THE PLANCHEREL FORMULA FOR HOMOGENEOUS SPACES WITH POLYNOMIAL SPECTRUM

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Abstract. The distribution-theoretic version of the Plancherel formula—known as the Penney-Fujiwara Plancherel Formula—for the decomposition of the quasi-regular representation of a Lie group G on $L^2(G/H)$ is considered. Attention is focused on the case that the spectrum consists of irreducible representations induced from a finite-dimensional representation. This happens with great regularity for *Strichartz* homogeneous spaces wherein G and H are semidirect products of normal abelian subgroups by a reductive Lie group. The results take an especially simple form if G/H is symmetric. Criteria for finite multiplicity and for multiplicity-free spectrum are developed. In the case that G is a motion group—the original situation stressed by Strichartz—the results are particularly striking.

1991 *Mathematics Subject Classification.* 22E45, 22E30.
Supported by NSF # DMS-90-02642

0. Introduction. This paper is a sequel to and generalization of [7]. In that paper we derived the Penney-Fujiwara version of the Plancherel formula for the quasi-regular representation of a homogeneous space with monomial spectrum. That is, for G a (connected) Lie group, $H \subset G$ a closed subgroup, and $\tau = Ind_H^G 1$ with the property that *a.a.* of the irreducibles that appear in the spectrum of τ are induced from a *character*, we derived a distribution-theoretic version of the Plancherel formula for τ. We remark that (as observed in [7] or [6]) such a formula gives us the explicit intertwining operator for the direct integral decomposition of τ, as well as a determination of the Plancherel measure. A specific situation to which the results of [7] apply is that of abelian symmetric spaces (defined originally in [5]).

The nature of the generalization of [7] in this paper is two-fold. First we will replace the hypothesis of monomial spectrum by that of polynomial spectrum—meaning that *a.a.* the representations in the spectrum of τ are induced from *finite-dimensional* representations. Second we shall expand the abelian symmetric space application to a much broader family of symmetric spaces introduced and studied by Strichartz. Regarding the first extension, it is quite a natural step to take. Approaching either the minimal principal series of a semisimple Lie group or the generic representations of a type I amenable group, one encounters representations that are polynomial, not monomial. As for the second extension, we obtain a much more encompassing collection of symmetric spaces—including semisimple symmetric spaces as well as the generalized Grassmannian bundles of Strichartz [11]. These spaces, various aspects of whose harmonic analysis we derive in this paper, are of the form $W \cdot H \backslash V \cdot G$, where $V \cdot G$ is a semidirect product of a normal vector subgroup V by a reductive Lie group G and $W \cdot H$ is of the same species. (These are the Strichartz homogeneous spaces.)

The paper is divided into four parts. In part I we review the Penney-Fujiwara formulation of the Plancherel formula (§1), and describe the distributions and matrix coefficients associated to polynomial representations (§2). Part II contains an abstract decomposition (§3) of the quasi-regular representation $Ind_{W \cdot H}^{V \cdot G} 1$. In §4 we describe a conjugacy problem that acts as an obstruction to computing the multiplicity function in the preceding direct integral decomposition. In §5 we specialize to symmetric spaces, show that the conjugacy problem disappears in that context, and obtain some results on the multiplicity. In part III we present our proof (§6) of the distribution-theoretic Plancherel formula for Strichartz homogeneous spaces with polynomial spectrum. The results draw upon parts I and II and generalize those of [7]. §7 contains a short list of examples and questions that complement the paper's main results. Finally in part IV we supply an Appendix (§8) which treats the special case of compact G. The main results of the paper are as follows: Theorem 2.2 which gives the matrix coefficients of an arbitrary polynomial representation; Theorem 3.1 which gives a direct integral decomposition into irreducibles of any Strichartz homogeneous space $W \cdot H \backslash V \cdot G$; Proposition 5.1 which characterizes the involutions on a semidirect product $V \cdot G$; Theorem 5.3 which specifies the multiplicity function for Strichartz symmetric spaces; Theorem 5.5 which details some special situations that manifest finite multiplicity or even multiplicity one; Theorem 6.2 which is the Penney- Fujiwara Plancherel formula for Strichartz homogeneous spaces with polynomial spectrum; Corollary 6.3 which specializes Theorem 6.2 to Strichartz symmetric spaces; and Theorem 8.1 which gives a reciprocity result in the special case of compact G.

I. The Plancherel Formula and Polynomial Representations

In this part we recall—and generalize—the results of [7,§§2 and 3]. G is a Lie group and $H \subset G$ a closed subgroup. We fix choices of right Haar measure dg, dh on G, H, and we assume that the quasi-regular representation $\tau = Ind_H^G 1$ is type I. Then there is a uniquely determined direct integral decomposition

$$\tau = \int_{\hat{G}(H)}^{\oplus} n_\tau(\pi)\, \pi \, d\mu_\tau(\pi), \tag{I.1}$$

where μ_τ is a smooth Borel measure on $\hat{G}$, $n_\tau(\pi)$ is the multiplicity function, and $\hat{G}(H) \subset \hat{G}$ is the set of irreducibles weakly contained in τ. ($\hat{G}(H)$ is a minimal closed μ_τ-co-null subset.) As discussed in [7], [6], [8] only the class of μ_τ is uniquely determined and the intertwining operator that effects the decomposition (I.1) is not evident. Both of these deficiencies are addressed by the distribution-theoretic Penney-Fujiwara refinement of this decomposition presented in the first section.

1. The Penney-Fujiwara Plancherel formula. If $\pi \in \hat{G}$ is realized in a Hilbert space $\mathcal{H}_\pi$, we write $(\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}}$ for the space of anti-distributions—conjugate linear functionals on the space of C^∞ vectors—which transform under H by $q^{-1/2}, q = q_{H,G} = \Delta_{H,G} = \Delta_H/\Delta_G$, the quotient of modular functions. One knows [9] that

$$n_\tau(\pi) \leq \dim(\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}} (\mu_\tau - a.e.) \tag{1.1}$$

The question of equality in (1.1) is an interesting one that we shall address in §§7 and 8. The Penney-Fujiwara Plancherel formula (PFPF for short) is obtained by locating $n_\tau(\pi)$ elements $\alpha_\pi^1, \dots \alpha_\pi^{n_\tau(\pi)} \in (\mathcal{H}_\pi^{-\infty})^{H,q^{-1/2}}$ and a unique (up to scalar) choice (in the class) of μ_τ which satisfy

$$< \tau(\omega)\alpha_\tau, \alpha_\tau >= \int_{\hat{G}(H)} \sum_{j=1}^{n_\tau} < \pi(\omega)\alpha_\pi^j, \alpha_\pi^j > d\mu_\tau(\pi), \omega \in \mathcal{D}(G), \tag{1.2}$$

where $\mathcal{D}(G) = C_c^\infty(G)$ is the space of test functions and α_τ is the canonical cyclic distribution associated to τ (see formulas (2.2)-(2.4))). Then the map

$$\tau(\omega)\alpha_\tau \hookrightarrow \{\pi(\omega)\alpha_\pi^j\} \tag{1.3}$$

is an isometry that extends uniquely to an intertwining operator effecting the direct integral decomposition (I.1).

Remark 1.1. Although the formulation makes sense without restriction on the multiplicity function, experience has shown that the PFPF is really only useful in the case of finite multiplicity (see [7,Prop. 3.2]). When n_τ is infinite on a set of positive measure, one must employ the Bonnet form of the Plancherel formula [2] to overcome the difficulties indicated in the opening paragraph of part I.

We need to recall another result from [7,§3] which will be used in the following. Typically one has, for $\mu_\tau - a.a.\quad \pi \in \hat{G}(H)$, a distinguished dense locally convex space $(\mathcal{H}_\pi^\infty)_c \subset \mathcal{H}_\pi^\infty$, which has a finer topology than the relative topology. If we denote the anti-dual by $(\mathcal{H}_\pi)_c^{-\infty}$, then we have natural continuous inclusions

$$(\mathcal{H}_\pi)_c^\infty \subset \mathcal{H}_\pi^\infty \subset \mathcal{H}_\pi \subset \mathcal{H}_\pi^{-\infty} \subset (\mathcal{H}_\pi)_c^{-\infty}.$$

One knows that $\pi(\mathcal{D})\mathcal{H}_\pi^{-\infty} \subset \mathcal{H}_\pi^\infty$, but usually $\pi(\mathcal{D})(\mathcal{H}_\pi)_c^{-\infty} \not\subset (\mathcal{H}_\pi)_c^\infty$. However for the polynomial representations π and distributions $\beta \in (\mathcal{H}_\pi)_c^{-\infty}$ we shall encounter, we will prove that $\pi(\mathcal{D}^+)(\beta) \in \mathcal{H}_\pi^{-\infty}, \mathcal{D}^+ =$ positive linear combinations of functions of the form $\omega = \omega_1^* * \omega_1, \omega_1 \in \mathcal{D}(G)$. The results of [7,§3], in particular Proposition 3.3, say that this is sufficient to prove (1.2). Indeed, the conclusion is that *a.a.* of the β must have a unique extension to an element of $\mathcal{H}_\pi^{-\infty}$, (1.2) holds for $\omega \in \mathcal{D}(G)$, and furthermore

$$\alpha_\tau = \int \sum_j n_\tau(\pi)\, \beta_\pi^j \, d\mu_\tau(\pi)$$

in the sense of [9]. Finally formula (1.3) supplies the intertwining operator.

2. Polynomial representations, relatively invariant distributions and matrix coefficients. We are primarily interested here in quasi-regular representations whose spectrum is polynomial—i.e. the irreducibles that occur are induced from finite-dimensional representations. We now describe the relatively invariant distributions and matrix coefficients of polynomial representations that appear in the PFPF.

Let $B \subset G$ be a closed subgroup with a choice db of right Haar measure. If σ is a finite-dimensional unitary representation of B, whose Hilbert space realization is denoted $\mathcal{H}_\sigma$, the induced representation $Ind_B^G \sigma$ acts in the space

$$C_c^\infty(G,B,\sigma) = \{f : G \to \mathcal{H}_\sigma, f \text{ is } C^\infty, f(bg) = \sigma(b)f(g), b \in B, g \in G,$$
$$||f|| \text{ compactly supported mod } B\}$$

by the formula

$$\pi_\sigma(g)f(x) = f(xg)[q_{B,G}(xg)/q_{B,G}(x)]^{1/2}. \tag{2.1}$$

[Here q is a smooth function on G satisfying $q(1) = 1, q(bg) = \Delta_{B,G}(b)q(g)$. It uniquely specifies a quasi-invariant measure $d\dot{g}$ on $B\backslash G$ which obeys

$$\int_G f(g)q(g)dg = \int_{B\backslash G}\int_B f(bg)\, db\, d\dot{g}.$$

The reader is referred to [7,§2] or [4] for more details on this relationship.] The extension of (2.1) to a unitary action of G on $L^2(B\backslash G, d\dot{g}) = L^2(G,B,\sigma)$ is as in [7]. Clearly (by [10] e.g.) we have

$$C_c^\infty(G,B,\sigma) \subset L^2(G,B,\sigma)^\infty \subset C^\infty(G,B,\sigma).$$

The space $C_c^\infty(G,B,\sigma)$ will play the role of $(\mathcal{H}_\pi)_c^\infty$ in the notation of §1 and [7].

If $B = H$ and $\sigma = 1$, then we obtain the quasi-regular representation τ. Proposition 2.2 of [7] remains valid. The distribution

$$\alpha_\tau : f \to \bar{f}(1) \tag{2.2}$$

defines the canonical cyclic distribution on $\mathcal{H}_\tau = L^2(G,H,1) \equiv L^2(G,H)$, and

$$\tau(\omega)\alpha_\tau(g) = \omega_H(g) \tag{2.3}$$

$$< \tau(\omega)\alpha_\tau, \alpha_\tau >= \omega_H(1) \tag{2.4}$$

where

$$\omega_H(g) = \Delta_G(g)^{-1} q_{H,G}^{-1/2}(g) \int_H \omega(g^{-1}h^{-1})\Delta_G(h)^{-1} q_{H,G}^{-1/2}(h)\, dh. \tag{2.5}$$

Now suppose we have a quasi-regular representation τ with polynomial spectrum, i.e. $\mu_\tau - a.a \quad \pi \in \hat{G}(H)$ are induced from finite-dimensional representations. We focus attention on one of these representations $\pi = Ind_B^G \sigma$ temporarily. In fact in order to get started in [7], we needed three additional assumptions. Here are their analogs in the current setting:

(a) $\sigma|_{H\cap B}$ contains a fixed vector
(b) BH is closed in G
(c) $q_{H\cap B,H}\, q_{H\cap B,B} \equiv 1$ on $H \cap B$.

Because of (b) any $f \in C_c(G)$ satisfies $f|_{BH} \in C_c(BH)$. Hence $f \to f|_H$ projects $C_c^\infty(G,B,\sigma)$ to $C_c^\infty(H, H\cap B, \sigma|_{H\cap B})$. Fix a right Haar measure on $H\cap B$ and let $d\dot{h}$ denote a quasi-invariant measure on $H\cap B\backslash H$. Finally let $\xi \in \mathcal{H}_\sigma$ be a $\sigma(H\cap B)$ - fixed vector. Then the antidistributions which play a central role in our theory are

$$\beta = \beta_\xi : f \to \int_{H\cap B\backslash H} < \xi, f(\cdot) > q_{B,G}^{1/2} q_{H\cap B,H}^{-1} q_{H,G}^{-1/2}\, d\dot{h}, f \in C_c^\infty(G,B,\sigma). \tag{2.6}$$

Here is the analog of [7,Thm. 2.1]

Theorem 2.1.

(i) *β is well-defined;*
(ii) *β is relatively invariant under the action of H with modulus $q_{H,G}^{-1/2}$;*
(iii) *$\pi(\mathcal{D}(G))\beta \in C^\infty(G,B,\sigma) \subset \mathcal{H}_\pi^{-\infty}$;*
(iv) *In fact for $\omega \in \mathcal{D}(G)$, the vector-valued function is given by the formula*

$$\pi(\omega)\beta(g) = \int_{H\cap B\backslash B} \omega_H(bg)\sigma(b)^{-1}\xi q_{B,G}^{-1/2}(bg) q_{H,G}^{1/2}(bg) q_{H\cap B,B}^{-1}(b)\, d\dot{b} \tag{2.7}$$

where ω_H is defined in (2.5);
(v) *For $\omega \in \mathcal{D}^+(G)$, the matrix coefficient of β is*

$$< \pi(\omega)\beta, \beta > \tag{2.8}$$

$$= \int_{H\cap B\backslash H} \int_{H\cap B\backslash B} \omega_H(bh) < \xi, \sigma(b)\xi > q_{B,G}^{-1/2}(b) q_{H,G}^{1/2}(h^{-1}bh) q_{H\cap B,B}^{-1}(b) q_{H\cap B,H}^{-1}(h)\, d\dot{b}\, d\dot{h}.$$

The matrix coefficient is a non-negative number, possibly equal to $+\infty$.

In fact there is really nothing further to prove. (i) still follows from (c). Furthermore the proof of Theorem 2.1 in [7] continues to hold virtually word-for-word and equation-by-equation. However it is important to make the

Remark 2.2. As in [7] we observe that the values of the function $\pi(\omega)\beta$ and the matrix coefficient $< \pi(\omega)\beta, \beta >$ are independent of the choice of the quasi-invariant measures $d\dot{b}, d\dot{h}$, but they do depend on the original choices of Haar measure on G, H, B and $H \cap B$.

II. Strichartz Homogeneous Spaces

In [11], Strichartz studied homogeneous spaces $\tilde{H}\backslash\tilde{G}$ of the following form: $\tilde{G}$ is a semidirect product $\tilde{G} = V \cdot G$ of a closed normal vector subgroup V by a (connected) reductive Lie group G; and $\tilde{H} = W \cdot H$ is a closed subgroup of the same species, where $W \subset V$ and $H \subset G$. He calls such spaces [11] generalized Grassmannian bundles, taking his cue from the case where $G = SO(n), V = \mathbb{R}^n, H = S(O(k) \times O(n-k))$ and $W = \mathbb{R}^k$. Geometrically one can think of $\tilde{H}\backslash\tilde{G}$ as a bundle as follows. First fix the vector bundle $\mathcal{V} = G \times_H V$ with base $X = H\backslash G$ and fiber V. Then consider the bundle $\mathcal{W}$ of affine (G, W) spaces in $\mathcal{V}$—i.e. the bundle with base X and fiber at $x = Hg$ the collection of affine subspaces of the form $W \cdot g + v, v \in V$. $\mathcal{W}$ is a transitive space for the natural action of $V \cdot G$ and the stability group at $(x_0, W), x_0 = H$, is clearly $W \cdot H$.

In part II we shall show how to decompose the quasi-regular representation $\tau = Ind_{\tilde{H}}^{\tilde{G}} 1$ into irreducibles. We shall discover that a very tricky conjugacy problem prevents us from precisely calculating the multiplicity in the most general situation. But then we shall see that when we specialize to symmetric spaces, the difficulty can be overcome.

3. A direct integral decomposition. The notation being as above, we wish to compute a direct integral decomposition of $\tau = Ind_{\tilde{H}}^{\tilde{G}} 1$ into irreducible representations. We recall the Mackey theory for semidirect products. Let $\lambda \in \hat{V}$ be a unitary character, G_λ the stability group in G. Then for any $\sigma \in \hat{G}_\lambda$ the induced representation

$$\pi_{\lambda,\sigma} = Ind_{V \cdot G_\lambda}^{G} \lambda \times \sigma \tag{3.1}$$

is irreducible. Two such representations $\pi_{\lambda,\sigma}, \pi_{\lambda',\sigma'}$, are equivalent iff $\exists g \in G \ni g \cdot \lambda = \lambda'$ and $g \cdot \sigma \cong \sigma'$. Moreover if the quotient space $\hat{V}/G$ is countably separated, all irreducible unitary representation classes of $\tilde{G} = V \cdot G$ are accounted for in this way. In fact we shall really only need the (usually) weaker assumption that $W^\perp/H$ is countably separated in order to decompose τ. (Here $W^\perp = \{\lambda \in \hat{V} : \lambda(W) = 1\}$. We shall also utilize [5]—namely in the special case that $W = \{1\}, H = G$ we know:

$$Ind_G^{V \cdot G} 1 = \int_{\hat{V}/G}^{\oplus} \pi_{\lambda,1} \, d\dot{\lambda} \tag{3.2}$$

where $\pi_{\lambda,1} = Ind_{V \cdot G_\lambda} \lambda \times 1$, and $d\dot{\lambda}$ is the push-forward of Lebesgue measure $d\lambda$ from $\hat{V}$ to $\hat{V}/G$. If we assume that for $a.a.$ $\lambda \in W^\perp$, the "little homogeneous spaces" are type I, i.e.

$$\tau_\lambda = Ind_{H_\lambda}^{G_\lambda} 1$$

is type I, then we can decompose

$$\tau_\lambda = \int_{\hat{G}_\lambda(H_\lambda)}^{\oplus} n_\lambda(\sigma) \sigma \, d\mu_\lambda(\sigma). \tag{3.3}$$

Putting these facts together with the theorems of induction in stages, and commutation of inducing and direct integrals, we can now compute

$$
\begin{aligned}
\tau &= Ind_{W\cdot H}^{V\cdot G} 1 \\
&= Ind_{V\cdot H}^{V\cdot G}\, Ind_{W\cdot H}^{V\cdot H} 1 && (stages) \\
&= Ind_{V\cdot H}^{V\cdot G} \int_{W^\perp/H}^{\oplus} Ind_{V\cdot H_\lambda}^{V\cdot H} \lambda \times 1\, d\dot\lambda && (3.2) \\
&= \int_{W^\perp/H}^{\oplus} Ind_{V\cdot H}^{V\cdot G} Ind_{V\cdot H_\lambda}^{V\cdot H} \lambda \times 1\, d\dot\lambda && (commutativity) \\
&= \int_{W^\perp/H}^{\oplus} Ind_{V\cdot H_\lambda}^{V\cdot G} \lambda \times 1\, d\dot\lambda && (stages) \\
&= \int_{W^\perp/H}^{\oplus} Ind_{V\cdot G_\lambda}^{V\cdot G}\, Ind_{V\cdot H_\lambda}^{V\cdot G_\lambda} \lambda \times 1\, d\dot\lambda && (stages) \\
&= \int_{W^\perp/H}^{\oplus} Ind_{V\cdot G_\lambda}^{V\cdot G} \int_{\hat G_\lambda(H_\lambda)}^{\oplus} \lambda \times n_\lambda(\sigma)\sigma\, d\mu_\lambda(\sigma)\, d\dot\lambda && (3.3) \\
&= \int_{W^\perp/H}^{\oplus} \int_{\hat G_\lambda(H_\lambda)}^{\oplus} Ind_{V\cdot G_\lambda}^{V\cdot G} (\lambda \times n_\lambda(\sigma)\sigma)\, d\mu_\lambda(\sigma)\, d\dot\lambda && (commutativity) \\
&= \int_{W^\perp/H}^{\oplus} \int_{\hat G_\lambda(H_\lambda)}^{\oplus} n_\lambda(\sigma)\pi_{\lambda,\sigma}\, d\mu_\lambda(\sigma)\, d\dot\lambda && (3.1).
\end{aligned}
$$

We have proven the following theorem of Strichartz.

Theorem 3.1. *Let $\tilde G = V \cdot G$ be a semidirect product of a closed normal vector group V by a (connected) (reductive) Lie group G. Let $\tilde H = W \cdot H \subset \tilde G$ be a closed subgroup of the same species, $W \subset V, H \subset G$. Suppose $W^\perp/H$ is countably separated, and suppose that generically on $W^\perp, Ind_{H_\lambda}^{G_\lambda} 1$ is type I. Then*

$$Ind_{\tilde H}^{\tilde G} 1 = \int_{W^\perp/H}^{\oplus} \int_{\hat G_\lambda(H_\lambda)}^{\oplus} n_\lambda(\sigma)\pi_{\lambda,\sigma}\, d\mu_\lambda(\sigma)\, d\dot\lambda, \tag{3.4}$$

where for a.a. $\lambda \in W^\perp$, we have (by the type I assumption) the description

$$Ind_{H_\lambda}^{G_\lambda} 1 = \int_{\hat G_\lambda(H_\lambda)}^{\oplus} n_\lambda(\sigma)\sigma\, d\mu_\lambda(\sigma).$$

Remarks 3.2.

(1) Our proof of Theorem 3.1 is conceptually the same as Strichartz', but the details are much more straightforward.
(2) Strichartz is primarily interested in the case that G is compact. Then all the separation and type hypotheses are automatically satisfied. Moreover the direct integral (3.3) is then a direct sum. This special case is also discussed in §8.
(3) In Theorem 3.1 it is sufficient to assume V is locally compact abelian and G is a Lie group. No further structure is required for the proof. However the interesting cases arise when V is a vector group and G is (connected) reductive. In the following

we always assume V is a vector group and G is reductive in the Harish Chandra class. In particular, it is really no loss of generality to assume that $\tilde{G} = V \cdot G$ is algebraic.

(4) A critical point—which is overlooked by Strichartz—is that the decomposition (3.4) may not specify the multiplicity. This is because two points in $W^\perp$, which lie in distinct H-orbits, may nonetheless be in the same G-orbit. We address this issue in the next section.

4. The conjugacy problem. Suppose $\lambda \in W^\perp$ and $G \cdot \lambda \cap W^\perp \supsetneq H \cdot \lambda$. Choose $g \in G$ so that $\lambda \neq \lambda_1 = g \cdot \lambda \in W^\perp$, but λ and λ_1 lie in distinct H-orbits. Then of course we have $G_{\lambda_1} = G_{g\cdot\lambda} = gG_\lambda g^{-1}$; but it does *not* follow that $H_{\lambda_1} = gH_\lambda g^{-1}$. It is not clear that H_λ and H_{λ_1} are conjugate. (However under the condition that G is compact or H is reductive, one can invoke a principal orbit type theorem to conclude that generically on $W^\perp$, the stability groups in H are conjugate. But even if $hH_\lambda h^{-1} = H_{\lambda_1}$, it cannot be that $h \cdot \lambda = \lambda_1$ since λ and λ_1 are not in the same H-orbit. Such principal orbit type theorems are of no real value in dealing with this particular conjugacy problem.) Thus it is extremely difficult to use (3.4) to derive the precise multiplicity for the representation $\pi_{\lambda,\sigma}, \lambda \in W^\perp, \sigma \in \hat{G}_\lambda(H_\lambda)$. Just to illustrate the difficulty, suppose G is compact and $\#[(G \cdot \lambda \cap W^\perp)/H] < \infty$ generically on $W^\perp$. Then we can refine (3.4) to

$$Ind_{H}^{\tilde{G}} 1 = \int_{G\cdot W^\perp/G}^{\oplus} \sum_{\hat{G}_\lambda} n(\lambda, \sigma) \pi_{\lambda,\sigma} \, d\ddot{\lambda}; \tag{4.1}$$

where $d\ddot{\lambda}$ is the push-forward to $G \cdot W^\perp/G$ of Lebesgue measure on $W^\perp$, and if $\lambda = \lambda_1, \lambda_2, \dots, \lambda_r$ are representatives of the H- orbits on $G \cdot \lambda \cap W^\perp$, and $\sigma = \sigma_1, \sigma_2, \dots, \sigma_r$ are given by $\sigma_j = g_j \cdot \lambda$, then

$$n(\lambda, \sigma) = \sum_{j=1}^{r} n(\sigma|_{H_{\lambda_j}}, 1). \tag{4.2}$$

Here $n(\nu, 1)$ is the dimension of the space of fixed vectors for the finite-dimensional representation ν. Formula (4.2) is not terribly useful if the multiplicities—which will likely be computed by complicated combinatorics— are varying with j. On the other hand, we have

Proposition 4.1. *Suppose g_j conjugates H_λ to $H_{\lambda_j}, 1 \leq j \leq r$. Then all the summands in (4.1) are equal and we obtain*

$$n(\lambda, \sigma) = rn(\sigma|_{H_\lambda}, 1) = \#[(G \cdot \lambda \cap W^\perp)/H] n_\lambda(\sigma).$$

Proof. This is obvious.

Remark 4.2. There is really nothing special about the compact case vis-a-vis the conjugacy problem. If in the generality of Theorem 3.1 we know that, generically on $W^\perp$, the condition

$$\lambda, \lambda' \in W^\perp \text{ in the same } G \text{ orbit } \Rightarrow H_\lambda \text{ and } H_{\lambda'} \text{ are } G \text{ conjugate,} \tag{4.3}$$

holds, then we can rewrite (4.1) as

$$Ind_{\tilde{H}}^{\tilde{G}} 1 = \int_{G\cdot W^{\perp}/G}^{\oplus} n_\lambda \int_{\hat{G}_\lambda(H_\lambda)}^{\oplus} n_\lambda(\sigma)\pi_{\lambda,\sigma}\, d\mu_\lambda(\sigma)\, d\ddot{\lambda},$$

$$n_\lambda = \#[(G\cdot\lambda \cap W^{\perp})/H].$$

In the next section we prove that for symmetric spaces, the conjugacy problem disappears since (4.3) always holds.

5. Symmetric spaces. We now specialize to Strichartz homogeneous spaces which are symmetric. Suppose $\tilde{G} = V\cdot G, \tilde{H} = W\cdot H$. Then $\tilde{H}\backslash\tilde{G}$ is a symmetric space if there exist involutions η_1, η_2 of V, G respectively which satisfy $W = V^{\eta_1}, H = G^{\eta_2}$ and

$$\eta_1(gvg^{-1}) = \eta_2(g)\eta_1(v)\eta_2(g^{-1}). \tag{5.1}$$

The map

$$\eta(vg) = \eta_1(v)\eta_2(g) \tag{5.2}$$

then defines an involution of $\tilde{G}$ with $\tilde{G}^\eta = \tilde{H}$. But in fact it is possible to show that any involution of $\tilde{G}$ is essentially of this form.

Proposition 5.1. *Suppose η is an involution of $\tilde{G} = V\cdot G$. Then V is preserved by η and there exists a conjugate G_2 of G which is preserved by η.*

Note. If we replace G by G_2 and write $\eta_1 = \eta|_V, \eta_2 = \eta|_{G_2}$, then η is of the form (5.2).

Proof of Proposition 5.1. The subgroup V of $\tilde{G}$ is characteristic and so is presevED by η. If η also preserves G, there is nothing to prove. If not, write $G' = \eta(G)$. Now all the reductive co-factors of V in $\tilde{G}$ are conjugate by V. Choose $v \in V$ so that $G' = vGv^{-1}$.

Lemma 5.2. *Suppose $\eta(v) = v$, that is $\eta(G) = vGv^{-1}$ with $\eta(v) = v$. Then $\eta(G) = G$.*

Proof. We have $G = \eta^2(G) = \eta(\eta(G)) = \eta(vGv^{-1}) = v\eta(G)v^{-1} = v^2Gv^{-2}$. That is $u = v^2$ normalizes G. But V is also normal and so $G \ni ugu^{-1} = ugu^{-1}g^{-1}g \Rightarrow ugu^{-1}g^{-1} = 1, \forall g \in G$ That is u centralizes G. The action of G on V is linear, and the set of vectors fixed by G is a subspace. Hence v also must be fixed by G. That is $\eta(G) = G' = vGv^{-1} = G$.

Continuation of the proof of Proposition 5.1. We have $G' = \eta(G) = vGv^{-1}$ and we can write $v = u_1u_2$ uniquely where $\eta(u_1) = u_1, \eta(u_2) = u_2^{-1}$. Consider wGw^{-1} where $w^2 = u_2$. Then

$$\eta(wGw^{-1}) = w^{-1}u_1u_2Gu_2^{-1}u_1^{-1}w^{-1} = u_1wGw^{-1}u_1^{-1}.$$

Applying Lemma 5.2 to wGw^{-1} we obtain that wGw^{-1} is η- invariant.

So now we see it is really no loss of generality to assume that our involution on $\tilde{G} = V\cdot G$ arises from involutions on V and G separately which are related by (5.1). Then one of our main results is

Theorem 5.3.

(i) *For any $\lambda \in W^\perp$, the homogeneous space $H_\lambda \backslash G_\lambda$ is symmetric;*

(ii) *If $\lambda_1, \lambda_2 \in W^\perp$ are in the same G-orbit, then H_{λ_1} and H_{λ_2} are conjugate by an element of G;*

(iii) *Generically on $W^\perp$ we have $n_\lambda = \#[(G \cdot \lambda \cap W^\perp)/H]$ is finite.*

Proof. (i) It is obviously enough to show that G_λ is stabilized by η. Then $G_\lambda^\eta = G^\eta \cap G_\lambda = H \cap G_\lambda = H_\lambda$. If $\lambda = 1$, then $G_\lambda = G, H_\lambda = H$ and there is nothing to prove.

Lemma 5.4. *If $\lambda \in W^\perp$ and $\lambda \neq 1$, then $\eta\lambda = \lambda^{-1}$.*

Proof. Of course η acts on $\hat{V}$ by duality, $(\eta\lambda)(v) = \lambda(\eta(v)), v \in V$. We have already used that $V = W \oplus W'$ uniquely where $\eta|_W = 1, \eta|_{W'} = -1$. Now take $\lambda \in W^\perp$. Then for $w \in W$ we have $\lambda^{-1}(w) = \bar{\lambda}(w) = \bar{1} = 1$ and $\eta\lambda(w) = \lambda(\eta(w) = 1$, since η preserves W. On the other hand, if $w' \in W'$, then

$$\eta\lambda(w') = \lambda(\eta w') = \lambda((w')^{-1}) = \bar{\lambda}(w') = \lambda^{-1}(w').$$

This proves the lemma.

Now continuing with the proof of (i), we must show that if $g \in G_\lambda$, then $\eta g \in G_\lambda$, i.e.

$$\lambda(g \cdot v) = \lambda(v), \forall v \in V \Rightarrow \lambda(\eta(g) \cdot v) = \lambda(v), \forall v \in V.$$

But in fact (using $g \in G_\lambda$) we have

$$\lambda(\eta(g) \cdot v) = \eta\lambda(g \cdot \eta v) = \lambda^{-1}(g \cdot \eta v) = \bar{\lambda}(g \cdot \eta v) = \bar{\lambda}(\eta v) = \lambda(v).$$

(ii) Suppose $\lambda \in W^\perp$ and $g \cdot \lambda \in W^\perp$. Set $\lambda' = g \cdot \lambda$. Then by part (i) we have $H_{\lambda'} = G_{\lambda'}^\eta = G_{g\cdot\lambda}^\eta = (gG_\lambda g^{-1})^\eta = \eta(g)G_\lambda^\eta \eta(g)^{-1} = \eta(g)H_\lambda\eta(g)^{-1}$.

(iii) Since the groups we are dealing with are algebraic it is enough to prove that the generic H-orbits on the variety $G \cdot \lambda \cap W^\perp$ are open. For this it suffices to show

$$\dim G \cdot \lambda \cap W^\perp \leq \dim H \cdot \lambda, \text{generic } \lambda \in W^\perp.$$

But the dimension of $H \cdot \lambda$ is the same as $\dim \mathfrak{h} \cdot \lambda$. And the generic dimensions $\dim G \cdot \lambda \cap W^\perp$ are no bigger than $\dim \mathfrak{g} \cdot \lambda \cap W^\perp$. Thus to prove (iii) it will suffice to show

$$\mathfrak{g} \cdot \lambda \cap W^\perp = \mathfrak{h} \cdot \lambda, \lambda \in W^\perp.$$

We use the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ into ± 1 eigenspaces for the involution η considered (after differentiating) as an involution of $\mathfrak{g}$. Let $\lambda \in W^\perp$. Then if $X \in \mathfrak{h}, Y \in \mathfrak{q}$ and $(X + Y) \cdot \lambda \in W^\perp$, we have

$$\begin{aligned} -(X + Y) \cdot \lambda &= \eta[(X + Y) \cdot \lambda] \\ &= (\eta(X) + \eta(Y)) \cdot \eta\lambda \\ &= (X - Y) \cdot -\lambda. \end{aligned}$$

That is $Y \cdot \lambda = 0$ or $(X + Y) \cdot \lambda = X \cdot \lambda \in \mathfrak{h} \cdot \lambda$.

Note. Combining the results of §§4,5 we see that for a Strichartz symmetric space $\tilde{H}\backslash\tilde{G}$ the quasi-regular representation τ has finite multiplicity iff *a.a.* the little symmetric spaces do.

In part III we shall specialize our study to Strichartz symmetric spaces where the spectrum is generically polynomial. But before I do that I wish to single out several types of Strichartz symmetric spaces that merit special attention.

(1) *Abelian symmetric spaces.* That refers to the situation where $H = G$ and $W = \{e\}$, i.e. $\eta_1 = 1, \eta_2 = -1$. These have been studied extensively in [5], [7].
(2) *Motion symmetric spaces.* That refers to the case of compact G. These were the primary concern of Strichartz, although his results are somewhat more general. The corresponding quasi-regular representation has polynomial spectrum, and thus will be one of the examples covered by the results of part III.
(3) *Partially Riemannian symmetric spaces.* Here we mean that H should be compact. Then in that case $H\backslash G$ is a (perhaps non-compact) Riemannian symmetric space.
(4) *Takiff symmetric spaces.* This name is reserved for the situation when $V = \mathfrak{g}$ and $\eta_1 = d\eta_2$. The name is based upon the work of Torasso [12].

There are two general results we can prove pertaining to items #3 and #4.

Theorem 5.5. *(i) Any Takiff symmetric space has finite multiplicity.*
(ii) Any partially Riemannian symmetric space with G non-compact is multiplicity-free.

Proof. (i) G is reductive of Harish Chandra class with an involution η. G acts on $\mathfrak{g}$ by the adjoint action with η differentiating to $\mathfrak{g}$. We have $H = G^\eta, \mathfrak{h} = \mathfrak{g}^\eta, \tilde{G} = \mathfrak{g}\cdot G, \tilde{H} = \mathfrak{h}\cdot H$. By Theorem 5.3 (iii) it is enough to show that for generic $\lambda \in \mathfrak{h}^\perp$, the little symmetric space $H_\lambda\backslash G_\lambda$ has finite multiplicity. The ± 1 eigenspace decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ is H-equivariant, so we can identify $\mathfrak{h}^\perp$ to $\mathfrak{q}$. The semisimple elements in $\mathfrak{q}$ are generic, and for any such element its stabilizer in G is reductive. Thus the little symmetric spaces are again reductive (pseudo-Riemannian symmetric spaces), and therefore by van den Ban's theorem [1] have finite multiplicity.

(ii) Now suppose $\tilde{G} = V\cdot G, G$ non-compact semisimple and suppose $\theta = \eta_2$ is a Cartan involution. Then $\tilde{H} = W\cdot K, K$ a maximal compact subgroup of G. We show $L^2(\tilde{H}\backslash\tilde{G})$ is multiplicity-free. We must show that for generic $\lambda \in W^\perp$ we have

$$n_\lambda = \#[(G\cdot\lambda\cap W^\perp)/K] = 1$$

and

$$L^2(K_\lambda\backslash G_\lambda) \text{ is multiplicity-free.}$$

In fact we shall deduce both facts from the following lemma. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, the Cartan decomposition determined by θ.

Lemma 5.6. *The following implication is true for generic $\lambda \in W^\perp$:*
If $p = \exp X, X \in \mathfrak{p}$, satisfies $p^2\cdot\lambda = \lambda$, then $p\cdot\lambda = \lambda$.

Proof. Choose a maximal split abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ containing X. Then diagonalize the action of $A = \exp\mathfrak{a}$ on V^*:

$$V^* = \sum_\beta V^*_\beta$$

where $\beta : A \to \mathbb{R}_+^*$ is a homomorphism, $V_\beta^* = \{\mu \in V^* : a \cdot \mu = \beta(a)\mu\}$, and only finitely many eigenspaces are non-trivial. Any $\lambda \in W^\perp \subset V^*$ can be written

$$\lambda = \sum_\beta \lambda_\beta$$

and the condition $\lambda_\beta \neq 0, \forall$ weight β, determines a generic set. Now suppose $p^2 \cdot \lambda = \lambda$. Then

$$\lambda = p^2 \cdot \lambda = \sum \beta(p^2)\lambda_\beta = \sum \beta(p)^2 \lambda_\beta$$

implies $\beta(p) = 1, \forall \beta$. In particular

$$p \cdot \lambda = \sum \beta(p)\lambda_\beta = \sum \lambda_\beta = \lambda$$

Completion of the proof of Theorem 5.5(ii). Suppose λ is in the generic set determined by Lemma 5.6. Assume also $g \cdot \lambda \in W^\perp$. Write $g = k \exp X, k \in K, X \in \mathfrak{p}$. Then

$$-k \exp X \cdot \lambda = -g \cdot \lambda = \eta(g \cdot \lambda) = \theta(g) \cdot \eta(\lambda) = k \exp -X \cdot (-\lambda).$$

Hence $\exp 2X \cdot \lambda = \lambda$. By Lemma 5.6 we conclude $\exp X \cdot \lambda = \lambda \Rightarrow g \cdot \lambda = k \cdot \lambda$. That is $G \cdot \lambda \cap W^\perp = K \cdot \lambda$. As for the fact that $L^2(K_\lambda \backslash G_\lambda)$ is multiplicity-free, it is enough (e.g. by [3,IV.3.1 & V.3.5]) to show $G_\lambda = K_\lambda \exp \mathfrak{p}_\lambda$. But again if $g = k \exp X \in G_\lambda$, then $k \exp X \cdot \lambda = \lambda$. Applying the involution we get

$$-k \exp X \cdot \lambda = \eta(k \exp X \cdot \lambda) = \theta(k \exp X) \cdot \eta(\lambda) = k \exp -X \cdot (-\lambda).$$

That is $\exp 2X \cdot \lambda = \lambda$. As before we conclude $\exp X \cdot \lambda = \lambda \Rightarrow k \in K_\lambda, X \in \mathfrak{p}_\lambda$.

Remark 5.7. Theorem 5.5 (ii) may not be true if G is compact also—see Proposition 7.3.

III. Strichartz Spaces with Polynomial Spectrum

In this part we bring together the material from parts I and II. We deal with Strichartz homogeneous spaces $\tilde{H} \backslash \tilde{G} = W \cdot H \backslash V \cdot G$ satisfying the condition

(III.1) for generic $\lambda \in W^\perp$, the space $L^2(H_\lambda \backslash G_\lambda)$ has polynomial spectrum.

That is for *a.a.* $\lambda \in W^\perp$, it is the case that *a.e.* $\sigma \in \hat{G}_\lambda(H_\lambda)$ is finite-dimensional. Then the generic irreducible representations $\pi_{\lambda,\sigma} = Ind_{V \cdot G_\lambda}^{\tilde{G}} \lambda\sigma$ are polynomial, that is induced from a finite-dimensional representation. Of course we must always assume $W^\perp/H$ is countably separated. Under these conditions we derive the explicit PFPF for $\tau = Ind_{\tilde{H}}^{\tilde{G}} 1$ in §6. In §7 we gather a series of illustrative examples and open questions.

6. The PFPF for polynomial spectrum. We begin with some normalizations and elementary consequences of various choices of Haar and (quasi-)invariant measures. We fix once and for all choices dv, dg, dw, dh of right Haar measures on V, G, W, H respectively. A Haar measure $d\dot{v}$ on $W \backslash V$ is uniquely determined by

$$\int_V f(v)\, dv = \int_{W \backslash V} \int_W f(wv)\, dw\, d\dot{v}, f \in C_c(V).$$

The dual Haar measure on $W^\perp$ is denoted $d\lambda$. We fix a choice of a smooth function $q = q_{H,G}$ on G satisfying $q(1) = 1, q(hg) = \Delta_H(h)\Delta_G(h)^{-1}q(g)$. Then a quasi-invariant measure $d\dot{g}$ is uniquely determined on $H\backslash G$ by

$$\int_G f(g)q(g)\,dg = \int_{H\backslash G}\int_H f(hg)\,dh\,d\dot{g}, f \in C_c(G).$$

(See §2.) Similarly we choose a smooth function $\tilde{q} = q_{\tilde{H},\tilde{G}}$. Let us note that

$$\Delta_{W\cdot H}(wh) = \delta_{H,W}(h)\Delta_H(h)$$

where $\delta_{H,W}$ is the modulus for the action of H on W determined by

$$\delta_{H,W}(h)\int_W f(hwh^{-1})\,dw = \int_W f(w)\,dw, f \in C_c(W)$$

(see [4]). Similar formulae apply to $V \cdot G$. Then

$$\begin{aligned}\tilde{q}(wh) &= \frac{\delta_{H,W}(h)\Delta_H(h)}{\delta_{G,V}(h)\Delta_G(h)} = \frac{\delta_{H,W}}{\delta_{H,V}}(h)q(h)\\ &= \delta^{-1}_{H,W\backslash V}(h)q(h) = \delta_{H,W^\perp}(h)q(h)\end{aligned}$$

by duality. We shall abuse notation slightly and write $\delta_{H,W^\perp}(g)$ for the quotient $\tilde{q}(g)/q(g)$. It is a smooth extension of $\delta_{H,W^\perp}$ from H to G satisfying $\delta_{H,W^\perp}(hg) = \frac{\delta_{H,W}}{\delta_{H,V}}(h)\delta_{H,W^\perp}(g)$. In general it will not be a character unless G leaves W invariant.

Now disintegrate $d\lambda$ under the action of H on $W^\perp$. Fix a pseudo-image $d\dot{\lambda}$ on $W^\perp/H$. Then there are uniquely determined relatively invariant measures $d\nu_{\dot{\lambda}}$ of modulus $\delta_{H,W^\perp}$ on $H_\lambda\backslash H$ so that

$$\int_{W^\perp} f(\lambda)\,d\lambda = \int_{W^\perp/H}\int_{\lambda\cdot H} f(\lambda\cdot h)\,d\nu_{\dot{\lambda}}\,d\dot{\lambda}.$$

It follows that the quotient $\frac{\Delta_{H_\lambda}}{\Delta_H}$ has a unique extension from H_λ to H, and in fact it is $\delta^{-1}_{H,W^\perp}$,

$$\frac{\Delta_{H_\lambda}}{\Delta_H} = \delta^{-1}_{H,W^\perp}. \tag{6.1}$$

We also have a uniquely determined right Haar measure dh_λ on H_λ so that

$$\int_H f(h)\,dh = \int_{H_\lambda\backslash H}\int_{H_\lambda} f(h_\lambda h)\,dh_\lambda\,d\dot{h},$$

where we have abused the notation slightly by writing $d\dot{h} = d\nu_{\dot{\lambda}}$. At this point we choose a right Haar measure dg_λ on G_λ. Then a quotient measure on $H_\lambda\backslash G_\lambda$ is uniquely specified, and thus since $\tau_\lambda = Ind_{H_\lambda}^{G_\lambda}1$ is type I with finite multiplicity, so is a Plancherel measure class μ_λ on $\hat{G}_\lambda(H_\lambda)$. We shall choose a specific measure in that class momentarily. (Note we are using the fact that if $\dim \sigma < \infty a.e.$ in (3.3), then $n_\lambda(\sigma) < \infty a.e.$ This follows from (6.3) below for example.)

Next we turn to a structural result on the homogeneous space $\tilde{H}\backslash\tilde{G} = W \cdot H\backslash V \cdot G$ and a consequence for the quasi-invariant measure (class) that lives on it. First we observe that $W \cdot H\backslash V \cdot G \not\cong W\backslash V \times H\backslash G$. Indeed neither of the maps $(Wv, Hg) \to W \cdot Hvg, W \cdot Hvg \to Wv$ is well-defined. In fact $W \cdot H\backslash V \cdot G$ is a fiber space

$$\begin{array}{ccc} W\backslash V & \xrightarrow{\ i\ } & W \cdot H\backslash V \cdot G \\ & & \Big\downarrow p \\ & & H\backslash G \end{array}$$

The projection is $p : W \cdot Hvg \to Hg$, and the fiber maps are $i_{Hg}(Wv) = W \cdot Hvg$ injecting $W\backslash V$ to the fiber over Hg. (Of course the fiber map depends on the base point.) We can express a quasi-invariant measure $d(\dot{vg})$ in terms of this fiber description as follows. Fix a cross-section $s : H\backslash G \to G$. Having done that, the map

$$W\backslash V \times H\backslash G \to W \cdot H\backslash V \cdot G$$

$$(Wv, Hg) \to W \cdot Hvs(Hg)$$

is a Borel bijection. Then we can carry out the following computation

$$\begin{aligned}
&\int_{W\cdot H\backslash V\cdot G} \int_{W\cdot H} f(whvg)\, dw\, dh\, d(\dot{vg}) \\
&= \int_{V\cdot G} f(vg) q_{\tilde{H},\tilde{G}}(vg)\, dv\, dg \\
&= \int_{V\cdot G} f(vg) \frac{\delta_{H,W}}{\delta_{G,V}}(g) q_{H,G}(g)\, dv\, dg \\
&= \int_{H\backslash G} \int_{V\cdot H} f(vhg) \frac{\delta_{H,W}}{\delta_{G,V}}(hg)\, dv\, dh\, d\dot{g} \\
&= \int_{H\backslash G} \int_{V\cdot H} f(hvg) \delta_{H,W}(h) \frac{\delta_{H,W}}{\delta_{G,V}}(g)\, dv\, dh\, d\dot{g} \\
&= \int_{H\backslash G} \int_{W\backslash V} \int_{W\cdot H} f(hwvg) \delta_{H,W}(h) \frac{\delta_{H,W}}{\delta_{G,V}}(g)\, dw\, d\dot{v}\, dh\, d\dot{g} \\
&= \int_{H\backslash G} \int_{W\backslash V} \int_{W\cdot H} f(whvg) \frac{\delta_{H,W}}{\delta_{G,V}}(g)\, dw\, dh\, d\dot{v}\, d\dot{g}.
\end{aligned}$$

This proves

$$d(\dot{vg}) = \frac{\delta_{H,W}}{\delta_{G,V}}(g)\, d\dot{v}\, d\dot{g}.$$

But we already know that on H,

$$\frac{\delta_{H,W}}{\delta_{G,V}} = \frac{\delta_{H,W}}{\delta_{H,V}} = \delta^{-1}_{H,W\backslash V} = \delta_{H,W^\perp},$$

a function which we have extended to G. Thus

$$d(\dot{vg}) = \delta_{H,W^\perp}(g)\, d\dot{v}\, d\dot{g}. \tag{6.2}$$

Now we pass to the main situation. We have

$$\tau_\lambda = Ind_{H_\lambda}^{G_\lambda} 1 = \int_{\hat{G}_\lambda(H_\lambda)}^{\oplus} n_\lambda(\sigma)\sigma\, d\mu_\lambda(\sigma)$$

and $\mu_\lambda - a.a.$ σ are finite-dimensional. We know that

$$n_\lambda(\sigma) \le \dim \mathcal{H}_\sigma^\lambda, \mathcal{H}_\sigma^\lambda = \{\xi \in \mathcal{H}_\sigma : \sigma(h_\lambda)\xi = \xi, \forall h_\lambda \in H_\lambda\}. \tag{6.3}$$

Thus there exist an orthonormal family $\xi_1, \dots, \xi_{n_\lambda(\sigma)}$ of vectors in $\mathcal{H}_\sigma^\lambda$ and a unique choice of the measure in the Plancherel class so that

$$< \tau_\lambda(\omega)\alpha_\lambda, \alpha_\lambda >= \int_{\hat{G}_\lambda(H_\lambda)} \sum_{j=1}^{n_\lambda(\sigma)} < \sigma(\omega)\xi_j, \xi_j > d\mu_\lambda(\sigma), \omega \in \mathcal{D}(G_\lambda).$$

Here of course α_λ is the canonical distribution for τ_λ. If $H_\lambda \backslash G_\lambda$ is a direct product of an abelian and a compact group, then $n_\lambda(\sigma) = \dim \mathcal{H}_\sigma^\lambda$. It seems quite likely that is always the case (see §8).

Now in the terminology of part II, we have $\tilde{H} = W \cdot H$ playing the role of H there; $V \cdot G_\lambda$ playing the role of B; $W \cdot H \cap V \cdot G_\lambda = W \cdot H_\lambda$ playing the role of $H \cap B$. The representation

$$\tau = Ind_{\tilde{H}}^{\tilde{G}} 1 = \int_{W^\perp/H}^{\oplus} \int_{\hat{G}_\lambda(H_\lambda)}^{\oplus} n_\lambda(\sigma)\pi_{\lambda,\sigma}\, d\mu_\lambda(\sigma)\, d\dot{\lambda}$$

has polynomial spectrum. What about the three properties (a)-(c)?

(a) Clearly $\lambda\sigma|_{W\cdot H_\lambda} = \sigma|_{H_\lambda}$ contains an $n_\lambda(\sigma)-$dimensional space of fixed vectors;

(b) The group $(V \cdot G_\lambda)(W \cdot H) = V \cdot G_\lambda H$ is closed $\Leftrightarrow$ $G_\lambda H$ is closed;

(c)

$$q_{W\cdot H_\lambda, W\cdot H}\, q_{W\cdot H_\lambda, V\cdot G_\lambda} = \frac{\delta_{H_\lambda,W}\Delta_{H_\lambda}}{\delta_{H,W}\Delta_H} \cdot \frac{\delta_{H_\lambda,W}\Delta_{H_\lambda}}{\delta_{G_\lambda,V}\Delta_{G_\lambda}}.$$

On the subgroup $W \cdot H_\lambda$, the above becomes

$$\left(\frac{\Delta_{H_\lambda}}{\Delta_H} \frac{\delta_{H_\lambda,W}}{\delta_{H_\lambda,V}} \frac{\Delta_{H_\lambda}}{\Delta_{G_\lambda}} \right) |_{H_\lambda}$$

$$= \delta_{H,W^\perp}^{-1} \delta_{H,W^\perp} \frac{\Delta_{H_\lambda}}{\Delta_{G_\lambda}} = \frac{\Delta_{H_\lambda}}{\Delta_{G_\lambda}}.$$

The latter is $1 \Leftrightarrow H_\lambda \backslash G_\lambda$ has an invariant measure.

Thus we will reformulate our hypotheses as follows: for $a.a.$ $\lambda \in W^\perp$ we have

(d) $G_\lambda H$ is closed in G;

(e) $H_\lambda \backslash G_\lambda$ has an invariant measure.

(Comments on the propriety of these hypotheses are found in the next section.)

Now let $\xi_1, \ldots \xi_{n_\lambda(\sigma)}$ be the orthonormal family selected above.The recipe for the distributions that enter the PFPF is found in (2.6). We have

$$\beta_{\lambda,\sigma,\xi_j} = \beta_{\lambda,\sigma,j} : f \to \int_{H_\lambda \backslash H} < \xi_j, f > d\dot{h}, f \in C_c^\infty(H, H_\lambda), \tag{6.4}$$

where the simplicity of the integrand comes about because of the q evaluations already done, and the further computations:

$$q_{V\cdot G_\lambda, V\cdot G} = \frac{\delta_{G_\lambda,V}\Delta_{G_\lambda}}{\delta_{G,V}\Delta_G} = \frac{\Delta_{G_\lambda}}{\Delta_G} = \frac{\Delta_{H_\lambda}}{\Delta_G},$$

since $H_\lambda \backslash G_\lambda$ has an invariant measure; and

$$\begin{aligned} q^{1/2}_{V\cdot G_\lambda, V\cdot G} q^{-1}_{W\cdot H_\lambda, W\cdot H} q^{-1/2}_{W\cdot H, V\cdot G} &= \frac{\Delta^{1/2}_{H_\lambda}}{\Delta^{1/2}_G} \frac{\Delta^{-1}_{H_\lambda}}{\Delta^{-1}_H} \frac{\Delta^{-1/2}_H}{\Delta^{-1/2}_G} \delta^{-1/2}_{H,W} \\ &= \left(\frac{\Delta_{H_\lambda}}{\Delta_H}\right)^{-1/2} \delta^{-1/2}_{H,W^\perp} \equiv 1 \text{ on } H. \end{aligned}$$

(Actually it is not so unexpected if one compares [7,§4].)

We need one more auxiliary result before we can proceed to the PFPF. The following result will apply to the little homogeneous spaces.

Lemma 6.1. *Suppose one has a PFPF for $\tau = Ind_H^G 1$ in the form*

$$\omega_H(1) = \int_X < \pi(\omega)\alpha_\pi, \alpha_\pi > d\mu(\pi), \omega \in \mathcal{D}(G)$$

and suppose $H\backslash G$ has an invariant measure. Then one also has the equation

$$\Omega(1) = \int_X \int_{H\backslash G} \Omega(g) < \pi(g)^{-1}\alpha_\pi, \alpha_\pi > d\dot{g}\, d\mu(\pi), \Omega \in C_c^\infty(G; H).$$

Proof. Since $H\backslash G$ has an invariant measure, we have $\Delta_G|_H = \Delta_H$ and $q_{H,G} = 1$. Therefore, by definition

$$\omega_H(g) = \Delta_G(g)^{-1} \int_H \omega(g^{-1}h^{-1})\Delta_G(h)^{-1}\, dh.$$

Also the distributions α_π are H-invariant. Then we can compute

$$\begin{aligned} \omega_H(1) &= \int_X < \pi(\omega)\alpha_\pi, \alpha_\pi > d\mu_\pi = \int_X \int_G \omega(g) < \pi(g)\alpha_\pi, \alpha_\pi > dg\, d\mu(\pi) \\ &= \int_X \int_G \omega(g^{-1})\Delta_G(g^{-1}) < \pi(g^{-1})\alpha_\pi, \alpha_\pi > dg\, d\mu(\pi) \\ &= \int_X \int_{H\backslash G} \int_H \omega(g^{-1}h^{-1})\Delta_G(g^{-1}h^{-1}) < \pi(g^{-1})\alpha_\pi, \alpha_\pi > dh\, d\dot{g}\, d\mu(\pi) \\ &= \int_X \int_{H\backslash G} \omega_H(g) < \pi(g^{-1})\alpha_\pi, \alpha_\pi > d\dot{g}\, d\mu(\pi). \end{aligned}$$

Since $\omega \hookrightarrow \omega_H$ maps $C_c^\infty(G)$ onto $C_c^\infty(G,H)$, the lemma is proven.

Now we are ready for the main result of the section.

Theorem 6.2. *Let $\tilde{H}\backslash\tilde{G}$ be a Strichartz homogeneous space with polynomial spectrum. Then the distributions $\beta_{\lambda,\sigma,j}$ have unique extensions to $\mathcal{H}^\infty_{\pi_{\lambda,\sigma}} = \left(Ind_{V\cdot G_\lambda}^{V\cdot G}\lambda\sigma\right)^\infty$ so that for $\omega\in\mathcal{D}(\tilde{G})$ we have*

$$\omega_{\tilde{H}}(1) =< \tau(\omega)\alpha_\tau, \alpha_\tau >= \int_{W^\perp/H}\int_{\hat{G}_\lambda(H_\lambda)}\sum_{j=1}^{n_\lambda(\sigma)} < \pi_{\lambda,\sigma}(\omega)\beta_{\lambda,\sigma,j}, \beta_{\lambda,\sigma,j} > \, d\mu_\lambda(\sigma)\, d\dot{\lambda}.$$

Remarks 6.3. (1) Implicit in the statement is the realization of $\pi_{\lambda,\sigma}$ in $L^2(G, G_\lambda, \sigma)$. We know

$$C_c^\infty(G, G_\lambda, \sigma) \subset L^2(G, G_\lambda, \sigma)^\infty \subset C^\infty(G, G_\lambda, \sigma)$$

and any $f \in C_c^\infty(G, G_\lambda, \sigma)$, when restricted to $G_\lambda H$, can be viewed as an element of $C_c^\infty(H, H_\lambda, \sigma)$ as explained in §2.

(2) The same comment as in [7,Remarks 4.2 & 5.2(2)] obtains. Namely it seems likely that the distribution integrals (6.4) converge for **all** $f \in \mathcal{H}^\infty_{\pi_{\lambda,\sigma}}$, but I do not have a proof.

Proof of Theorem 6.2. We utilize formula (2.8) for the matrix coefficients derived in Theorem 2.1. In fact we have for $\omega \in \mathcal{D}(\tilde{G})$,

$$\begin{aligned}
< \pi_{\lambda,\sigma}(\omega)\beta_{\lambda,\sigma,j}, \beta_{\lambda,\sigma,j} >&= \int_{\tilde{H}\cap B\backslash\tilde{H}}\int_{\tilde{H}\cap B\backslash B}\omega_{\tilde{H}}(b\tilde{h}) < \xi_j, \lambda\sigma(b)\xi_j > \times\\
&\times q^{1/2}_{B,\tilde{G}}(b) q^{1/2}_{\tilde{H},\tilde{G}}(\tilde{h}^{-1}b\tilde{h}) q_{\tilde{H}\cap B,B}(b)^{-1} q_{\tilde{H}\cap B,\tilde{H}}(\tilde{h})^{-1}\, d\dot{b}\, d\dot{\tilde{h}}\\
&= \int_{H_\lambda\backslash H}\int_{H_\lambda\backslash G_\lambda}\int_{W\backslash V}\omega_{\tilde{H}}(vg_\lambda h) < \xi_j, \lambda\sigma(vg_\lambda)\xi_j > \times\\
\times [q^{1/2}_{G_\lambda,G}(g_\lambda)][\delta_W^\perp(g_\lambda) q_{H,G}(h^{-1}g_\lambda h)]^{1/2}&[\delta_{W^\perp}(g_\lambda)]^{-1}[q_{H_\lambda,H}(h)]^{-1}\delta_{W^\perp}(g_\lambda)\, d\dot{v}\, d\dot{g}_\lambda\, d\dot{h}\\
= \int_{H_\lambda\backslash H}\int_{H_\lambda\backslash G_\lambda}\int_{W\backslash V}&\omega_{\tilde{H}}(vg_\lambda h) < \xi_j, \lambda\sigma(g_\lambda)\xi_j > \overline{\lambda(v)}\rho_h(g_\lambda) q_{H_\lambda,H}(h)^{-1}\, d\dot{v}\, d\dot{g}_\lambda\, d\dot{h},
\end{aligned}$$

using (6.2) applied to $W\cdot H_\lambda\backslash V\cdot G_\lambda$ instead of $W\cdot H\backslash V\cdot G$, and setting

$$\rho_h(g_\lambda) = q^{1/2}_{G_\lambda,G}(g_\lambda)\delta_W^\perp(g_\lambda)^{1/2} q_{H,G}(h^{-1}g_\lambda h)^{1/2}.$$

The only property of this measuurable function we shall need is the obvious one that

$$\rho_h(1) = 1.$$

We now invoke the Plancherel formula and Lemma 6.1, applied to $H_\lambda\backslash G_\lambda$, to conclude

$$\begin{aligned}
&\int_{W^\perp/H}\int_{\hat{G}_\lambda(H_\lambda)}\sum_{j=1}^{n_\lambda(\sigma)} < \pi_{\lambda,\sigma}(\omega)\beta_{\lambda,\sigma,j}, \beta_{\lambda,\sigma,j} > \, d\mu_\lambda(\sigma)\, d\dot{\lambda}\\
&\qquad= \int_{W^\perp/H}\int_{H_\lambda\backslash H}\int_{W\backslash V}\omega_{\tilde{H}}(vh)\overline{\lambda(v)} q_{H_\lambda,H}(h)^{-1}\, d\dot{v}\, d\dot{h}\, d\dot{\lambda}\\
&\qquad= \int_{W^\perp/H}\int_{H_\lambda\backslash H}\int_{W\backslash V}\omega_{\tilde{H}}(v)\bar{\lambda}(hvh^{-1})\, d\dot{v}\, d\dot{h}\, d\dot{\lambda} \qquad \text{by (6.1))}\\
&\qquad= \int_{W^\perp}\int_{W\backslash V}\omega_{\tilde{H}}(v)\bar{\lambda}(v)\, d\dot{v}\, d\lambda = \omega_{\tilde{H}}(1).
\end{aligned}$$

If we further asume that $\tilde{H}\backslash\tilde{G}$ is symmetric, then we can combine Theorem 6.2 with Theorem 5.3 to obtain

Corollary 6.4. *Suppose $\tilde{H}\backslash\tilde{G}$ is a Strichartz symmetric space having polynomial spectrum, and satisfying (d) and (e). Then for $\omega \in \mathcal{D}(\tilde{G})$, we have*

$$\omega_{\tilde{H}}(1) = \int_{G\cdot W^{\perp}/G} n_\lambda \int_{\hat{G}_\lambda(H_\lambda)} \sum_{j=1}^{n_\lambda(\sigma)} < \pi_{\lambda,\sigma}(\omega)\beta_{\lambda,\sigma,j}, \beta_{\lambda,\sigma,j} > d\mu_\lambda(\sigma)\, d\ddot{\lambda}.$$

7. Examples and open questions. In this section we illustrate our results with several examples and some outstanding questions.

Examples 7.1. (1) The prime example, and the one which motivated Strichartz [11], is that of compact G. Then all additional conditions are automatically satisfied—i.e. the spectrum is generically polynomial, $H_\lambda\backslash G_\lambda$ has an invariant measure, $G_\lambda H$ is closed in G, and all q functions are identically one. In this case $W^\perp/H$ is countably separated, the measure $d\dot{\lambda}$ on $W^\perp/H$ is the canonical image of Lebesgue measure, and of course μ_λ is discrete. If $\tilde{H}\backslash\tilde{G}$ is symmetric then Proposition 4.1 and Theorem 5.3 hold, that is the multiplicity function is $n_\lambda n_\lambda(\sigma)$. Each little homogeneous space $H_\lambda\backslash G_\lambda$ is symmetric, but G_λ may not be connected. Nevertheless I do not know a single example where the little homogeneous space is not multiplicity free. This gives rise to our first question.

Questions 7.2. (i) Is it true that, for any Strichartz symmetric space $\tilde{H}\backslash\tilde{G} = W \cdot H\backslash V \cdot G$ with G compact connected, the little homogeneous spaces are multiplicity-free? Incidentally we note that for partially Riemannian spaces with G non-compact, the answer to the corresponding question is *yes* (see Thm. 5.3). The answer is yes also in a special case we consider next (see Proposition 7.3 below).

(2) *The Strichartz Symmetric Space associated to a Pseudo-Riemannian Symmetric Space.* Suppose $H\backslash G$ is a pseudo-Riemannian symmetric space, that is G is connected semisimple, and $H = G^\eta$ is the stabilizer of an involution. One knows that it is possible to choose a Cartan involution θ of G which commutes with η. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition, and let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ be the eigenspace decomposition corresponding to η. The motion group associated to θ is $\mathfrak{p}\cdot K$. Then η preserves K and $\mathfrak{p}$, since it commutes with θ. Hence it defines an involution of $\tilde{G} = \mathfrak{p}\cdot K$. The stabilizer is $\tilde{H} = \tilde{G}^\eta = (\mathfrak{p}\cap\mathfrak{h})\cdot(K\cap H)$. The symmetric space $\tilde{H}\backslash\tilde{G}$ is called the Strichartz symmetric space associated to $H\backslash G$.

Proposition 7.3. *The symmetric space $\tilde{H}\backslash\tilde{G}$ has uniform multiplicity equal to that of the most continuous series of the pseudo-Riemannian symmetric space $H\backslash G$.*

Proof. By Proposition 4.1 and Theorem 5.3 the multiplicity function for $\tilde{H}\backslash\tilde{G}$ is a product $n_\lambda n_\lambda(\sigma)$, where for $\lambda \in (\mathfrak{p}\cap\mathfrak{h})^\perp = \mathfrak{p}\cap\mathfrak{q}$

$$n_\lambda = \#[(K\cdot\lambda \cap \mathfrak{p}\cap\mathfrak{q})/(K\cap H)] \tag{7.1}$$

$$n_\lambda(\sigma) = \text{ mult. funct. for } L^2((K_\lambda\cap H)\backslash K_\lambda). \tag{7.2}$$

In fact for generic $\lambda \in \mathfrak{p}\cap\mathfrak{q}$ the symmetric spaces $(K_\lambda\cap H)\backslash K_\lambda$ are multiplicity-free and n_λ is the multiplicity of the most continuous series of the semisimple symmetric space $H\backslash G$. These facts are contained in as yet unpublished results of van den Ban and Schlichtkrull.

I thank them for pointing this out to me. I shall content myself with the remark that (7.1) represents an index of certain Weyl groups, and (7.2) is identically 1 because $K_\lambda = (K_\lambda \cap H)\exp(\mathfrak{k}_\lambda \cap \mathfrak{q})$, which is one of their results.

(3) There are many examples where the generic stability group $G_\lambda, \lambda \in W^\perp$, is compact, or perhaps finite or even trivial. This typically happens when $\dim V >> \dim G$. Any of these will guarantee polynomial spectrum and the validity of all the additional conditions. The multiplicity will be finite if the space is symmetric, but it need not be *one*. This follows already from the observation that finite symmetric spaces need not be multiplicity-free. For a simple example of that, take $G = SL(2, \mathbb{Z}_2)$ with involution $\eta(g) = CgC^{-1}, C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $H = G^\eta = CentG$, but $G/CentG$ is not abelian.

(4) *Takiff spaces.* Suppose again $H\backslash G$ is a semisimple symmetric space. Form the semidirect product $\tilde{G} = \mathfrak{g} \cdot G$. The involutuion η (corresponding to H) acts naturally on $\tilde{G}$ and $\tilde{H} = \mathfrak{h} \cdot H$.

Proposition 7.4. *The spectrum is polynomial and the multiplicity of $\tilde{H}\backslash\tilde{G}$ is finite.*

Proof. We identify $\mathfrak{h}$ to $\mathfrak{q}$. It is enough to prove that for generic $\lambda \in \mathfrak{q}$, the little symmetric space $H_\lambda\backslash G_\lambda$ has a spectrum consisting of finite-dimensional representations. The argument that follows is due to van den Ban. Let λ be regular semisimple. Then its centralizer $\mathfrak{a}$ in $\mathfrak{q}$ is a Cartan subspace. Moreover G_λ is the centralizer of $\mathfrak{a}$ in G. But then $H_\lambda\backslash G_\lambda$ has tangent space $\mathfrak{a} \subset \mathfrak{q} \approx \mathfrak{h}\backslash\mathfrak{g}$ at the origin. Let A be the analytic subgroup corresponding to $\mathfrak{a}$. A is abelian. Moreover A has finitely many orbits on $H_\lambda\backslash G_\lambda$ and they are all open. This implies that $L^2(H_\lambda\backslash G_\lambda)$ decomposes into finite-dimensional representations.

As for the second statement, it is already proven in Theorem 5.5 (i). But it does lead to a second question.

Questions 7.2. (ii). Are the symmetric spaces $L^2(H_\lambda\backslash G_\lambda)$ in the above proposition multiplicity-free? If so, the multiplicity of the Takiff symmetric space is precisely

$$n_\lambda = \#[(G \cdot \lambda \cap \mathfrak{q})/H].$$

Note that questions (i) and (ii) are the same—but applied to different situations.

Remarks 7.5. (1) Conditions (d) and (e) hold for Takiff spaces. But unlike partially Riemannian or Strichartz spaces associated to a pseudo-Riemannian space, they may not have uniform multiplicity. The simplest example is $\mathfrak{h} \cdot H\backslash\mathfrak{g} \cdot G$ where $G = SL(2, \mathbb{R})$, and the involution is conjugation by the matrix C above.

(2) Unlike Riemannian spaces, and more like pseudo-Riemannian spaces, the spectrum of Strichartz spaces will usually not consist of representations from the spectrum of the regular representation. To illustrate, consider the example which is both Takiff and partially Riemannian—$\mathfrak{k} \cdot K\backslash\mathfrak{g} \cdot G$. The representations in the spectrum of the regular representation are

$$\pi_{\lambda,\sigma} = Ind_{\mathfrak{g}\cdot G_\lambda}^{\mathfrak{g}\cdot G} \lambda\sigma,$$

where $\hat{\mathfrak{g}} = \mathfrak{g}$ via the Killing form, $\lambda \in \mathfrak{g}$ is regular integral semisimple, G_λ is a Cartan subgroup and σ is a character. (Under very mild conditions the group $\mathfrak{g} \cdot G$ has polynomial, or even monomial, spectrum.) However the spectrum of the symmetric space is quite different: $\mathfrak{k}^{\perp} = \mathfrak{p}$ and for $\lambda \in \mathfrak{p}$ regular, G_λ will be a Cartan subgroup only when G is quasi-split. (The spectrum is still monomial, but it is degenerate.)

All of the previous results and remarks lead to a third question.

Questions 7.2. (iii) Let $H\backslash G$ be *any* symmetric space with G a connected Lie group, $H = G^\eta$. Suppose the quasi-regular representation τ is type I. Then is it true that τ has (bounded) finite multiplicity? Such a result was conjectured in [8]. The results of this paper provide further evidence for a positive response.

Our final question was raised in the discussion in §6.

Questions 7.2 (iv) Suppose $H\backslash G$ has an invariant measure, and the quasi-regular representation $\tau = Ind_H^G 1$ is type I and has finite-dimensional spectrum. Thus

$$\tau = \int_{\hat{G}(H)} n_\tau(\sigma)\sigma \, d\mu_(\sigma), n_\tau(\sigma) < \infty \quad a.e.$$

If we let $\mathcal{H}_\sigma^H$ denote the space of invariant vectors, then is it true that

$$n_\tau(\sigma) = \dim \mathcal{H}_\sigma^H,$$

for $\mu - a.a \quad \sigma$? for all σ?

IV. Appendix

We have already noted that in the direct integral decomposition

$$\text{(IV.1)} \qquad \tau = Ind_H^G 1 = \int_{\hat{G}(H)} n_\tau(\pi)\pi \, d\mu(\pi),$$

the multiplicity satisfies

$$n_\tau(\pi) \le \dim(\mathcal{H}_\pi^{-\infty})^{H, q^{-1/2}}.$$

In general, the inequality is strict. But in this appendix we demonstrate equality for Strichartz symmetric spaces $\tilde{H}\backslash\tilde{G}$ when G is compact.

8. A reciprocity result. Adopting the point of view of Penney [9], we think of the equality

$$n_\tau(\pi) = \dim(\mathcal{H}_\pi^{-\infty})^{H, q^{-1/2}}$$

as a generalization of Frobenius reciprocity. Now take $\tilde{G} = V \cdot G, G$ compact. For a subgroup $\tilde{H} = W \cdot H$, we know

$$\tau = \int_{W^\perp/H}^{\oplus} \sum_{\hat{G}_\lambda(H_\lambda)}^{\oplus} n_\lambda(\sigma)\pi_{\lambda,\sigma} d\dot{\lambda}$$

provided

$$Ind_{H_\lambda}^{G_\lambda} 1 = \sum_{\hat{G}_\lambda(H_\lambda)}^{\oplus} n_\lambda(\sigma)\sigma.$$

Since G_λ is compact, we know by Frobenius reciprocity that

$$n_\lambda(\sigma) = \dim\{\xi \in \mathcal{H}_\sigma : \sigma(h)\xi = \xi, \forall h \in H_\lambda\}.$$

Of course in this case $\mathcal{H}_\sigma^{-\infty} = \mathcal{H}_\sigma^{\infty} = \mathcal{H}_\sigma$ by finite-dimensionality.

Theorem 8.1. *Let* $\pi = \pi_{\lambda,\sigma} \in \hat{\tilde{G}}$. *Then*

(i) $(\mathcal{H}_\pi^{-\infty})^H = \{0\}$ *unless* $\lambda \in G \cdot W^\perp$ *and* $\sigma \in \hat{G}_\lambda(H_\lambda)$;

(ii) *Suppose for* $\lambda \in W^\perp, g \cdot \lambda \in W^\perp$, *we have* $gH_\lambda g^{-1} = H_{g\cdot\lambda}$ *for example if* $\tilde{H}\backslash\tilde{G}$ *is symmetric. Then for any* $\sigma \in \hat{G}_\lambda(H_\lambda)$, *we have*

$$\dim\left(\mathcal{H}_\pi^{-\infty}\right)^H = n_\lambda(\sigma)n_\lambda.$$

Proof. The representations $\pi_{\lambda,\sigma}$ are as usual realized on $L^2(G, G_\lambda, \sigma)$. The action is:

$$\pi_{\lambda,\sigma}(v)f(x) = \lambda(xvx^{-1})f(x), v \in V, x \in G$$

$$\pi_{\lambda,\sigma}(g)f(x) = f(xg), x, g \in G.$$

(i) Suppose $\lambda \notin G \cdot W^\perp$ and $\beta \in \mathcal{H}_\pi^{-\infty}$ is $W \cdot H$-invariant. Then $\forall w \in W, \forall f \in C_c^\infty(G, G_\lambda, \sigma)$ we have

$$< \beta, (\lambda(gwg^{-1}) - 1)f(g) >= 0.$$

But $\forall g, \exists w$ such that $\lambda(g\cdot w) \neq 1$. In fact we can do that locally uniformly in g. Hence $\forall g, \exists$ a neighborhood on which β kills every function with support in that neighborhood. Hence $\beta \equiv 0$. Thus $\lambda \in G \cdot W^\perp$. Now suppose there is an H-invariant function in $L^2(G, G_\lambda, \sigma)$. If there were such an f, then

$$f(gh) = f(g), \forall g, h.$$

Therefore if $f \not\equiv 0$, then $f(1) \neq 0$. But then for $h \in H_\lambda$,

$$f(1) = f(1h) = f(h) = \sigma(h)f(1).$$

That is $\sigma \in \hat{G}_\lambda(H_\lambda)$. We complete the proof of part (i) by showing that the existence of a non-zero $W \cdot H$-invariant distribution forces the existence of a non-zero H-invariant function. Define the projection

$$P : C^\infty(G, G_\lambda, \sigma) \to C^\infty(G, G_\lambda, \sigma)$$

by

$$Pf(g) = \int_H f(gh)\, dh.$$

Now if β is a non-zero H-invariant distribution, then $\forall f \in C^\infty(G, G_\lambda, \sigma)$

$$< \beta, f >=< \beta, Pf > .$$

Choose $f_0 \in C^\infty(G, G_\lambda, \sigma)$ such that $< \beta, f_0 >\neq 0$. Then

$$0 \neq< \beta, f_0 >=< \beta, Pf_0 >$$

$\Rightarrow Pf_0$ is a non-zero H-invariant function.

(ii) Let β be a $W \cdot H$-invariant distribution in $\mathcal{H}_\pi^{-\infty}$. Exactly as in part (i) we conclude

$$< \beta, (\lambda(gwg^{-1}) - 1)f(g) >= 0, \forall f, \forall w.$$

Hence $Supp\beta \subset \{g \in G : g \cdot \lambda \cap W^\perp \neq 0\}$. But $G \cdot \lambda \cap W^\perp$ is a finite disjoint union of n_λ open H-orbits and it is enough to consider one H-orbit at a time. Thus we are looking at distributions supported on $G_\lambda H$, the functions there satisfying $f(g_\lambda h) = \sigma(g_\lambda)f(h)$. It is clear from the H-invariance that the only possibilities are

$$f \to \int_{H_\lambda\backslash H} < \xi, f(h) > \, d\dot{h}, \xi \in \mathcal{H}_\sigma^{H_\lambda}.$$

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