$$\langle \tilde{\beta}, \phi_h \rangle = \Delta_H^{1/2}(h) \Delta_G^{1/2}(h) \langle \tilde{\beta}, \phi \rangle \tag{3.2}$$

$$\langle \tilde{\beta}, \phi^h \rangle = q^{1/2}(h) \langle \tilde{\beta}, \phi \rangle \tag{3.3}$$

where as usual $\phi_h(x) = \phi(h^{-1}x)$, $\phi^h(x) = \phi(xh)$. (We have used the easily verified equations $\tilde{\phi}_h = \Delta_H^{1/2}(h)\Delta_G^{1/2}(h)\tilde{\phi}$, $(\phi^g)^\sim = \tau(g)\tilde{\phi}$.)

Now, given any intertwining operator A for τ, we can define a distribution β_A on G by

$$\langle \beta_A, \phi \rangle = (A\tilde{\phi})(1), \phi \in \mathcal{D}(G).$$

But β_A satisfies the same equivariance conditions (3.2), (3.3). Indeed (3.2) is obvious, and (3.3) follows from the computaion

$$\begin{aligned} \langle \beta_A, \phi^h \rangle &= A(\phi^h)^\sim(1) = A\tau(h)\tilde{\phi}(1) \\ &= \tau(h) A\tilde{\phi}(1) = A\tilde{\phi}(h) q^{1/2}(h) \\ &= A\tilde{\phi}(1) q^{1/2}(h) = q^{1/2}(h) \langle \beta_A, \phi \rangle. \end{aligned}$$

Now the point of [21, Theorem 2] is that the set of distributions on G satisfying (3.2) and (3.3) is in 1-1 correspondence with the set of intertwining operators. Then we invoke the irreducibility of τ to know that the only intertwining operators are scalars. Therefore, every distribution satisfying (3.2) and (3.3) is a scalar multiple of $\beta_I : \phi \to \tilde{\phi}(1) = \int_H q^{1/2}(h)\phi(h)\,dh$. In particular, for $\beta \in (\mathcal{H}_\tau^{-\infty})^{H,q^{-1/2}}$, we must have $\tilde{\beta} = c\beta_I, i.e. \langle\tilde{\beta},\phi\rangle = \langle\beta,\tilde{\phi}\rangle = c\tilde{\phi}(1) = c\langle\bar{\alpha},\phi\rangle$.

Now let π be an irreducible unitary representation of G, inequivalent to τ. Let $\beta \in (\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}}$. We shall show that $\beta = 0$. First of all, since π is irreducible, β must be cyclic. Then we can use the canonical realization $(\tilde{\pi},\tilde{\beta})$ of (π,β) (see [20,p. 183]). In that realization the Hilbert space $\tilde{\mathcal{H}}_\pi$ of π is a space of distributions on $\mathcal{D}(G)$. Moreover, according to [20,Prop.II.3], the C^∞ vectors in that realization will consist of a collection of smooth locally integrable functions, and $\tilde{\beta}$ is evaluation at the identity. But since $\tilde{\beta}$ is H-equivariant, the functions in the Hilbert space $\tilde{\mathcal{H}}_\pi$ must be H-invariant. We also know (from [20,Theorem A]) that the action of π on $\tilde{\mathcal{H}}_\pi^\infty$ is irreducible. All of this together implies that, since $\pi \ncong \tau$, the space $\tilde{\mathcal{H}}_\pi$ is trivial, and therefore $\beta = 0$.

In fact we can also prove the converse.

Theorem 3.4. *Suppose $H \subset G$ (not necessarily symmetric) and $\tau = Ind_H^G 1$ is type I. Suppose there exists an irreducible unitary representation $\pi_1 \in \hat{G}$ such that*

$$\dim\left(\mathcal{H}_\pi^{-\infty}\right)^{H,q^{-1/2}} = \begin{cases} 0 & \pi \ncong \pi_1 \\ 1 & \pi \cong \pi_1. \end{cases}$$

Then τ is irreducible and equivalent to π_1.

Proof. If τ is not irreducible, the type I condition means that $\tau = \int_X^\oplus \pi\,d\mu(\pi)$, a direct integral of irreducibles. By [20] we also have

$$\alpha_\tau = \int_X^\oplus \alpha_\pi\,d\mu(\pi).$$

Moreover, the H-equivariance condition is preserved by the direct integral, so $\alpha_\pi \in (\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}}, \mu - a.a.\,\pi$. But by hypothesis, no distribution space other than the one corresponding to π_1, contains any non-zero elements. Therefore it must be that $\pi \cong \pi_1$ for $\mu - a.a.\,\pi \in X$. That is, τ is a multiple of π_1. But if $\tau = n\pi_1$, $n \geq 1$, then again by (3.1), $n \leq \dim(\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}} = 1$. ☐

4. The orbital interpretation. In order to justify the name, we would like to bring the family of orbital symmetric spaces under the rubric of the Orbit Method. In particular, we would like to characterize — in orbital terms — these spaces. In this section we formulate precise orbital criteria for G/H to be orbital or quasi-orbital. In the next two sections we set up the inductive scheme required for the proof of the criteria.

We take our cue from prior work on orbital formulations of harmonic analysis on homogeneous spaces([12] [15]). These papers suggest the orbital integral formulae

$$\tau = \mathrm{Ind}_H^G 1 = \int_{\mathfrak{h}^\perp/H}^\oplus \pi_\phi\,d\dot{\phi} = \int_{G\cdot\mathfrak{h}^\perp/G}^\oplus n_\phi\pi_\phi\,d\ddot{\phi}, \tag{4.1}$$

where $\mathfrak{h}^\perp = \{\phi \in \mathfrak{g}^* : \phi(\mathfrak{h}) = 0\}$, $n_\phi = \#(G\cdot\phi \cap \mathfrak{h}^\perp)/H$, and $d\dot{\phi}, d\ddot{\phi}$ are pseudo-images of Lebesgue measure $d\phi$ on $\mathfrak{h}^\perp$. Thus for orbital symmetric spaces one would expect that $\mathfrak{h}^\perp/H$ should be one point measure-theoretically; and finitely many points for quasi-orbital spaces. Taking into account that the dual of the component group $G_\phi/(G_\phi)^0$ also contributes parameters (in the Orbit Method [1] [5]), we are led to the following

Theorem 4.1. *Let G be algebraic, σ an involution of G, $H = G^\sigma$. Then G/H is*

(i) *orbital iff there exists a Zariski-open H-orbit on $\mathfrak{h}^\perp$ and for any point ϕ in that orbit, $G_\phi \subset H$;*

(ii) *quasi-orbital iff there exists finitely many open H-orbits in $\mathfrak{h}^\perp$ whose union is conull.*

We prove this result by the inductive technique established in the next section. As explained in the introduction, the technique is just as important as the result, and we shall use it to get another important theorem later in the paper. But first we make several elementary observations. In fact, the existence of an open H-orbit on $\mathfrak{h}^\perp$ being related to irreducibility of τ is quite plausible — even aside from the motivation supplied by the orbital integral formulae (4.1). For we have

Proposition 4.2. *Let ϕ be in an open H-orbit on $\mathfrak{h}^\perp$. Then $\mathfrak{h}$ is maximal totally isotropic with respect to the bilinear form $B_\phi(X,Y) = \phi[X,Y]$, i.e. $\mathfrak{h}$ is a real polarization for ϕ.*

Proof. Since $H\cdot\phi$ is open in $\mathfrak{h}^\perp$ we must have $\mathfrak{h}\cdot\phi = \mathfrak{h}^\perp$. Then $\dim \mathfrak{h}/\mathfrak{h}_\phi = \dim \mathfrak{g}/\mathfrak{h} = \dim \mathfrak{q}$, where $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ is the ± 1 eigenspace decomposition of $\mathfrak{g}$ with respect to σ.

Lemma 4.3. *For any $\phi \in \mathfrak{h}^\perp$, we have $\mathfrak{g}_\phi = \mathfrak{h}_\phi + \mathfrak{q}_\phi$.*

Proof. Let $X \in \mathfrak{g}_\phi$, $X\cdot\phi = 0$. Write $X = Y + Z \in \mathfrak{h} + \mathfrak{q}$. Then $(Y+Z)\cdot\phi = 0$. Apply σ to obtain $(Y-Z)\cdot -\phi = 0$. Combining these equations we get $Y\cdot\phi = Z\cdot\phi = 0$.

We observe that $\mathfrak{h}$ is clearly totally isotropic for B_ϕ (since $\phi \in \mathfrak{h}^\perp$). Thus we can complete the proof of Proposition 4.2 by showing that $\mathfrak{h}_\phi = \mathfrak{g}_\phi$ when $H\cdot\phi$ is open. If that is not the case, then $\mathfrak{h} + \mathfrak{q}_\phi$ is a subordinate subspace for B_ϕ containing $\mathfrak{g}_\phi$. Therefore

$$\dim(\mathfrak{h} + \mathfrak{q}_\phi)/\mathfrak{g}_\phi \le \dim \mathfrak{g}/(\mathfrak{h} + \mathfrak{q}_\phi).$$

Applying Lemma 4.3, we obtain

$$\dim \mathfrak{h}/\mathfrak{h}_\phi \le \dim \mathfrak{q}/\mathfrak{q}_\phi.$$

But we already saw that $\dim \mathfrak{h}/\mathfrak{h}_\phi = \dim \mathfrak{q}$. Hence $\mathfrak{q}_\phi = 0$.

The theory of real polarizations and induced representations suggests that the representation $\pi_\phi = \mathrm{Ind}_H^G \chi_\phi$, $\chi_\phi(\exp X) = e^{i\phi(X)}$, which is precisely τ since $\phi(\mathfrak{h}) = 0$, should be irreducible — at least if H includes the stability group. Taken together, these remarks provide further corroboration for the plausibility of Theorem 4.1. Also, included in the above proof is a result which will be needed in the proof of Theorem 6.1.

Corollary 4.4. *If $\phi \in \mathfrak{h}^\perp$ lies in an open H-orbit, then $\mathfrak{g}_\phi = \mathfrak{h}_\phi$.*

I close this section with two remarks about differential operators. The first is that if G/H is orbital, then $D(G/H)$ — the algebra of G-invariant differential operators on G/H — must be trivial. But, unlike in the Riemannian or nilpotent situations, G/H may not be diffeomorphic to $\exp \mathfrak{q}$ and in fact the algebra of H-invariants $D(\mathfrak{q})^H$ may be non-trivial. When H is not reductive, the existence of an open H-orbit on $\mathfrak{h}^\perp$ does *not* imply the same on $\mathfrak{q}$. The reader can verify these claims by means of Example 2.1(iia). Now the fact that the algebra of unbounded operators $D(G/H)$ is trivial when τ is irreducible, although strongly suggested by the triviality of the commutatant (of bounded operators) of τ, is not obvious. The referee has supplied a proof which is found in section 9.

5. The inductive technique. The immediate goal is a proof of Theorem 4.1. Suppose $G = R \ltimes N$ is algebraic, σ an involution of G. By Theorem 3.1, it is no loss of generality to assume R is σ- invariant. Then $H = G^\sigma = S \ltimes M$, $S = R^\sigma, M = N^\sigma$. Now the strategy is to apply the Mackey Machine to the analysis of τ. We shall see that it is possible to develop an inductive procedure or scheme which relates issues for the symmetric space G/H to corresponding matters on the "little homogeneous spaces." We shall construct the scheme by stirring the Mackey Machine output into the analysis of τ in situations of successively greater generality. At each stage we add some new ingredients into the mixture (mainly from [1] [6] [7] [13] and [19]) that allow the structure to take shape. The stages are marked by the weaker and weaker assumptions of: (a) abelian unipotent radical, (b) no Mackey obstructions, (c) no central twist, and (d) full generality. Stages (a) and (b) can be thought of as "first approximations." Stage (c) contains the guts of the computation, but falls slightly short of the goal since the little homogeneous spaces that appear do not have exactly the same structure as the original. This is remedied in stage (d), where very little new computation is required, but a reformulation of the original problem is critical to rendering the situation completely hereditary.

5a. Abelian unipotent radical. This computation appears in [19,§3] (see also Example 2.1(ii)). To emphasize that N is abelian, we temporarily write V in place of N and $U = V^\sigma$. The computation proceeds as follows.

$$\begin{aligned}
\tau &= \operatorname{Ind}_{SU}^{RV} 1 = \operatorname{Ind}_{SV}^{RV} \operatorname{Ind}_{SU}^{SV} 1 \\
&= \operatorname{Ind}_{SV}^{RV} \int_{U^\perp/S}^{\oplus} \operatorname{Ind}_{S_\lambda V}^{SV} 1 \times \lambda \, d\dot{\lambda} \qquad \text{(by [11])} \\
&= \int_{U^\perp/S}^{\oplus} \operatorname{Ind}_{S_\lambda V}^{RV} 1 \times \lambda \, d\dot{\lambda} \\
&= \int_{U^\perp/S}^{\oplus} \operatorname{Ind}_{R_\lambda V}^{RV} \operatorname{Ind}_{S_\lambda V}^{R_\lambda V} 1 \times \lambda \, d\dot{\lambda} \\
&= \int_{U^\perp/S}^{\oplus} \operatorname{Ind}_{R_\lambda V}^{RV} \int_{\hat{R}_\lambda(S_\lambda)}^{\oplus} n_\lambda(\nu)\nu \times \lambda \, d\dot{\lambda}
\end{aligned}$$

$$= \int_{U^\perp/S}^{\oplus} \int_{\hat{R}_\lambda(S_\lambda)}^{\oplus} n_\lambda(\nu)\pi_{\lambda,\nu} \, d\mu_\lambda(\nu) d\dot{\lambda} \tag{5a.1}$$

$$= \int_{R\cdot U^\perp/R}^{\oplus} \int_{\hat{R}_\lambda(S_\lambda)}^{\oplus} n_\lambda n_\lambda(\nu)\pi_{\lambda,\nu} \, d\mu_\lambda(\nu) d\ddot{\lambda} \tag{5a.2}$$

Here: $d\dot{\lambda}, d\ddot{\lambda}$ are the measure classes of the pseudo-images of Lebesgue measure $d\lambda$; $(\hat{R}_\lambda(S_\lambda), n_\lambda, d\mu_\lambda)$ are the Plancherel constituents of the "little" quasi-regular representation

$$\tau_\lambda = \operatorname{Ind}_{S_\lambda}^{R_\lambda} 1 = \int_{\hat{R}_\lambda(S_\lambda)}^{\oplus} n_\lambda(\nu)\nu \, d\mu_\lambda(\nu); \tag{5a.3}$$

and $n_\lambda = \#(R \cdot \lambda \cap U^\perp)/S$. (See [19,§§3-5] for a discussion of how the symmetry of G/H simplifies the computation.) The spaces R_λ/S_λ are symmetric (see [19,Thm.5.3]), and they have lower dimension than that of G/H. So as we said in §2, the contours of the inductive technique begin to be evident. That is, if we wish to derive some property of τ, we can assume by induction that all the τ_λ's have it (usually manifested in (5a.3)), and then try to control the outer integral in (5a.1) or (5a.2).

5b. No Mackey obstructions. Now we want to extend to non-abelian unipotent radical. To see how the computation of part (a) might generalize, we temporarily make two dramatic, and ultimately untenable, assumptions. The first of them is that: generically on $\hat{N}(M)$, any representation γ extends to an ordinary representation $\tilde{\gamma}_R \times \gamma$ of $R_\gamma N$. In that case we can do the following. We know by the nilpotent theory that

$$\tau_N = \mathrm{Ind}_M^N 1 = \int_{\hat{N}(M)}^{\oplus} \gamma \, d\mu_M(\gamma),$$

a multiplicity-free decomposition ([6] [12]). In fact: $\hat{N}(M) \leftrightarrow \mathfrak{m}^\perp/M$; $\mu_M \leftrightarrow d\dot{\theta}$; the multiplicity-free condition arises because $N \cdot \theta \cap \mathfrak{m}^\perp = M \cdot \theta$ for any $\theta \in \mathfrak{m}^\perp$; and

$$\tau_N = \int_{\mathfrak{m}^\perp/M}^{\oplus} \gamma_\theta \, d\dot{\theta}.$$

Now in attempting to mimic the computation in (a), we have

$$\tau = \mathrm{Ind}_H^G 1 = \mathrm{Ind}_{SM}^{RN} 1 = \mathrm{Ind}_{SN}^{RN} \mathrm{Ind}_{SM}^{SN} 1.$$

The inside representation $\mathrm{Ind}_{SM}^{SN} 1$ is exactly the subject matter of [13,Thm.5.1]. The result is that for $\mu_M - a.a.\,\gamma$, there is a canonical unitary intertwining action $\tilde{\gamma}_S$ of S_γ on the space of γ so that

$$\mathrm{Ind}_{SM}^{SN} 1 = \int_{\hat{N}(M)/S}^{\oplus} \mathrm{Ind}_{S_\gamma N}^{SN} \tilde{\gamma}_S \times \gamma \, d\dot{\mu}_M(\gamma).$$

The second of our excessive assumptions is that $\tilde{\gamma}_R|_{S_\gamma} = \tilde{\gamma}_S$ (so that in particular we can drop the subscript and write unambiguously $\tilde{\gamma}$). But let us be quite clear: $\tilde{\gamma}$ exists naturally on S_γ; it is the extension to R_γ that is entirely too much to expect. Nevertheless, we continue with the computation

$$\begin{aligned}\tau = \mathrm{Ind}_H^G 1 &= \mathrm{Ind}_{SN}^{RN} \int_{\hat{N}(M)/S}^{\oplus} \mathrm{Ind}_{S_\gamma N}^{SN} \tilde{\gamma}_S \times \gamma \, d\dot{\mu}_M(\gamma) \\ &= \int_{\hat{N}(M)/S}^{\oplus} \mathrm{Ind}_{S_\gamma N}^{RN} \tilde{\gamma} \times \gamma \, d\dot{\mu}_M(\gamma) \\ &= \int_{\hat{N}(M)/S}^{\oplus} \mathrm{Ind}_{R_\gamma N}^{RN} \mathrm{Ind}_{S_\gamma N}^{R_\gamma N} \tilde{\gamma} \times \gamma \, d\dot{\mu}_M(\gamma).\end{aligned}$$

Now it is not too difficult to show: suppose that the quasi-regular representations $\tau_\gamma = \mathrm{Ind}_{S_\gamma}^{R_\gamma} 1$ are type I and have direct integral decompositions

$$\tau_\gamma = \mathrm{Ind}_{S_\gamma}^{R_\gamma} 1 = \int_{\hat{R}_\gamma(S_\gamma)}^{\oplus} n_\gamma(\nu) \nu \, d\mu_\gamma(\nu); \tag{5b.1}$$

then

$$\mathrm{Ind}_{S_\gamma N}^{R_\gamma N} \tilde{\gamma} \times \gamma \cong \int_{\hat{R}_\gamma(S_\gamma)}^{\oplus} n_\gamma(\nu) \nu \otimes \tilde{\gamma} \times \gamma \, d\mu_\gamma(\nu). \tag{5b.2}$$

Since these computations are all made presuming unrealistic hypotheses, and since I shall not actually need (5b.2) in the sequel, I omit the proof. But I do include the proof of

Lemma 5.1. *R_γ/S_γ is symmetric.*

Proof. Any $\gamma \in \hat{N}(M)$ can be associated to some $\theta \in \mathfrak{m}^\perp$ (we write $\gamma = \gamma_\theta$). Then $\sigma\theta = -\theta$ and σ preserves G_θ. Indeed if $g \in G_\theta$, then $\theta(g \cdot X) = \theta(X), \forall X \in \mathfrak{n}$. Replacing θ by $-\sigma\theta$ we get $-\theta(\sigma(g \cdot X)) = -\theta(\sigma(X))$. That is, $\theta(\sigma(g) \cdot \sigma(X)) = \theta(\sigma(X))$, which says $\sigma(g) \in G_\theta$. But $G_\gamma = G_\theta N$, so σ preserves G_γ. Then $R_\gamma = G_\gamma \cap R$, and the σ-invariance of R implies that R_γ is also invariant. Clearly $R_\gamma^\sigma = R_\gamma \cap R^\sigma = R_\gamma \cap S = S_\gamma$. We note for future reference that we have also proven that the spaces G_θ/H_θ, $\theta \in \mathfrak{m}^\perp$, are symmetric.

Now we can conclude the computation of the direct integral decomposition:

$$\tau = \mathrm{Ind}_H^G 1 = \int_{\hat{N}(M)/S}^{\oplus} \int_{\hat{R}_\gamma(S_\gamma)}^{\oplus} n_\gamma(\nu)\pi_{\gamma,\nu}\, d\mu_\gamma(\nu) d\dot{\mu}_M(\gamma), \tag{5b.3}$$

where $\pi_{\gamma,\nu}$ are the Mackey parameters for the irreducible $\pi_{\gamma,\nu} = \mathrm{Ind}_{R_\gamma N}^G \nu \otimes \tilde{\gamma} \times \gamma$. Finally, the proof of Lemma 5.1 shows that if $r \cdot \gamma_1 = \gamma_2$, then $H_{\gamma_2} = \sigma(r) H_{\gamma_1} \sigma(r)^{-1}$. (See also [19,Thm.5.3(ii)].) Therefore, the final expression is

$$\tau = \int_{R\cdot\hat{N}(M)/R}^{\oplus} \int_{\hat{R}_\gamma(S_\gamma)}^{\oplus} n_\gamma n_\gamma(\nu)\pi_{\gamma,\nu}\, d\mu_\gamma(\nu) d\ddot{\mu}_M(\gamma), \tag{5b.4}$$

where $n_\gamma = \#(R \cdot \gamma \cap \hat{N}(M))/S$.

Hence, as in part (a), we have the makings of an inductive technique — i.e. a passage from the symmetric space G/H to the lower dimensional little symmetric spaces R_γ/S_γ. In fact a formal proof of Theorem 4.1 could be built around (5b.3) and (5b.4). But unfortunately we must acknowledge the unlikelihood of the assumptions we made on $\tilde{\gamma}$ being satisfied. Basically they only hold when the Mackey obstructions to $\tilde{\gamma}_R$ existing as an ordinary representation of R_γ vanish. Typically the obstructions don't vanish — they have two components: an obstruction to extending the character χ_θ (see below) from N_θ to G_θ; and the possible failure of the existence of a G_θ-invariant polarization in $Sp(\mathfrak{n}/\mathfrak{n}_\theta)$ — which causes a 2-fold obstruction to appear. The means of dealing with these obstructions, as well as their integration into the Orbit Method framework, were accomplished simultaneously by Duflo in [7]. It is our purpose to further integrate that mechanism with the symmetric space structure.

5c. No central twist. Henceforth, we set $K = SN$. Let $\theta \in \mathfrak{n}^*$ and $\gamma = \gamma_\theta \in \hat{N}$ the corresponding representation. γ_θ is constructed as follows. Starting from the character χ_θ of N_θ defined by $\chi_\theta(\exp X) = e^{i\theta(X)}, X \in \mathfrak{n}_\theta$, it is induced (ordinarily or holomorphically) via a (real or complex) polarization for the bilinear form $B_\theta(X,Y) = \theta[X,Y]$ (see [10]). Then $G_\gamma = R_\gamma N = G_\theta N$, and $K_\gamma = S_\gamma N = K_\theta N$. Of course G_θ and K_θ may not split. There is a *canonical* 2-fold covering of these groups defined as follows. The group G_θ preserves the symplectic form B_θ on $\mathfrak{n}/\mathfrak{n}_\theta$, so there is a natural map

$$G_\theta \to Sp(\mathfrak{n}/\mathfrak{n}_\theta).$$

Let $Mp(\mathfrak{n}/\mathfrak{n}_\theta)$ be the metaplectic cover of $Sp(\mathfrak{n}/\mathfrak{n}_\theta)$, with p the covering map. Then there is a commutative diagram

$$\begin{array}{ccc} \tilde{G}_\theta & \longrightarrow & Mp(\mathfrak{n}/\mathfrak{n}_\theta) \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ G_\theta & \longrightarrow & Sp(\mathfrak{n}/\mathfrak{n}_\theta), \end{array}$$

where we can take $\tilde{G}_\theta = \{(g_\theta, m) \in G_\theta \times Mp(\mathfrak{n}/\mathfrak{n}_\theta)$: the components have the same image in $Sp(\mathfrak{n}/\mathfrak{n}_\theta)\}$. Alternatively, we have $\tilde{G}_\theta = \{(g_\theta, s) \in G_\theta \times \mathbb{C} : s^2 = \delta(g_\theta) := \det_{\mathfrak{n}_c/\mathfrak{b}}(g_\theta)/|\det_{\mathfrak{n}_c/\mathfrak{b}}(g_\theta)|\}$, where $\mathfrak{b}$ is *any* polarization for θ, stable under g_θ. We set $\delta^{1/2}(g_\theta, s) = s$. The notation is valid since $\delta^{1/2}(g_\theta, s)^2 = \delta(g_\theta)$. It is evident now that $\tilde{G}_\theta$ is a 2-fold cover of G_θ. We set $\epsilon =$ the non-trivial element in $\ker p$, $\delta^{1/2}(\epsilon) = -1$. $\delta^{1/2}$ may or may not be a character of $\tilde{G}_\theta$. It is if there exists a single polarization invariant under all of G_θ.

The group $\tilde{G}_\theta$ acts on $\mathfrak{n}$ (by factoring through p), and a fundamental result of Duflo [7,Thm.6.1] asserts that there is a *canonical* unitary representation $\tilde{\gamma}_\theta$ of $\tilde{G}_\theta$ on the space of γ so that

$$\tilde{\gamma}_\theta(g_\theta, s)\gamma(n)\tilde{\gamma}_\theta(g_\theta, s)^{-1} = \gamma(g_\theta n g_\theta^{-1}) \tag{5c.1}$$

$$\tilde{\gamma}_\theta(n_\theta, s) = \tilde{\chi}_\theta(n_\theta, s)^{-1}\gamma(n_\theta), \tag{5c.2}$$

where $\tilde{\chi}_\theta(n_\theta, s)$ is the unique character of $\tilde{N}_\theta$ satisfying $\tilde{\chi}_\theta|_{N_\theta} = \chi_\theta$, $\tilde{\chi}_\theta(\epsilon) = -1$. $\tilde{\gamma}_\theta$ is defined as follows. For any $g_\theta \in \tilde{G}_\theta$ we may choose (as above) a positive complex polarization $\mathfrak{c}$ for θ which is invariant by $\tilde{g}_\theta$. We realize γ_θ by holomorphic induction via $\mathfrak{c}$, i.e. $\gamma_\theta = \mathfrak{c} - \mathrm{Ind}_{N_\theta}^{N}\chi_\theta$, which acts in the subspace of functions (in the space of $\mathrm{Ind}_{N_\theta}^{N}\chi_\theta$) which satisfy $X * f = i\theta(X)f, X \in \mathfrak{c}$. Then $\tilde{\gamma}_\theta(\tilde{g}_\theta)$ is defined by

$$\tilde{\gamma}_\theta(\tilde{g}_\theta)f(n) = \delta^{-1/2}(\tilde{g}_\theta)q_{D,N}^{-1/2}(\tilde{g}_\theta)f(g_\theta^{-1}ng_\theta),$$

where $\mathfrak{d} = \mathfrak{c} \cap \mathfrak{n}$. Let us not forget that $\mathfrak{c}$ may vary with g_θ. (But the fundamental result of [1,Thm.II.3.1] or [7,Thm.3.1] is that if we *fix* a single realization of γ and conjugate all the operators $\tilde{\gamma}_\theta(\tilde{g}_\theta)$ over to that realization, then $\tilde{\gamma}_\theta$ is uniquely determined.)

Note that G_θ usually does not split in $\tilde{G}_\theta$, but that N_θ always splits in $\tilde{N}_\theta$ (since $\delta(n_\theta) \equiv 1$). Then the representations of G lying over $\gamma = \gamma_\theta$ are obtained by

$$\pi_{\gamma,\nu} = \mathrm{Ind}_{G_\theta N}^{G}\nu \otimes \tilde{\gamma}_\theta \times \gamma_\theta,$$

where ν is any irreducible representation of $\tilde{G}_\theta$ satisfying $\nu(n_\theta, s) = \tilde{\chi}(n_\theta, s)\mathrm{Id}$. Thus the little group spectrum is parameterized by the spectrum of $\mathrm{Ind}_{\tilde{N}_\theta}^{\tilde{G}_\theta}\tilde{\chi}_\theta$ — which we write as $\tilde{G}_\theta(\tilde{N}_\theta, \tilde{\chi}_\theta)^\wedge$.

Now we bring in the symmetric space structure. Indeed, we do not need the entire set $\tilde{G}_\theta(\tilde{N}_\theta, \tilde{\chi}_\theta)^\wedge$. We only need the part that contributes to the quasi-regular representation τ. To see exactly what that is we need another

Lemma 5.2. *Let $\theta \in \mathfrak{m}^\perp$. Then*

(1) $K_\theta = H_\theta N_\theta$
(2) $\tilde{K}_\theta = \tilde{H}_\theta \tilde{N}_\theta = \tilde{H}_\theta N_\theta$.

Proof. Since $\theta \in \mathfrak{m}^\perp$, it satisfies $\sigma\theta = -\theta$ and $N \cdot \theta \cap \mathfrak{m}^\perp = M \cdot \theta$.

(i) For $k \in K_\theta$, we have $k = sn \in SN$ so that $n \cdot \theta = s^{-1} \cdot \theta \in N \cdot \theta \cap \mathfrak{m}^\perp$. Therefore $n \cdot \theta = m \cdot \theta$ for some $m \in M$. But then $k = sn = smm^{-1}n \in H_\theta N_\theta$.

(ii) The first equation is clear from part (i). The second follows from the fact that N_θ splits in $\tilde{N}_\theta$. It is also evident that $\tilde{H}_\theta \cap \tilde{N}_\theta = \tilde{M}_\theta$ and that $H_\theta \cap N_\theta = M_\theta$.

We already observed in the proof of Lemma 5.1 that σ preserves G_θ. It is also clear that σ extends canonically to $\tilde{G}_\theta$ — namely, $\sigma(g_\theta, s) = (\sigma g_\theta, s)$. Obviously $\tilde{G}_\theta^\sigma = \tilde{H}_\theta$ and the symmetric space $\tilde{G}_\theta/\tilde{H}_\theta$ is the same as G_θ/H_θ.

Next we observe that χ_θ has a canonical extension to K_θ by setting it equal to 1 on H_θ. That is a consistent thing to do since on $H_\theta \cap N_\theta = M_\theta$, the character χ_θ is identically 1. Also, $\tilde{\chi}_\theta$ has a canonical extension to $\tilde{K}_\theta$. That is because, by [13], the representation γ has an extension to S_γ. By the Duflo theory, the only way that can happen is if $\delta^{1/2}$ is a character of $\tilde{K}_\theta$. That does not assure the existence of a single K_θ-invariant polarization — just that $\delta^{1/2}$ is a character. Then we get $\tilde{\chi}_\theta$ on $\tilde{K}_\theta$ by defining

$$\tilde{\chi}_\theta(\tilde{h}_\theta \tilde{n}_\theta) = \delta^{1/2}(\tilde{h}_\theta)\tilde{\chi}_\theta(\tilde{n}_\theta).$$

Lemma 5.3. *$\mathrm{Ind}_{SM}^{SN} 1 = \int_{\hat{N}(M)/S}^{\oplus} \mathrm{Ind}_{S_\gamma N}^{SN} \tilde{\chi}_\theta \tilde{\gamma}_\theta \times \gamma_\theta \, d\dot{\mu}_M(\gamma_\theta)$.*

Proof. Note that $\tilde{\chi}_\theta\tilde{\gamma}_\theta$ is an ordinary representation of $\tilde{K}_\theta$ that passes to K_θ. So since $S_\gamma N = K_\theta N$, and since $\tilde{\chi}_\theta\tilde{\gamma}_\theta|_{N_\theta} = \gamma|_{N_\theta}$, the representation $\tilde{\chi}_\theta\tilde{\gamma}_\theta \times \gamma_\theta$ is a well-defined representation of $K_\theta N$.

This lemma is proven in [13] in the case that SN is completely solvable — which translates here to S being split abelian. For the more general S we have here, the argument is somewhat lengthy. If we examine [12,§7] and [13,§5], we see from the general arguments there that the following is true:

$$\tau = \mathrm{Ind}_{SM}^{SN} 1 = \int_{\hat{N}(M)/S}^{\oplus} \mathrm{Ind}_{S_\gamma N}^{SN} \tilde{\gamma} \times \gamma \, d\dot{\mu}_M(\gamma) \tag{5c.3}$$

where $\tilde{\gamma}$ is the representation (referred to in §5b as $\tilde{\gamma}_S$) of S_γ on the space $\mathcal{H}_\gamma$ of γ determined as follows. We have: (i) a *multiplicity-free* direct integral decomposition

$$\tau_N = \mathrm{Ind}_M^N 1 = \int_{\hat{N}(M)}^{\oplus} \gamma \, d\mu_M(\gamma); \tag{5c.4}$$

an action of S on τ_N determined by

$$\left((\tilde{\tau}_N(s)f\right)(n) = q_{N,M}^{-1/2}(s) f(s^{-1}ns), \; f \in L^2(N;M); \tag{5c.5}$$

and therefore a uniquely determined family of operators

$$\tilde{\tau}_{N,\gamma}(s) : \mathcal{H}_{s\cdot\gamma} \to \mathcal{H}_\gamma$$

satisfying

$$\gamma(n)\tilde{\tau}_{N,\gamma}(s) = \tilde{\tau}_{N,\gamma}(s)(\gamma\cdot s)(s^{-1}ns), \; s \in S, \, n \in N.$$

Therefore $\tilde{\gamma} := \tilde{\tau}_{N,\gamma}|_{S_\gamma}$ is a representation of S_γ; the representation $\tilde{\gamma} \times \gamma$ is the canonical extension of γ to $S_\gamma N$ specified by [13,Thm.5.1]; the representation $\pi_\gamma = \mathrm{Ind}_{S_\gamma N}^{SN} \tilde{\gamma} \times \gamma$ is irreducible; and we have a multiplicity-free decomposition (5c.3). All that we have to do is prove that

$$\tilde{\tau}_{N,\gamma} = \tilde{\chi}_\theta \tilde{\gamma}_\theta. \tag{5c.6}$$

We recall that $S_\gamma N = K_\theta N$ and so it will be enough to prove (5c.6) on K_θ. Furthermore, we know from Lemma 5.2 that $K_\theta = H_\theta N_\theta$, so it will be sufficient to examine $\tilde{\gamma}$ on $\tilde{H}_\theta$. Looking closely at the definition of $\tilde{\chi}_\theta$ on $\tilde{H}_\theta$, as well as the definition of $\tilde{\gamma}_\theta$ in [7], we see that the proof reduces to showing the following: Let

$$\Phi : f \to \{f_\theta\}_{\theta \in \mathfrak{m}^\perp / M}$$

be the intertwining operator for the direct integral decomposition of τ_N (5c.4). The operator Φ is described explicitly in [16] — which description we shall use momentarily. For any $\theta \in \mathfrak{m}^\perp$, let S be a group of automorphisms of N that fix θ. For $s \in S$ define $\tilde{\tau}_N(s)$ according to (5c.5), i.e.

$$\left(\tilde{\tau}_N(s) f\right)(n) = q_{N,M}^{-1/2}(s) f(s^{-1} n s).$$

Then the infinitesimal action of $\tilde{\tau}_N$ at the place θ, call it $\tilde{\tau}_\theta(s)$, is precisely $\Phi_\theta(\tilde{\tau}(s) f)$. We must prove that

$$\tilde{\tau}_\theta(s) = \delta^{1/2}(s) \tilde{\gamma}_\theta(s). \tag{5c.7}$$

We do it by induction on $\dim N$. Low dimensional cases (say less than three) correspond to abelian N and the argument is trivial. The induction argument itself is patterned after [7,pp. 102,3]. But before starting it, we examine the case of an invariant *real* polarization $\mathfrak{b}$. So suppose $\theta \in \mathfrak{m}^\perp$ is left invariant by a group of automorphisms S (of N), and that there also exists a real polarization $\mathfrak{b}$ for θ preserved by S.

In that case the representation γ_θ is realized by $\gamma_\theta = \mathrm{Ind}_B^N \chi_\theta$ and the explicit formula for $\Phi_\theta(f)$ is given in [16,§3] (see also [9]), namely

$$\Phi_\theta(f)(n) = \int_{M \cap B \backslash B} f(bn) \bar{\chi}_\theta(b)\, d\dot{b}. \tag{5c.8.}$$

But then from [7] and (5c.8), we see we must show

$$\Phi_\theta(q_{N,M}^{-1/2}(s) f(s^{-1} \cdot s)) = q_{N,B}^{-1/2}(s)(\Phi_\theta f)(s^{-1} \cdot s). \tag{5c.9}$$

From (5c.8), the left side of (5c.9) is

$$\begin{aligned} &q_{N,M}^{-1/2}(s) \int_{M \cap B \backslash B} f(s^{-1} b n s) \bar{\chi}_\theta(b)\, d\dot{b} \\ &= q_{N,M}^{-1/2}(s) q_{B,M\cap B}^{-1}(s) \int_{M \cap B \backslash B} f(b s^{-1} n s) \bar{\chi}_\theta(b)\, d\dot{b} \quad (S \text{ preserves } \mathfrak{b} \text{ and } \theta). \end{aligned}$$

Thus proving (5c.9) comes down to demonstrating $q_{N,M}^{-1/2}(s) q_{B,M\cap B}^{-1}(s) q_{N,B}^{1/2}(s) \equiv 1$. In [19] it is shown that the latter equals $q_{M,M\cap B}^{-1/2}(s) q_{B,M\cap B}^{-1/2}(s)$. Moreover, in the exponential solvable case one can choose $\mathfrak{b}$ so that this product is 1. I believe the same is true in general, but do not have an airtight proof. In any event, there may not be an S-invariant real polarization for θ — at best only a complex polarization, say $\mathfrak{c}$, which may vary with $s \in S$. Then there is an intertwining operator

$$\gamma_{\mathfrak{b},\mathfrak{c}} : \text{space of } \mathrm{Ind}_B^N \chi_\theta \to \text{space of } \mathfrak{c} - \mathrm{Ind}_{N_\theta}^N \chi_\theta.$$

A fundamental theorem in [1,Thm.III.3.1] asserts that the operator

$$\gamma_{\mathfrak{b},\mathfrak{c}}^{-1} \circ \tilde{\gamma}_\theta(s) \circ \gamma_{\mathfrak{b},\mathfrak{c}}$$

does not depend on $\mathfrak{c}$. (Think of $\mathfrak{b}$ as fixed and $\mathfrak{c}$ as varying.) Therefore if we set

$$\Phi_\theta^{\mathfrak{c}}(f) = \gamma_{\mathfrak{b},\mathfrak{c}}^{-1} \circ \Phi_\theta^{\mathfrak{c}}(s) \circ \gamma_{\mathfrak{b},\mathfrak{c}},$$

and examine [7] again, we see that our argument is reduced to showing

$$\Phi_\theta^{\mathfrak{c}} \left(q_{N,M}^{-1/2}(s) f(s^{-1} \cdot s) \right) = q_{N,D}^{-1/2}(s) (\Phi_\theta^{\mathfrak{c}} f)(s^{-1} \cdot s). \tag{5c.10}$$

Moreover, it is enough to consider $f \in \mathcal{D}(N, M)$, the smooth, compactly supported (mod M) functions on N. In line with the previous remark, one should be able to do this by a close examination of the intertwining operators $\gamma_{\mathfrak{b},\mathfrak{c}}$ (see [7]). The actual argument we employ is induction on $\dim N$ — to which we at last proceed.

So now let $\mathfrak{z} = \mathrm{Cent}\ \mathfrak{n}$ and consider $\mathfrak{a} = \mathfrak{z} \cap \mathfrak{m}$. Suppose $\dim \mathfrak{a} > 0$. Such an ideal will be S- and σ-invariant, and τ_N and γ_θ will be trivial on it. Moreover $A \subset M$ and $A \subset D$. Thus the modulus for the action of S on A disappears from both sides of (5c.10). By induction, we may divide it out. So we may assume $\mathfrak{z} \cap \mathfrak{m} = \{0\}$. Let $\mathfrak{a} = \mathfrak{z} \cap \ker \theta$, again S- and σ-invariant, and suppose $\dim \mathfrak{a} > 0$. Consider the group $N_1 = N/A$, $p : N \to N_1$, the canonical projection. Set $M_1 = p(M)$ and note that $M_1 \cong M$. Of course γ_θ is trivial on A, but τ_N is not. Clearly we have

$$q_{N,M} = q_{N_1,M} q_A \qquad q_{N,D} = q_{N_1,D_1},$$

where $D_1 = p(D)$. But we can reason as follows. We have

$$\mathrm{Ind}_M^N 1 = \mathrm{Ind}_{MA}^N \mathrm{Ind}_M^{MA} 1 = \mathrm{Ind}_{MA}^N \int_{\hat{A}}^{\oplus} \chi \, d\chi.$$

Moreover the intertwining operator for the latter direct integral decomposition is easily written down. Furthermore, the component at the place corresponding to $\chi = 1$, which maps left M-invariant functions on N to left MA-invariant functions on N converts $\tilde{\tau}_N(s)$ into $q_A^{-1/2}(s) \tilde{\tau}_{N_1}(s)$. Thus if equation (5c.10) is proven for N_1, it is also true for N.

So we may assume $\mathfrak{z} \cap \mathfrak{m} = \{0\}$, $\dim \mathfrak{z} = 1$ and $\theta|_{\mathfrak{z}} \neq 0$. In that case, we consider the characteristic ideal $\mathfrak{a} = \mathrm{Cent}\ [\mathfrak{n}, \mathfrak{n}]$. Either $\mathfrak{n}$ is Heisenberg, or $\mathfrak{a}$ is an abelian ideal strictly bigger than $\mathfrak{z}$. Set $\mathfrak{n}_1 = \mathrm{Cent}_{\mathfrak{n}}(\theta|_{\mathfrak{a}})$, a σ- and S-invariant proper ideal in $\mathfrak{n}$. Set $\theta_1 = \theta|_{\mathfrak{n}_1}$. We consider two possibilities:

(a) $\mathfrak{n}_1 + \mathfrak{m} \subsetneq \mathfrak{n}$ (b) $\mathfrak{n}_1 + \mathfrak{m} = \mathfrak{n}$.

(a) We will apply the induction assumption to the smaller algebra $\mathfrak{n}_0 = \mathfrak{n}_1 + \mathfrak{m}$ which is σ- and S- invariant. It contains both $\mathfrak{m}$ and $\mathfrak{d}$. The latter is because (from the Mackey Machine) we know $\gamma_\theta = \mathrm{Ind}_{N_1}^N \gamma_{\theta_1}$. Hence $M \subset N_0$ and $D \subset N_0$. Now $\tau_N = \mathrm{Ind}_{N_0}^N \mathrm{Ind}_M^{N_0} 1$. Write $\tau_0 = \mathrm{Ind}_M^{N_0} 1$. By the induction assumption, the result is true for N_0 — i.e. (5c.10) holds with N replaced by N_0 and $f \in \mathcal{D}(N_0, M)$. But the equivalence $\tau_N = \mathrm{Ind}_{N_0}^N \tau_0$ means in part that we can view any $f \in L^2(N; M)$ as an element $f_0 \in L^2(N; L^2(N_0; M))$ by $f_0(n)(n_0) = f(n_0 n)$. Combining this observation with the common factor $q_{N,N_0}^{-1/2}$ on both sides, we see that (5c.10) also holds for N.

(b) In this case set $M_1 = M \cap N_1 = N_1^\sigma$. So N_1/M_1 is a symmetric space and $\dim N_1 < \dim N$. Then $N/M = N_1 M/M \approx N_1/M_1$, and so $q_{N,M} = q_{N_1,M_1}$. But by the Subgroup Theorem

$$\tau_N|_{N_1} = \mathrm{Ind}_M^N 1|_{N_1} = \mathrm{Ind}_{M_1}^{N_1} 1 = \tau_{N_1}.$$

Thus the left side of (5c.10) is unchanged when we view functions $f \in \mathcal{D}(N, M)$ as elements of $\mathcal{D}(N_1, M_1)$. By induction, that is the same as the right side with N replaced by N_1. But now we use the reasoning on the top of p. 103 in [7]. For any $s \in S$ choose an s-invariant polarization $\mathfrak{c}$ which contains $\mathfrak{a}$. (This can be done since $\mathfrak{a}$ is clearly isotropic.) Then $\mathfrak{c} \subset (\mathfrak{n}_1)_c$ and is a polarization for θ_1 — reflecting the fact that we still have $\gamma_\theta = \mathrm{Ind}_{N_1}^N \gamma_{\theta_1}$. But when we view functions $f \in \mathcal{D}(N, M)$ as M_1-invariant functions on N_1, this introduces a factor of $q_{N,N_1}^{-1/2}(s)$ into the formula for $\Phi_\phi^{\mathfrak{c}} f$. Replacing $q_{N_1,D}^{-1/2}$ by $q_{N,D}^{-1/2} q_{N,N_1}^{1/2}$, we see the two factors involving $q_{N,N_1}(s)$ cancel, and the result is proven for N.

We are reduced finally to the case that $\mathfrak{n}$ is Heisenberg. But then N/M can be symmetric (when $\mathfrak{z} \cap \mathfrak{m} = \{0\}$) only when $\mathfrak{m}/\mathfrak{z}$ is totally isotropic inside $\mathfrak{n}/\mathfrak{z}$. This restricts S to be in a Borel subgroup of $Sp(\mathfrak{n}/\mathfrak{z})$. In that case, it leaves invariant some real polarization for any $\theta \neq 0$ on $\mathfrak{z}$ — namely $\mathfrak{m}$ itself. Therefore, the proof of Lemma 5.3 is done by the fact that, when $M = B$, the real polarization argument given earlier constitutes a proof.

Now we redo the Mackey Machine analysis of τ from §5b — but without the extraneous assumptions and with the incorporation of the Mackey obstructions as outlined above.

$$\begin{aligned}
\tau = \mathrm{Ind}_H^G 1 = \mathrm{Ind}_K^G \mathrm{Ind}_H^K 1 &= \mathrm{Ind}_K^G \int_{\hat{N}(M)/S}^{\oplus} \mathrm{Ind}_{S_\gamma N}^{SN} \tilde{\chi}_\theta \tilde{\gamma}_\theta \times \gamma_\theta \, d\dot{\mu}_M(\gamma_\theta) \\
&= \int_{\hat{N}(M)/S}^{\oplus} \mathrm{Ind}_{S_\gamma N}^{G} \tilde{\chi}_\theta \tilde{\gamma}_\theta \times \gamma_\theta \, d\dot{\mu}_M(\gamma_\theta) \\
&= \int_{\hat{N}(M)/S}^{\oplus} \mathrm{Ind}_{R_\gamma N}^{G} \mathrm{Ind}_{S_\gamma N}^{R_\gamma N} \tilde{\chi}_\theta \tilde{\gamma}_\theta \times \gamma_\theta \, d\dot{\mu}_M(\gamma_\theta).
\end{aligned}$$

So now we have to analyze

$$\mathrm{Ind}_{S_\gamma N}^{R_\gamma N} \tilde{\chi}_\theta \tilde{\gamma}_\theta \times \gamma_\theta = \mathrm{Ind}_{K_\theta N}^{G_\theta N} \tilde{\chi}_\theta \tilde{\gamma}_\theta \times \gamma_\theta$$

in analogy with (5b.1) and (5b.2).

Lemma 5.4. *Suppose* $\mathrm{Ind}_{\tilde{K}_\theta}^{\tilde{G}_\theta} \tilde{\chi}_\theta$ *is a direct integral*

$$\mathrm{Ind}_{\tilde{K}_\theta}^{\tilde{G}_\theta} \tilde{\chi}_\theta = \int_{\tilde{G}_\theta(\tilde{K}_\theta, \tilde{\chi}_\theta)^\wedge}^{\oplus} n_\theta(\nu) \nu \, d\mu_\theta(\nu). \tag{5c.11}$$

Then

$$\mathit{Ind}_{K_\theta N}^{G_\theta N}\tilde{\chi}_\theta\tilde{\gamma}_\theta \times \gamma_\theta = \int_{\tilde{G}_\theta(\tilde{K}_\theta,\tilde{\chi}_\theta)^\wedge}^{\oplus} n_\theta(\nu)\nu\otimes\tilde{\gamma}_\theta\times\gamma_\theta\, d\mu_\theta(\nu); \tag{5c.12}$$

and so

$$\mathit{Ind}_{S_\gamma N}^{G}\tilde{\chi}_\theta\tilde{\gamma}_\theta \times \gamma_\theta = \int_{\tilde{G}_\theta(\tilde{K}_\theta,\tilde{\chi}_\theta)^\wedge}^{\oplus} n_\theta(\nu)\pi_{\theta,\nu}\, d\mu_\theta(\nu), \tag{5c.13}$$

where

$$\pi_{\theta,\nu} = \mathit{Ind}_{G_\theta N}^{G}\nu\otimes\tilde{\gamma}_\theta\times\gamma_\theta.$$

Proof. It is enough to prove (5c.12) from which (5c.13) follows immediately. The induced representation on the left in (5c.12) acts in $L^2(G_\theta N; K_\theta N; \mathcal{H}_\gamma)$ by right translation. (We write $L^2(A;B;\mathcal{H}_\omega)$ for the realization of the representation $\mathrm{Ind}_B^A\omega$ on $\{f : A \to \mathcal{H}_\omega,\ f(ba) = \omega(b)f(a),\ \int_{B\backslash A}||f||^2 < \infty\}$.) The latter Hilbert space is naturally identified to $L^2(\tilde{G}_\theta N; \tilde{K}_\theta N; \mathcal{H}_\gamma)$. We then employ the unitary map given by restricting functions to $\tilde{G}_\theta$, thereby obtaining the Hilbert space $L^2(\tilde{G}_\theta; \tilde{K}_\theta; \mathcal{H}_\gamma)$. $\tilde{G}_\theta$ still acts by right translation; but N now acts by

$$n\cdot F(\tilde{g}_\theta) = \gamma(\tilde{g}_\theta n\tilde{g}_\theta^{-1})F(\tilde{g}_\theta).$$

Next we employ the unitary isomorphism specified by

$$f\otimes\xi \to F_{f,\xi}(g_\theta) = f(g_\theta)\tilde{\gamma}(g_\theta)\xi, \tag{5c.14}$$

which converts the previous Hilbert space into $L^2(\tilde{G}_\theta;\tilde{K}_\theta)\otimes\mathcal{H}_\gamma$. It is straightforward to check that $F_{f,\xi}$ satisfies the equivariance relation

$$F_{f,\xi}(\tilde{k}_\theta\tilde{g}_\theta) = (\tilde{\chi}_\theta\tilde{\gamma}_\theta)(\tilde{k}_\theta)F_{f,\xi}(\tilde{g}_\theta).$$

Moreover, if we compute the transformed group action in the last Hilbert space we find that the N action is specified by

$$f\otimes\xi \to F_{f,\xi} \to n\cdot F_{f,\xi}(\tilde{g}_\theta) = \gamma(\tilde{g}_\theta n\tilde{g}_\theta^{-1})F_{f,\xi}(\tilde{g}_\theta) = \gamma(\tilde{g}_\theta)\gamma(n)f(\tilde{g}_\theta)\xi = F_{f,\gamma(n)\xi};$$

and the $\tilde{G}_\theta$-action is determined by

$$f\otimes\xi \to F_{f,\xi} \to F_{f,\xi}(\tilde{g}'_\theta\tilde{g}_\theta) = f(\tilde{g}'_\theta\tilde{g}_\theta)\tilde{\gamma}(\tilde{g}'_\theta\tilde{g}_\theta)\xi = f^{\tilde{g}_\theta}(\tilde{g}'_\theta)\otimes\tilde{\gamma}(\tilde{g}'_\theta)\tilde{\gamma}(\tilde{g}_\theta)\xi = F_{f^{\tilde{g}_\theta},\gamma(\tilde{g}_\theta)\xi}.$$

But this proves precisely that we have a unitary equivalence

$$\mathrm{Ind}_{\tilde{K}_\theta N}^{\tilde{G}_\theta N}(\tilde{\chi}_\theta\tilde{\gamma}_\theta\times\gamma_\theta) \cong (\mathrm{Ind}_{\tilde{K}_\theta}^{\tilde{G}_\theta}\tilde{\chi}_\theta)\otimes\tilde{\gamma}_\theta\times\gamma_\theta$$

and the lemma is proven.

Now we put it all together to obtain

$$\tau = \mathrm{Ind}_H^G 1 = \int_{\hat{N}(M)/S}^{\oplus}\int_{\tilde{G}_\theta(\tilde{K}_\theta,\tilde{\chi}_\theta)^\wedge}^{\oplus} n_\theta(\nu)\pi_{\theta,\nu}\, d\mu_\theta(\nu)d\dot{\mu}_M(\gamma_\theta). \tag{5c.15}$$

Thus we have reduced the study of the symmetric space quasi-regular representation $\tau = \mathrm{Ind}_H^G 1$ to the study of the induced representations $\mathrm{Ind}_{\tilde{K}_\theta}^{\tilde{G}_\theta} \tilde{\chi}_\theta$. This falls short of our goal for two reasons: the latter are not quasi-regular representations ($\tilde{\chi}_\theta \neq 1$); and also $\tilde{G}_\theta / \tilde{K}_\theta$ is not symmetric. How can there be an inductive scheme if the inherited structure differs from the original — in two ways yet? Indeed, we shall overcome both difficulties with a single change of reference.

5d. Full generality. Set $\mathfrak{q}_\theta = \ker \theta|_{\mathfrak{n}_\theta}$, Q_θ the corresponding analytic subgroup of N_θ. It is clear that $[\mathfrak{g}_\theta, \mathfrak{q}_\theta] \subset \mathfrak{n}_\theta$, and obviously $\theta[\mathfrak{g}_\theta, \mathfrak{q}_\theta] = 0$. That is, $\mathfrak{q}_\theta$ is an ideal in $\mathfrak{g}_\theta$. Furthermore $\mathfrak{n}_\theta / \mathfrak{q}_\theta$ is central in $\mathfrak{g}_\theta / \mathfrak{q}_\theta$. This is because $[\mathfrak{g}_\theta, \mathfrak{n}_\theta] \subset \mathfrak{n}_\theta$ and $\theta[\mathfrak{g}_\theta, \mathfrak{n}_\theta] = 0$, so that $[\mathfrak{g}_\theta, \mathfrak{n}_\theta] \subset \mathfrak{q}_\theta$. In fact these properties lift to the groups. The reasoning follows [5,p. 207]. For $g \in G_\theta$ and $X \in \mathfrak{n}_\theta$, it is clear that $g \cdot X - X \in \mathfrak{q}_\theta$. That is, the adjoint action of G_θ on $\mathfrak{n}_\theta / \mathfrak{q}_\theta$ is trivial. It follows that $[G_\theta, N_\theta] \subset Q_\theta$. It is then further evident that $[\tilde{G}_\theta, \tilde{N}_\theta] \subset \tilde{Q}_\theta$, that is $\tilde{N}_\theta / Q_\theta$ is central in $\tilde{G}_\theta / Q_\theta$. (We note that the preceding idea originated in [1].)

We also know from Lemma 5.2 that $\tilde{K}_\theta / Q_\theta = \tilde{H}_\theta \cdot \tilde{N}_\theta / Q_\theta$. Moreover, the character $\tilde{\chi}_\theta$ satisfies $\sigma \tilde{\chi}_\theta = (\tilde{\chi}_\theta)^-$. Therefore, in place of considering the quasi-regular representation τ of the symmetric space G/H, we should consider the following scenario:

$$Z \subset \mathrm{Cent}\, G \cap N, \text{ connected and } \sigma - \text{invariant},$$

$$\chi \in \hat{HZ}, \sigma\chi = \bar{\chi},$$

$$\tau_\chi = \mathrm{Ind}_{HZ}^G \chi.$$

If we redo the entire computation of section 5c starting with τ_χ in place of τ, the key point is that the inner integral in (5c.15) is still totally governed by (5c.11), i.e. by

$$\tau_\theta = \mathrm{Ind}_{\tilde{K}_\theta}^{\tilde{G}_\theta} \tilde{\chi}_\theta = \mathrm{Ind}_{\tilde{K}_\theta / Q_\theta}^{\tilde{G}_\theta / Q_\theta} \tilde{\chi}_\theta = \mathrm{Ind}_{\tilde{H}_\theta N_\theta / Q_\theta}^{\tilde{G}_\theta / Q_\theta} \tilde{\chi}_\theta.$$

Since $\tilde{G}_\theta / \tilde{H}_\theta = (\tilde{G}_\theta / Q_\theta) / (\tilde{H}_\theta / Q_\theta)$ is symmetric, $\tilde{N}_\theta / Q_\theta$ is central in $\tilde{G}_\theta / Q_\theta$, and $\sigma \tilde{\chi}_\theta = (\tilde{\chi}_\theta)^-$, the structure of the little homogeneous space representation τ_θ is an exact replica of that of τ_χ. The induction scheme is now completely practical and the final result is

Theorem 5.5. *Let G be algebraic, σ an involution, N the unipotent radical of G, R a σ-invariant Levi factor. Set $H = G^\sigma, S = R^\sigma, M = N^\sigma$, and let Z be a connected central unipotent σ-invariant subgroup of G together with a character $\chi \in \hat{HZ}$ satisfying $\sigma\chi = \bar{\chi}$. Let $\psi \in \mathfrak{g}^*$ be any functional satisfying $\sigma\psi = -\psi$ and $\psi|_{\mathfrak{h}+\mathfrak{z}} = \frac{1}{i} d\chi$. Suppose that for $\theta \in \psi + (\mathfrak{m} + \mathfrak{z})^\perp$ we have the direct integral decomposition*

$$Ind_{\tilde{K}_\theta}^{\tilde{G}_\theta} \tilde{\chi}_\theta = Ind_{\tilde{K}_\theta / Q_\theta}^{\tilde{G}_\theta / Q_\theta} \tilde{\chi}_\theta = \int_{\tilde{G}_\theta(\tilde{K}_\theta, \tilde{\chi}_\theta)^\wedge}^{\oplus} n_\theta(\nu) \nu \, d\mu_\theta(\nu). \tag{5d.1}$$

Then we also have

$$\begin{aligned} \tau_\chi &= Ind_{HZ}^G \chi \\ &= \int_{[\psi + (\mathfrak{m}+\mathfrak{z})^\perp]/H}^{\oplus} \int_{\tilde{G}_\theta(\tilde{K}_\theta, \tilde{\chi}_\theta)^\wedge}^{\oplus} n_\theta(\nu) \pi_{\theta,\nu} \, d\mu_\theta(\nu) d\dot{\theta} && \text{(5d.2)} \\ &= \int_{G \cdot [\psi + (\mathfrak{m}+\mathfrak{z})^\perp]/G}^{\oplus} n_\theta \int_{\tilde{G}_\theta(\tilde{K}_\theta, \tilde{\chi}_\theta)^\wedge}^{\oplus} n_\theta(\nu) \pi_{\theta,\nu} \, d\mu_\theta(\nu) d\ddot{\theta}, && \text{(5d.3)} \end{aligned}$$

where $n_\theta = \#\{G \cdot \theta \cap [\psi + (\mathfrak{m} + \mathfrak{z})^\perp)]\}/H$.

Proof. In fact there is not much more to do beyond what we have already done. We only need to review and redo the computations in section 5c, and make some observations:

(1) The set $\psi + (\mathfrak{m} + \mathfrak{z})^\perp$ is independent of the choice of ψ.
(2) $\psi + (\mathfrak{m} + \mathfrak{z})^\perp \subset \mathfrak{m}^\perp$.
(3) For any $\theta \in \psi + (\mathfrak{m} + \mathfrak{z})^\perp$, we have $\sigma\theta = -\theta$.
(4) The representation $\mathrm{Ind}_{MZ}^N \chi$ is still multiplicity-free because if $\theta \in \psi + (\mathfrak{m} + \mathfrak{z})^\perp$, then $N \cdot \theta \cap [\psi + (\mathfrak{m} + \mathfrak{z})^\perp] = M \cdot \theta$. Indeed if $n \cdot \theta \in \psi + (\mathfrak{m} + \mathfrak{z})^\perp$, then since $\psi + (\mathfrak{m} + \mathfrak{z})^\perp \subset \mathfrak{m}^\perp$, we have $n \cdot \theta \in \mathfrak{m}^\perp \Rightarrow n \cdot \theta = m \cdot \theta$ for some $m \in M$ (since N/M is symmetric).
(5) The word-for-word analog of the argument in [19,Thm.5.3] shows that: If θ_1, $\theta_2 \in \psi + (\mathfrak{m}+\mathfrak{z})^\perp$ are G-conjugate, say $g \cdot \theta_1 = \theta_2$, then (replacing g by $\sigma(g)$ if necessary), we have $g \cdot H_{\theta_1} = H_{\theta_2}$. Since N is characteristic, we also have $g \cdot N_{\theta_1} = N_{\theta_2}$. Therefore K_{θ_1} and K_{θ_2} are g-conjugate. Furthermore, the numbers n_θ must be finite.
(6) The character χ must be of order two on H — since on H, $\sigma\chi = \bar{\chi} = \chi \Rightarrow \chi^2 = 1$. In particular $\chi(H^0) = 1$.

For the record, here is the formal derivation of the direct integral decomposition:

$$\begin{aligned}
\tau_\chi &= \mathrm{Ind}_{HZ}^G \chi \\
&= \mathrm{Ind}_K^G \mathrm{Ind}_{HZ}^K \chi \\
&= \mathrm{Ind}_K^G \mathrm{Ind}_{SMZ}^{SN} \chi \\
&= \mathrm{Ind}_K^G \int_{[\psi+(\mathfrak{m}+\mathfrak{z})^\perp]/H}^{\oplus} \mathrm{Ind}_{K_\theta N}^K \tilde{\chi}_\theta \tilde{\gamma}_\theta \times \gamma_\theta \, d\dot{\theta}
\end{aligned}$$

$$\Big(\text{from } \mathrm{Ind}_{MZ}^N \chi = \int_{[\psi+(\mathfrak{m}+\mathfrak{z})^\perp]/H}^{\oplus} \gamma_\theta \, d\dot{\theta} \quad \text{(by [6]) and the evident (and easily proven) analog of Lemma 5.3}\Big)$$

$$\begin{aligned}
&= \int_{[\psi+(\mathfrak{m}+\mathfrak{z})^\perp]/H}^{\oplus} \mathrm{Ind}_{K_\theta N}^G \tilde{\chi}_\theta \tilde{\gamma}_\theta \times \gamma_\theta \, d\dot{\theta} \\
&= \int_{[\psi+(\mathfrak{m}+\mathfrak{z})^\perp]/H}^{\oplus} \mathrm{Ind}_{G_\theta N}^G \mathrm{Ind}_{K_\theta N}^{G_\theta N} \tilde{\chi}_\theta \tilde{\gamma}_\theta \times \gamma_\theta \, d\dot{\theta} \\
&= \int_{[\psi+(\mathfrak{m}+\mathfrak{z})^\perp]/H}^{\oplus} \int_{\tilde{G}_\theta(\tilde{K}_\theta,\tilde{\chi}_\theta)^\wedge}^{\oplus} \mathrm{Ind}_{G_\theta N}^G n_\theta(\nu)\nu \otimes \tilde{\gamma}_\theta \times \gamma_\theta \, d\mu_\theta(\nu) d\dot{\theta} \quad \text{(Lemma 5.4)} \\
&= \int_{[\psi+(\mathfrak{m}+\mathfrak{z})^\perp]/H}^{\oplus} \int_{\tilde{G}_\theta(\tilde{K}_\theta,\tilde{\chi}_\theta)^\wedge}^{\oplus} n_\theta(\nu) \pi_{\theta,\nu} \, d\mu_\theta(\nu) d\dot{\theta} \\
&= \int_{G\cdot[\psi+(\mathfrak{m}+\mathfrak{z})^\perp]/G}^{\oplus} n_\theta \int_{\tilde{G}_\theta(\tilde{K}_\theta,\tilde{\chi}_\theta)^\wedge}^{\oplus} n_\theta(\nu) \pi_{\theta,\nu} \, d\mu_\theta(\nu) d\ddot{\theta}.
\end{aligned}$$

Formulas (5d.1-3) embody the inductive scheme that we have envisioned. We use them critically in the next two sections.

Remark. We note that the entire scheme of this section (and so its consequences in the next two sections) relies critically on G being algebraic. Without that, there is no Levi decomposition and one cannot even get started in setting up an inductive technique.

6. The orbital criterion. We reformulate Theorem 4.1 so as to accommodate the more general representation τ_χ in place of the quasi-regular representation τ. We prove the reformulated form of the theorem using the inductive scheme from §5 (see Theorem 5.5). Then we obtain Theorem 4.1 as a special case. In fact, it is quite clear how the orbital characterizations of Theorem 4.1 should generalize to τ_χ. The result is

Theorem 6.1. *Let G be algebraic, σ an involution, Z a connected central unipotent σ-invariant subgroup with a character $\chi \in \hat{H Z}$ satisfying $\sigma\chi = \bar{\chi}$. Select ψ as in Theorem 5.5. Then the representation $\tau_\chi = \mathrm{Ind}_{HZ}^G \chi$ is*

- (i) *irreducible iff there is a Zariski-open H-orbit on $\psi + (\mathfrak{h} + \mathfrak{z})^\perp$, and for any ϕ in that orbit, $G_\phi \subset HZ$;*
- (ii) *a finite sum of irreducibles iff there is a finite family of open H-orbits in $\psi + (\mathfrak{h} + \mathfrak{z})^\perp$ whose union is conull.*

Remark. It is obvious that Theorem 4.1 is a special case of Theorem 6.1 (taking $Z = \{1\}, \chi = 1$). As in that theorem, either condition in (i) forces Cent $G \subset HZ$.

Proof. First we make an important clarification on notation. In the preceding, all uses of the perp symbol ($\perp$) have been self-evident — namely, $\mathfrak{h}^\perp$ is in $\mathfrak{g}^*$, $\mathfrak{m}^\perp$ and $(\mathfrak{m} + \mathfrak{z})^\perp$ are in $\mathfrak{n}^*$, and $U^\perp$ is in $\hat{V}$. But in this section and the next we shall have to distinguish more closely the set in which the perps are taken. We use the same device as in [12,§2] — namely, if $\mathfrak{a} \subset \mathfrak{b}$ we set $\mathfrak{a}^\perp(\mathfrak{b}) = \{\phi \in \mathfrak{b}^* : \phi(\mathfrak{a}) = 0\}$. Then if $\mathfrak{a} \subset \mathfrak{b} \subset \mathfrak{c}$, the symbols $\mathfrak{a}^\perp(\mathfrak{b})$ and $\mathfrak{a}^\perp(\mathfrak{c})$ are unambiguous.

Now suppose there is a Zariski-open H-orbit $\mathcal{O}$ on $\psi + (\mathfrak{h} + \mathfrak{z})^\perp(\mathfrak{g})$ for which $G_\phi \subset HZ$ if $\phi \in \mathcal{O}$. The projection $p : \mathfrak{g}^* \to \mathfrak{n}^*, \phi \to \phi|_\mathfrak{n}$ supplies us with a Zariski-open H-orbit on $\psi + (\mathfrak{m} + \mathfrak{z})^\perp(\mathfrak{n})$. (We abuse notation slightly by using ψ for $\psi|_\mathfrak{n}$.) Therefore measure-theoretically, the set $[\psi + (\mathfrak{m} + \mathfrak{z})^\perp(\mathfrak{n})]/H$ is a singleton. Hence the outer integral in formula (5d.2) reduces to an atom. Next we show that the same hypotheses apply to the inner integral, or little homogeneous spaces in (5d.1). Let $\phi \in \mathcal{O}, \theta = \phi|_\mathfrak{n}$. Consider $\mathcal{O} \cap p^{-1}(\theta)$, Zariski-open and non-empty in $p^{-1}(\theta)$. Let $\phi_1, \phi_2 \in \mathcal{O} \cap p^{-1}(\theta)$. By hypothesis there is $h \in H$ such that $h \cdot \phi_1 = \phi_2$. Restricting to $\mathfrak{n}$, we see that $h \in H_\theta$. That is, H_θ has a single orbit on $\mathcal{O} \cap p^{-1}(\theta)$. Then project

$$p^{-1}(\theta) \to \theta + \mathfrak{k}_\theta^\perp(\mathfrak{g}_\theta).$$

The image of $\mathcal{O} \cap p^{-1}(\theta)$ will be a Zariski-open subset which is a single H_θ-orbit. Moreover, if ξ is in that image, it is classical (see [1,Prop.II.1.3]) that $(G_\theta)_\xi = G_\phi N_\theta$. Therefore $(G_\theta)_\xi / Q_\theta = G_\phi N_\theta / Q_\theta \subset (HZ)_\phi N_\theta / Q_\theta \subset H_\theta N_\theta / Q_\theta$. Hence, we may apply an induction hypothesis to obtain irreducibility in (5d.1), and so also for the inner integral in (5d.2). From Theorem 5.5 we obtain the irreducibility of τ_χ.

If we moderate the assumption to the condition in statement (ii), then essentially the same argument shows that $[\psi + (\mathfrak{m} + \mathfrak{z})^\perp(\mathfrak{n})]/H$ is finite and $n_\theta < \infty$. The induction hypothesis applies to the inner integral again, and we get a finite sum of irreducibles. Combining these observations, we see that τ_χ is also a finite sum of irreducibles.

This completes the proof of the sufficiency of the orbit condition in statements (i) and (ii). In order to prove the converse, we require another

Lemma 6.2. *Let* $\phi \in \psi + (\mathfrak{h} + \mathfrak{z})^\perp(\mathfrak{g}), \theta = \phi|_\mathfrak{n}$. *Then*

$$M_\theta \cdot \phi = \phi + (\mathfrak{g}_\theta + \mathfrak{n} + \mathfrak{h})^\perp(\mathfrak{g}).$$

Remark. This generalizes the classical ([15,Lemma 2.4] [22,Lemma 2]), and oft-used, fact that

$$N_\theta \cdot \phi = \phi + (\mathfrak{g}_\theta + \mathfrak{n})^\perp(\mathfrak{g}).$$

Proof. First $\mathfrak{m}_\theta \cdot \phi \in (\mathfrak{g}_\theta + \mathfrak{n} + \mathfrak{h})^\perp$ because

$$\phi[\mathfrak{g}_\theta, \mathfrak{m}_\theta] \subset \theta[\mathfrak{g}_\theta, \mathfrak{n}] = 0,$$

$$\phi[\mathfrak{m}_\theta, \mathfrak{n}] \subset \theta[\mathfrak{m}_\theta, \mathfrak{n}] = 0,$$

$$\begin{aligned}\phi[\mathfrak{m}_\theta, \mathfrak{h}] &= \theta[\mathfrak{m}_\theta, \mathfrak{m}] + \phi[\mathfrak{m}_\theta, \mathfrak{s}] \\ &= \phi[\mathfrak{m}_\theta, \mathfrak{s}] \subset \theta(\mathfrak{m}) = 0.\end{aligned}$$

Thus $M_\theta \cdot \phi \subset \phi + (\mathfrak{g}_\theta + \mathfrak{n} + \mathfrak{h})^\perp(\mathfrak{g})$. But since M is unipotent its orbits are closed and connected. So it is enough to show equal dimension, that is to show

$$\dim \mathfrak{m}_\theta/\mathfrak{m}_\phi = \dim \mathfrak{g} - \dim(\mathfrak{g}_\theta + \mathfrak{n} + \mathfrak{h}). \tag{6.1}$$

Proof of (6.1). The right side has dimension

$$\begin{aligned}&\dim \mathfrak{g} - [\dim \mathfrak{g}_\theta + \dim \mathfrak{n} + \dim \mathfrak{h} - \dim \mathfrak{g}_\theta \cap \mathfrak{n} - \dim \mathfrak{g}_\theta \cap \mathfrak{h} - \dim \mathfrak{n} \cap \mathfrak{h} + \dim \mathfrak{g}_\theta \cap \mathfrak{n} \cap \mathfrak{h}] \\ &= \dim \mathfrak{g} - \dim \mathfrak{g}_\theta - \dim \mathfrak{n} - \dim \mathfrak{h} + \dim \mathfrak{n}_\theta + \dim \mathfrak{h}_\theta + \dim \mathfrak{m} - \dim \mathfrak{m}_\theta.\end{aligned}$$

To compute the dimension of the left side of (6.1) we observe that (using $p : \mathfrak{n}^* \to \mathfrak{m}^*$ for the projection)

$$\begin{aligned}\mathfrak{m}_\phi &= \{X \in \mathfrak{m} : \phi[\mathfrak{g}, X] = 0\} = \{X \in \mathfrak{m} : \theta[\mathfrak{g}, X] = 0\} \\ &= \{X \in \mathfrak{m} : \mathfrak{g} \cdot \theta(X) = 0\} = (p(\mathfrak{g} \cdot \theta))^\perp(\mathfrak{m}).\end{aligned}$$

Therefore

$$\begin{aligned}\dim \mathfrak{m}_\phi = \dim \mathfrak{m} - \dim p(\mathfrak{g} \cdot \theta) &= \dim \mathfrak{m} - [\dim \mathfrak{g} \cdot \theta - \dim(\mathfrak{g} \cdot \theta \cap \mathfrak{m}^\perp)(\mathfrak{n})] \\ &= \dim \mathfrak{m} - \dim \mathfrak{g} \cdot \theta + \dim \mathfrak{h} \cdot \theta.\end{aligned}$$

Similarly we have $\mathfrak{m}_\theta = (p(\mathfrak{n} \cdot \theta))^\perp(\mathfrak{m})$, which leads to

$$\dim \mathfrak{m}_\theta = \dim \mathfrak{m} - \dim \mathfrak{n} \cdot \theta + \dim \mathfrak{m} \cdot \theta.$$

Combining, we obtain

$$\dim \mathfrak{m}_\theta/\mathfrak{m}_\phi = \dim \mathfrak{m} \cdot \theta + \dim \mathfrak{g} \cdot \theta - \dim \mathfrak{h} \cdot \theta - \dim \mathfrak{n} \cdot \theta$$

which is the same as the right side of (6.1).

Now, proceeding to the proof of the necessity of the orbital condition in (i), suppose that τ_χ is irreducible. Then from (5d.2), $[\psi + (\mathfrak{m} + \mathfrak{z})^\perp(\mathfrak{n})]/H$ must be a singleton. In particular, H has a Zariski-open orbit on $\psi + (\mathfrak{m} + \mathfrak{z})^\perp(\mathfrak{n})$. Also from (5d.1-2), $\mathrm{Ind}_{\tilde{K}_\theta}^{\tilde{G}_\theta} \tilde{\chi}_\theta$ must be irreducible. Apply an induction hypothesis. The conclusion is that K_θ — and so H_θ — has a Zariski-open orbit on $\theta + \mathfrak{k}_\theta^\perp(\mathfrak{g}_\theta)$, and for ξ in that orbit $(G_\theta)_\xi \subset K_\theta$. But $K_\theta = H_\theta N_\theta$ and $(G_\theta)_\xi = G_\phi N_\theta$. Moreover, Corollary 4.4 generalizes easily to the fact that for any ϕ in an open H-orbit on $\psi + (\mathfrak{m} + \mathfrak{z})^\perp$, we have $G_\phi^0 = H_\phi^0 Z$. Since $Z \subset N_\theta$ and N_θ is simply connected, it is clear that $G_\phi \subset HZ$. Therefore, it remains to show that there is a Zariski-open orbit. But this is a simple consequence of three facts that we have at our disposal, namely: (i) H_θ has a Zariski-open orbit on $\theta + \mathfrak{k}_\theta^\perp(\mathfrak{g}_\theta)$; (ii) $M_\theta \cdot \phi = \phi + (\mathfrak{g}_\theta + \mathfrak{n} + \mathfrak{h})^\perp(\mathfrak{g})$ (Lemma 6.2); and (iii) H has a Zariski-open orbit on $\psi + (\mathfrak{m} + \mathfrak{z})^\perp(\mathfrak{n})$.

We leave the details of the similar — but much easier — argument for necessity in item (ii) to the reader.

7. Algebraic symmetric spaces have finite multiplicity. We now reap an enormous benefit from the inductive scheme set up in §5. Namely, we settle the conjecture from [18] — at least in the algebraic case. We prove that algebraic symmetric spaces always have finite multiplicity. In fact we prove the slightly more general

Theorem 7.1. *Let G be algebraic, σ an involution, $H = G^\sigma$, Z a connected unipotent central σ-invariant subgroup, with $\chi \in \hat{HZ}$ satisfying $\sigma\chi = \bar{\chi}$. Then the representation*

$$\tau_\chi = \mathit{Ind}_{HZ}^G \chi$$

has finite multiplicity.

Of course we have as a corollary (by taking $Z = \{1\}, \chi = 1$)

Theorem 7.2. *Let G be algebraic, σ an involution, $H = G^\sigma$. Then the quasi-regular rep $\tau = \mathit{Ind}_H^G 1$ has finite multiplicity.*

Proof of Theorem 7.1. The proof, which is by induction on dimension, makes critical use of formula (5d.3) from Theorem 5.5. The key point is that the "little representations"

$$\tau_\theta = \mathrm{Ind}_{\tilde{K}_\theta/Q_\theta}^{\tilde{G}_\theta/Q_\theta} \tilde{\chi}_\theta$$

have the same structural properties as τ_χ. Now the result is patently true for groups of dimension 1. To implement the induction argument, we observe that the only way the numbers $\dim \tilde{G}_\theta/Q_\theta$ could fail to be smaller than $\dim G$ is if Q_θ is trivial and $G = G_\theta$. But then $N = N_\theta$ which implies γ_θ is trivial on the commutator subgroup of N — that is, it is a character. But then if Q_θ is trivial, N must be 1-dimensional and G is reductive. The structure then devolves to

$$\tau_\chi = \mathrm{Ind}_H^G \chi,\ \sigma\chi = \bar{\chi}.$$

But we have seen that this forces χ to be trivial on H^0, and the finite multiplicity of τ_χ follows from that of

$$\mathrm{Ind}_{H^0}^G 1,$$

which fact is proven in [2].

Therefore by (5d.3) and the induction assumption, to prove finite multiplicity it is enough to prove that the numbers n_θ are finite. The proof of that fact is almost the same argument as in [18], which is the heuristic argument motivating the original finite multiplicity conjecture. Specifically, we must show that generically on $\psi + (\mathfrak{m}+\mathfrak{z})^\perp(\mathfrak{n})$ the number of H-orbits is finite. But we have

$$\begin{aligned} \dim H\cdot\theta &\le \dim G\cdot\theta\cap[\psi+(\mathfrak{m}+\mathfrak{z})^\perp(\mathfrak{n})] \\ &\le \dim \mathfrak{g}\cdot\theta\cap(\mathfrak{m}+\mathfrak{z})^\perp(\mathfrak{n}) \\ &= \dim\mathfrak{h}\cdot\theta \qquad \text{(see below)} \\ &= \dim H\cdot\theta. \end{aligned} \tag{7.1}$$

That is H has open orbits on $\psi + (\mathfrak{m}+\mathfrak{z})^\perp(\mathfrak{n})$, and by the algebraic assumption there can only be finitely many.

Proof of (7.1). It suffices to show: $\theta\in\mathfrak{m}^\perp$ and $X\cdot\theta\in\mathfrak{m}^\perp$, $X\in\mathfrak{g}\Rightarrow X\cdot\theta\in\mathfrak{h}\cdot\theta$. Write $X\in\mathfrak{g}$ uniquely as $X=Y+W, Y\in\mathfrak{h}, W\in\mathfrak{q}$. Then, since $\sigma(\theta)=-\theta$ and $\sigma(X\cdot\theta)=-X\cdot\theta$, we have $(Y-W)\cdot-\theta=\sigma(X)\cdot\sigma\theta=\sigma(X\cdot\theta)=-X\cdot\theta=-(Y+W)\cdot\theta$. Hence $W\cdot\theta=0$. □

Now I consider the possible boundedness of the finite multiplicity in τ_χ or in the quasi-regular representation $\tau=\mathrm{Ind}_H^G 1$. The heuristic argument in [18] was also evidence for the boundedness (in addition to the finiteness) of the multiplicity. The argument made use of the fact that the Orbit Method suggested that the multiplicities were intimately related to the numbers

$$n_\phi = \#(G\cdot\phi\cap\mathfrak{h}^\perp)/H,\ \phi\in\mathfrak{h}^\perp.$$

That these numbers are finite for symmetric spaces G/H was in fact proven in [19,Thm.5.3]. Of course $n_\phi\le\#$ components of $(G\cdot\phi\cap\mathfrak{h}^\perp)$, and then a simple application of [17,in the special case $k=0$] shows that

$$n_{G,H} := \sup_{\phi\in\mathfrak{h}^\perp} n_\phi \le \sup_{\phi\in\mathfrak{h}^\perp}\#\text{ components of }(G\cdot\phi\cap\mathfrak{h}^\perp)<\infty.$$

We can now state

Theorem 7.3. *Suppose that for any reductive symmetric space G/H the multiplicity function in $\tau_\chi=\mathrm{Ind}_H^G\chi$, $\sigma\chi=\bar\chi$ is bounded above by $n_{G,H}$. Then for any algebraic symmetric space the multiplicity function in τ_χ (Theorem 7.2) or τ (Theorem 7.1) is uniformly bounded.*

Proof. We incorporate the twist into the argument. In fact the induction assumption is precisely that

$$\text{multiplicity in }\tau_\chi \le \sup_{\phi\in\psi+(\mathfrak{h}+\mathfrak{z})^\perp(\mathfrak{g})}\#\{G\cdot\phi\cap[\psi+(\mathfrak{h}+\mathfrak{z})^\perp(\mathfrak{g})]\}/H.$$

In analogy with the proof of Theorem 7.2, the reductive case comes down to precisely that the multiplicity is dominated by $n_{G,H}$ — which unfortunately we can only postulate at this point. As usual we use (5d.3). Set

$$\begin{aligned}(\mathfrak{g}^*/G,\mathfrak{h},\psi) &= \text{ set of } G\text{-orbits in } \mathfrak{g}^* \text{ that meet } \psi+(\mathfrak{h}+\mathfrak{z})^\perp(\mathfrak{g})\\ (\mathfrak{n}^*/G,\mathfrak{m},\psi) &= \text{ set of } G\text{-orbits in } \mathfrak{n}^* \text{ that meet } \psi+(\mathfrak{m}+\mathfrak{z})^\perp(\mathfrak{n})\\ (\mathfrak{g}_\theta^*/\tilde{G}_\theta,\mathfrak{h}_\theta,\theta) &= \text{ set of } \tilde{G}_\theta\text{-orbits in } \mathfrak{g}_\theta^* \text{ that meet } \theta+\mathfrak{k}_\theta^\perp(\mathfrak{g}_\theta).\end{aligned}$$

Then by (5d.3) and the induction hypothesis, the theorem will be proven if we can show that — for $\phi\in\psi+(\mathfrak{h}+\mathfrak{z})^\perp$, $\theta=\phi|_\mathfrak{n}$, $\xi=\phi|_{\mathfrak{g}_\theta}$, corresponding to which

$$G\cdot\phi\in(\mathfrak{g}^*/G,\mathfrak{h},\psi),\ G\cdot\theta\in(\mathfrak{n}^*/G,\mathfrak{m},\psi),\ \tilde{G}_\theta\cdot\xi\in(\mathfrak{g}_\theta^*/\tilde{G}_\theta,\mathfrak{h}_\theta,\theta)$$

— we have

$$\#\{G\cdot\phi\cap[\psi+(\mathfrak{h}+\mathfrak{z})^\perp(\mathfrak{g})]\}/H=n_\theta\cdot\#\{\tilde{G}_\theta\cdot\xi\cap[\theta+\mathfrak{k}_\theta^\perp(\mathfrak{g}_\theta)]\}/\tilde{H}_\theta. \tag{7.2}$$

Consider first the surjective map

$$p:\Omega\in(\mathfrak{g}^*/G,\mathfrak{h},\psi)\to\omega\in(\mathfrak{n}^*/G,\mathfrak{m},\psi)$$

$$p(\phi)=\theta=\phi|_\mathfrak{n}.$$

For fixed $\theta\in\mathfrak{m}^\perp$, we show there is a bijection between the sets

$$(\mathfrak{g}_\theta^*/G_\theta,\mathfrak{h}_\theta,\theta)\text{ and }\{\Omega\in(\mathfrak{g}^*/G,\mathfrak{h},\psi):\theta\in p(\Omega)\}.$$

Let $\xi\in\mathfrak{g}_\theta^*,\xi(\mathfrak{h}_\theta)=0,\xi|_{\mathfrak{n}_\theta}=\theta|_{\mathfrak{n}_\theta}$. Define $\Omega_\xi=G\cdot\phi$ where ϕ is any extension of $\xi+\theta$ that is 0 on $\mathfrak{h}$. This is well-defined as follows: if $\xi_1=g_1\cdot\xi,g_1\in G_\theta,\xi_1(\mathfrak{h}_\theta)=0$, then let ϕ_1 be any extension of $\xi_1+\theta$ that is 0 on $\mathfrak{h}$. Then $g_1\cdot\phi|_\mathfrak{n}=\theta=\phi_1|_\mathfrak{n}$, $\quad g_1\cdot\phi|_{\mathfrak{g}_\theta}=\xi_1=\phi_1|_{\mathfrak{g}_\theta}\Rightarrow g_1\cdot\phi=n\cdot\phi_1$ some $n\in N_\theta\Rightarrow G\cdot\phi=G\cdot\phi_1$. Claim: the map $T:G_\theta\cdot\xi\to\Omega_\xi$ is a bijection.

Onto: Let $\phi\in\mathfrak{g}^*,\phi(\mathfrak{h})=0,\phi|_\mathfrak{n}=\theta$. Let $\xi=\phi|_{\mathfrak{g}_\theta}$. Then $\xi|_{\mathfrak{n}_\theta}=\theta|_{\mathfrak{n}_\theta}$ and $\xi(\mathfrak{h}_\theta)=\phi(\mathfrak{h}_\theta)\subset\phi(\mathfrak{h})=0$.

1-1: Suppose $\xi_1,\xi_2\in\mathfrak{g}_\theta^*,\xi_j|_{\mathfrak{n}_\theta}=\theta|_{\mathfrak{n}_\theta},\xi_j(\mathfrak{h}_\theta)=0$. Let ϕ_1,ϕ_2 be extensions and suppose $g\cdot\phi_1=\phi_2$. Then $g\cdot\theta=\theta\Rightarrow g\in G_\theta$. But then $g\cdot\xi_1=\xi_2$ for $g\in G_\theta$.

Now fix a G-orbit $\Omega=G\cdot\phi$ that meets $\psi+(\mathfrak{h}+\mathfrak{z})^\perp(\mathfrak{g})$. It maps to a G-orbit $\omega=G\cdot\theta$ that meets $\psi+(\mathfrak{m}+\mathfrak{z})^\perp(\mathfrak{n})$. Moreover it is clear that H-orbits in the former map to H-orbits in the latter. So we get a surjective map

$$\{G\cdot\phi\cap[\psi+(\mathfrak{h}+\mathfrak{z})^\perp(\mathfrak{g})]\}/H\to\{G\cdot\theta\cap[\psi+(\mathfrak{m}+\mathfrak{z})^\perp(\mathfrak{n})]\}/H.$$

In general the map fails to be injective. Now there is a unique orbit $\Lambda=G_\theta\cdot\xi$ in $(\mathfrak{g}_\theta^*/G_\theta,\mathfrak{h}_\theta,\theta)$ that maps to Ω under the map T. Fix an H-orbit in $\omega\cap[\psi+(\mathfrak{m}+\mathfrak{z})^\perp(\mathfrak{n})]$. Let $\mathcal{C}_1,\ldots,\mathcal{C}_r$ be the H-orbits in $\Omega\cap[\psi+(\mathfrak{h}+\mathfrak{z})^\perp(\mathfrak{g})]$ that map to it. Let $\mathcal{C}_1',\ldots,\mathcal{C}_s'$ be the H_θ-orbits in $\Lambda\cap(\theta+k_\theta^\perp(\mathfrak{g}_\theta)$. We need to show: $r=s$ and the H-orbits correspond under the map T.

So let

$$\phi=\phi_1,\phi_2,\ldots,\phi_r\text{ be in distinct }H\text{-orbits in }G\cdot\phi\cap[\psi+(\mathfrak{h}+\mathfrak{z})^\perp(\mathfrak{g})]$$

while

$$\theta = \theta_1, \theta_2, \ldots, \theta_r \text{ are in the same } H\text{-orbit in } G \cdot \theta \cap [\psi + (\mathfrak{m} + \mathfrak{z})^\perp(\mathfrak{n})].$$

We show there are exactly r distinct H_θ-orbits in $G_\theta \cdot \xi \cap [\theta + \mathfrak{k}_\theta^\perp]$. First, there exists at least r orbits. To see that, choose $g_j \in G$ so that $\phi_j = g_j \cdot \phi$. Then $\exists h_j \ni h_j \cdot \theta = g_j \cdot \theta$ (because all θ_j are in the same H-orbit). Then $h_j^{-1} g_j \in G_\theta$. Let $\xi_j = h_j^{-1} g_j \cdot \xi$. Claim: these are all in distinct H_θ-orbits in $G_\theta \cdot \xi \cap [\theta + \mathfrak{k}_\theta^\perp]$. If not, we have $h_\theta h_1^{-1} g_1 \cdot \xi = h_2^{-1} g_2 \cdot \xi$. Then $h_\theta h_1^{-1} g_1 \cdot \phi$ and $h_2^{-1} g_2 \cdot \phi$ have the same restriction to $\mathfrak{n}$ and $\mathfrak{g}_\theta$, and both vanish on $\mathfrak{h}$. Then we use $M_\theta \cdot \phi = \phi + (\mathfrak{g}_\theta + \mathfrak{n} + \mathfrak{h})^\perp(\mathfrak{g})$ to get $h_\theta h_1^{-1} g_1 \cdot \phi = m_\theta h_2^{-1} g_2 \cdot \phi \Rightarrow \phi_1, \phi_2$ are in the same H-orbit, a contradiction.

Now why can't there be more than r orbits? Suppose $\xi_1, \ldots, \xi_s$, $s \geq r$, lie in distinct H_θ-orbits in $G_\theta \cdot \xi \cap [\theta + \mathfrak{k}_\theta^\perp]$. Take ϕ_j' any extension of $\xi_j + \theta$ which is 0 on $\mathfrak{h}$. These lie in distinct H-orbits because if $h \cdot \phi_1' = \phi_2' \Rightarrow h \in H_\theta$, contradicting the fact that the ξ_j lie in distinct H_θ-orbits. But they all have the same restriction to $\mathfrak{n}$, contradicting that there are exactly r distinct H-orbits in $G \cdot \phi \cap [\psi + (\mathfrak{h} + \mathfrak{z})^\perp(\mathfrak{g})]$ lying over the single H-orbit in $G \cdot \theta \cap [\psi + (\mathfrak{m} + \mathfrak{z})^\perp(\mathfrak{n})]$.

Finally it is clear from the preceding paragraph that the map $T : \Lambda \cap (\theta + \mathfrak{k}_\theta^\perp) \to \Omega \cap [\psi + (\mathfrak{h} + \mathfrak{z})^\perp]$ is a bijection of the H_θ-orbits in the former to the H-orbits in the latter. $\square$

8. Another example and two more questions. The point of this concluding section is to demonstrate that (quasi-)orbital symmetric spaces are actually quite common; and to raise two questions on orbital symmetric spaces that I have not been able to settle.

Consider the semidirect product $G_n = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$ with the natural action. Identify G_n to a subgroup of $GL(n+1, \mathbb{R})$ in the usual way

$$G_n = \left\{ \begin{pmatrix} r & v \\ 0 & 1 \end{pmatrix} : r \in GL(n, \mathbb{R}),\ v \in \mathbb{R}^{n \times 1}, \right\}.$$

Let $\epsilon = (\epsilon_1, \ldots, \epsilon_n; 1)$ denote a diagonal matrix in $GL(n+1, \mathbb{R})$ with $\epsilon_j = \pm 1, 1 \leq j \leq n$. Define an involution σ_ϵ on G_n by $\sigma_\epsilon(g) = \epsilon g \epsilon$. Note that any standard parabolic subgroup of $GL(n+1, \mathbb{R})$ is σ_ϵ-invariant. In particular the maximal parabolic G_n is σ_ϵ-invariant.

Proposition 8.1. *For any ϵ, the symmetric space $G_n / G_n^{\sigma_\epsilon}$ is orbital.*

Proof. First note that it is no loss of generality to assume ϵ is of the form

$$\epsilon = (+1, \ldots, +1, -1, \ldots, -1; 1), \tag{8.1}$$

that is r plus ones followed by s minus ones, $r + s = n$. Indeed, given any ϵ we pick a representative w of the element in the Weyl group (in $GL(n+1, \mathbb{R})$) that conjugates the original ϵ into the one of the form (8.1). But since the last diagonal entry is $+1$, w is actually in $GL(n, \mathbb{R}) \subset G_n$.

Now it is easy to see that with ϵ given by (8.1), we have

$$H = G_n^{\sigma_\epsilon} = \left\{ \begin{pmatrix} A & 0 & x \\ 0 & D & 0 \\ 0 & 0 & 1 \end{pmatrix} : A \in GL(r, \mathbb{R}), D \in GL(s, \mathbb{R}), x \in \mathbb{R}^{r \times 1} \right\} \cong G_r \times GL(s, \mathbb{R}).$$

If $s = 0$, then $G_n^{\sigma_\epsilon} = G_n$ and there is nothing to do. If $r = 0$, we are in the case of an abelian symmetric space with a Zariski-open orbit and the result follows from [11]. Otherwise, the set $\hat{U}(V)$ is naturally identified to

$$\left\{ \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & y & 1 \end{pmatrix}, y \in \mathbb{R}^{1 \times s} \right\}$$

on which $S = \begin{pmatrix} A & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has a Zariski-open orbit. Furthermore, for $y = (0, \ldots, 0, 1)$, the stability group in R is $\cong G_{n-1}$ and the little homogeneous space is isomorphic to

$$G_{n-1}/G_{n-1}^{\sigma_{\epsilon'}}, \; \epsilon' = (-1_1, \ldots, -1_r, +1_1, \ldots, +1_{s-1}; 1).$$

But the latter is conjugate-isomorphic to the symmetric space $G_{n-1}/G_{n-1}^{\sigma_{\epsilon''}}$, where $\epsilon'' = (+1_1, \ldots, +1_{s-1}, -1_1, \ldots, -1_r; 1)$. Applying an induction assumption, we see that eventually we are reduced to an abelian symmetric space with a Zariski-open orbit. □

One could play the same game over $\mathbb{C}$, or with $SL(n, \cdot)$ instead of $GL(n, \cdot)$, and with other split groups. I have a great deal of computational evidence for the general result suggested by Proposition 8.1 — although at the moment, no complete proof. So I formulate

Question 8.2. *Let G be a split connected reductive Lie group, σ an involution of G which is not a Cartan involution. Let P be any parabolic subgroup of G which is σ-invariant. Then is it true that P/P^σ is quasi-orbital?*

Here is the second unresolved issue. Suppose $\tau = \mathrm{Ind}_H^G 1$ is orbital. This says that τ is monomial in a very strong sense. The question is whether the monomial nature of τ is accidental or intrinsic — that is, if an irreducible arises from an orbital symmetric space, then must its Mackey Machine realization also be monomial? The latter can be examined through the lens of Theorem 5.5

$$\tau = \int_{\hat{N}(M)/S}^{\oplus} \mathrm{Ind}_{\tilde{K}_\theta N}^G \tilde{\chi}_\theta \tilde{\gamma}_\theta \times \gamma_\theta.$$

The irreducibility of τ forces S to have a Zariski-open orbit on $\hat{N}(M)$ and $\mathrm{Ind}_{\tilde{K}_\theta}^{\tilde{G}_\theta} \tilde{\chi}_\theta$ to be irreducible. Applying the inductive scheme, we would deduce that the latter is "Mackey-monomial"; and then to conclude the same for τ it would be enough to have a positive resolution of

Question 8.3. *Suppose that H has a Zariski-open orbit on $\mathfrak{m}^\perp$. Does it follow that there exists a G_θ-invariant* **real** *polarization for θ?*

9. Appendix. Here is a proof, supplied by the referee, of the fact that if $\tau = \mathrm{Ind}_H^G 1$ is irreducible, then $D(G/H)$ is trivial. First we observe that if $L \in D(G/H)$, then its formal adjoint L^* (on $C_c^\infty(G/H)$) is also in $D(G/H)$. Thus the problem reduces to showing that a formally self-adjoint operator in $D(G/H)$ must be scalar. This conclusion is a consequence of the following two facts:

(i) $L|_{C_c^\infty(G/H)}$ is essentially self-adjoint

(ii) $\tau(G)$ commutes with the spectral projections of the self-adjoint extension $\bar{L}$ of $L|_{C_c^\infty(G/H)}$.

Point (ii) is fairly standard — *cf.* Kato, *Perturbation Theory for Linear Operators*, Chapter 6, Section 5 (1966 ed). But point (i) is delicate. It follows from a result due to E. Thomas which is found in the article *Commutativity of invariant differential operators on nilpotent homogeneous spaces* $K\backslash G$ *with finite multiplicity*, **Comm. Pure & Applied Math.**, vol. 45 (1992), 681-748 (see Theorem 7.2 in the preprint version of the article for a fuller discussion). That result asserts that: if τ is a type I representation of a Lie group G whose multiplicities are finite, and if L is a closed symmetric operator on a dense domain $\mathcal{D} \subset \mathcal{H}_\tau$ satisfying

(a) $\tau(g)\mathcal{D} = \mathcal{D}$, $\forall g \in G$

(b) $\tau(g)Lv = L\tau(g)v$, $\forall g \in G$, $\forall v \in \mathcal{D}$,

then L is self-adjoint. Thomas's result applies to an irreducible quasi-regular representation τ since such a representation has *a fortiori* finite multiplicity.

References

1. L. Auslander & B. Kostant, *Polarization and unitary representations of solvable Lie groups*, Inventiones Math. **14** (1971), 255-254.
2. E. van den Ban, *Invariant differential operators on a semisimple symmetric space and finite multiplicities in a Plancherel formula*, Arkiv för Matematik **25** (1987), 175-187.
3. E. van den Ban, *The principal series for a reductive symmetric space I*, Ann. Sci. École Norm. Sup. **21** (1988), 359-412.
4. Y. Benoist, *Multiplicité un pour les éspaces symétriques exponentiel*, Mem. Math. Soc. France **15** (1984), 1-37.
5. P. Bernat et al, *Représentations des Groupes de Lie Résolubles*, Dunod, Paris, 1972.
6. L. Corwin, F. Greenleaf & G. Grélaud, *Direct integral decompositions and multiplicities for induced representations of nilpotent Lie groups*, Trans. Amer. Math. Soc. **304** (1987), 549-583.
7. M. Duflo, *Sur les extensions des répresentations irréductibles des groupes de Lie nilpotents*, Ann. Sci. École Norm. Sup. **5** (1972), 71-120.
8. M. Duflo, *Théorie de Mackey pour les groupes de Lie algébriques*, Acta Math. **149** (1982), 152-213.
9. H. Fujiwara, *Représentations monomiales d'un groupe de Lie nilpotente*, Pacific J. Math. **127** (1987), 329-352.
10. H. Fujiwara, *Représentations monomiales des groupes de Lie résolubles exponentiels*, The Orbit Method in Representation Theory, Birkhäuser, 1990, pp. 61-84.
11. R. Lipsman, *Harmonic analysis on non-semisimple symmetric spaces*, Israel J. Math. **54** (1986), 335-350.
12. R. Lipsman, *Orbital parameters for induced and restricted representations*, Trans. Amer. Math. Soc. **313** (1989), 433-473.
13. R. Lipsman, *Harmonic analysis on exponential solvable homogeneous spaces*, Pacific J. Math. **140** (1989), 117-147.
14. R. Lipsman, *The Penney-Fujiwara Plancherel formula for symmetric spaces*, The Orbit Method in Representation Theory, Birkhäuser, 1990, pp. 135-145.
15. R. Lipsman, *Induced representations of completely solvable Lie groups*, Annali della Scuola Norm. Pisa **17** (1990), 127-164.
16. R. Lipsman, *The Penney-Fujiwara Plancherel formula for abelian symmetric spaces and completely solvable homogeneous spaces*, Pacific J. Math. **151** (1991), 265-295.
17. R. Lipsman, *The multiplicity function on exponential and completely solvable homogeneous spaces*, Geometriae Dedicata **39** (1991), 155-161.
18. R. Lipsman, *The Penney-Fujiwara Plancherel formula for homogeneous spaces*, Representation Theory of Lie Groups and Lie Algebras, World Scientific Press, 1992, pp. 120-139.
19. R. Lipsman, *The Plancherel formula for homogeneous spaces with polynomial spectrum*, Pacific J. Math. **159** (1993), 351-377.
20. R. Penney, *Abstract Plancherel theorems and a Frobenius reciprocity theorem*, J. Functional Analysis **18** (1975), 177-190.

21. R. Penney, *Spherical distributions on Lie groups and C^∞ vectors*, Trans. Amer. Math. Soc. **223** (1976), 367-384.
22. L. Pukanszky, *Unitary representations of Lie groups with co-compact radical and applications*, Trans. Amer. Math. Soc. **236** (1978), 1-49.