PROPER ACTIONS AND A COMPACTNESS CONDITION

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1. Introduction

Suppose $X = G/H$ is a homogeneous space, with G a connected Lie group and H a closed connected subgroup. If $\Gamma \subset G$ is a discrete subgroup, then several classical problems that have been studied are : to characterize when $\Gamma \backslash X$ is a manifold; when it is a compact manifold; and, in either of these situations, to delineate the structural implications for Γ. If G is algebraic, one can replace Γ by its Zariski-closure $L = \bar{\Gamma}$ and consider analogous problems concerning L. In a basic paper [4], Kobayashi has done exactly that and he has discovered strong parallels between the two sets of problems. He then initiates an intense investigation of the latter set. He begins by noting the standard criteria for $\Gamma \backslash X$ to be a manifold—namely iff Γ acts both properly discontinuous and freely on X. He then develops continuous analogs of these properties for L—one of them well-known, another less so. We state these properties here. The action of a closed (connected) subgroup $L \subset G$ on $X = G/H$ is said to be:

(1.1) *proper* iff $\forall$ compact subset $S \subset X$ the set $L_S := \{\ell \in L : \ell \cdot S \cap S \neq \emptyset\}$ is compact;

(1.2) *free* iff $\forall x \in X$, the stability group L_x is trivial.

Kobayashi singles out an intermediate property. The action of L is said to have the

(1.3) *compact intersection property*, denoted CI, iff $\forall x \in X$, L_x is compact.

The name is meaningful since if $x = gH \in G/H$, then $L_x = L \cap gHg^{-1}$.

It is well-known that neither of the properties "properly discontinuous" nor "free" implies the other for the action of Γ on X. On the other hand, in the case of a torsionless discrete group, the analogs of (1.2) and (1.3) are clearly equivalent and evidently (1.1)$\Rightarrow$(1.3). In general, by passing to groups of finite index, a group Γ can be rendered torsionless (see *e.g.*[4, Fact 1]) and the freeness becomes much less important than the properness. In the same way, in the context of continuous actions (as Kobayashi observed), it is property (1.3) that is likely to be verifiable rather than (1.1), or perhaps even (1.2). As we observed, (1.1)$\Rightarrow$(1.3); it is the converse that is problematic. With these facts in mind, Kobayashi poses the following basic problems in [4]:

(I) Given $X = G/H$, is there an L acting properly on X? Is there an L acting properly with compact quotient?
(II) When is it true that CI$\Rightarrow$proper?
(III) What structural implications are forced on L by the proper or CI conditions?

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Kobayashi's main attention is focused on the case that G is reductive, and he obtains some interesting results. He also examines another important case—namely when G is a semidirect product of a reductive group H by a normal vector subgroup V. This case is important because of a classical conjecture due to Auslander [1] that applies to it. In this paper we shall study questions (I)–(III) for two structural situations: the above semidirect product situation, and the case that G is a simply connected nilpotent Lie group. We shall show that the first part of question I always has an affirmative answer in both cases. The second part also has an affirmative answer in the semidirect product case, and we characterize when it has an affirmative answer in the nilpotent case. The implication in II is false in general (see [4,Example 5.2]), but we shall explore two important cases where we can prove it is true. Then we shall show that, to a remarkable degree, the answers to questions II and III for the semidirect product situation are completely determined by the answers in the nilpotent case. The proof of that assertion will involve a new theorem that strengthens a theorem of Kobayashi, which is a continuous analog of Auslander's Conjecture.

Here is an outline of the paper. In section 2, we show that in the semidirect product case, both parts of question I have an easily proven positive answer. Then in the nilpotent case, we show that the first part has a positive answer and we characterize the structures for which the answer to the second part is positive. Question II is much more subtle and difficult, as is revealed in Kobayashi's paper [4]. We show in section 3 that it has a positive response in the semidirect product case if $H = PSL(2,\mathbb{R})$. Then in section 4 we also demonstrate a positive answer if $H = N_3(\mathbb{R})$, the group of 3x3 unipotent matrices. Finally, in section 5, we connect the two structures by showing that, as a consequence of an extension (of a continuous analog) of Auslander's Conjecture (which of course addresses question III), the truth of II in the semidirect product case follows from its truth in the nilpotent case—even when H is reductive.

2. Existence of Proper Actions

We dispose of the easy case first. Let $G = H \ltimes V$ be a semidirect product of a (connected) (reductive) group H by a normal vector group V. Let $X = G/H$. Then it is obvious that $L = V$ acts properly on X and $L\backslash X$ is a singleton. This is because the action of $L = V$ on $X = G/H$ is nothing more than the action of V on itself by left translation. The result is clear from

2.1 Proposition. *If a Lie group G acts on a manifold X simply transitively, then the action is proper.*

Proof. Let $S \subset X$ be compact. Fix any point $x \in S$. Then the map $G \to X$, $g \to g \cdot x$, is a homeomorphism. Hence $G^S = \{g \in G : g \cdot x \in S\}$, being the inverse image of S under the homeomorphism, is compact. Suppose $g \cdot S \cap S \neq \emptyset$. Then for some $x_1, x_2 \in S$, $g \cdot x_1 = x_2$. But there are unique choices of elements $g_1, g_2 \in G$ such that $x_1 = g_1 \cdot x$, $x_2 = g_2 \cdot x$. Clearly $g_1, g_2 \in G^S$. Then $g \cdot x_1 = x_2 \Rightarrow gg_1 \cdot x = g_2 \cdot x \Rightarrow g_2^{-1} g g_1 \in G_x = \{e\}$; that is $g = g_2 g_1^{-1} \in (G^S)(G^S)^{-1}$. The latter is compact, so $G_S = \{g \in G : g \cdot S \cap S \neq \emptyset\}$ is compact. $\square$

The corresponding issue for G nilpotent is a little more subtle. Here is the basic result.

2.2 Theorem. *Let G be a simply connected nilpotent Lie group, H a closed connected (therefore simply connected) subgroup. Then there is a non-trivial closed (simply) connected* *subgroup $L \subset G$ acting properly on G/H with compact quotient iff there is a subalgebra $\mathfrak{l} \subset \mathfrak{g}$ which is a complement to $\mathfrak{h}$, i.e. $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{h}$, a vector space direct sum.*

Proof. We first observe that if $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{h}$ as in the statement of the theorem, then the product manifold LH is open in G, and since G is nilpotent, it is also closed. Therefore $G = LH$. Furthermore, since the exponential map is a diffeomorphism we have $L \cap H = \{e\}$. Conversely, if $G = LH$ with L and H closed connected subgroups satisfying $L \cap H = \{e\}$, then $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{h}$. So the statement of the theorem comes down to showing that L can exist iff H has a group complement, *i.e.* iff there is a closed subgroup L satisfying $L \cap H = \{e\}$, $G = LH$. (Note of course then that the compact manifold $L\backslash G/H$ is a singleton.) In one direction, this is obvious: if $G = LH$, $L \cap H = \{e\}$, then L acts simply transitively on $X = G/H$ and Proposition 2.1 says that the action is proper. Conversely, suppose $L \subset G$ acts properly on $X = G/H$ with $L\backslash X$ compact. Then L satisfies CI—in particular $L \cap H$ is compact. Its connected component of the identity is then a compact connected subgroup of the simply connected nilpotent Lie group G, thus trivial. That is $L \cap H$ is compact and discrete, which is to say finite. But G has no torsion, so $L \cap H = \{e\}$.

It remains to show that $G = LH$. Since L acts properly on $X = G/H$, the variety $L\backslash G/H$ is a pseudo-manifold. Furthermore, it is, by assumption, compact. Therefore, we can find a compact variety $S \subset G/H$ which is a cross-section for the action of L. But for any $x \in X = G/H$, the stability group L_x is precisely $L \cap gHg^{-1}$ if $x = gH$. Once again, since proper$\Rightarrow$CI, the latter is compact—which by the same reasoning as in the previous paragraph means that L_x is trivial (showing in addition that $L\backslash G/H$ is actually a manifold). Therefore the manifold G/H is fibered over the compact base manifold S by fibers all of which are isomorphic to L. But G/H and L are Euclidean spaces, therefore the above fibration can happen only if S is trivial. In particular L is transitive on $X = G/H$. □

2.3 Corollary. *Let G be a simply connected nilpotent Lie group, H a closed connected subgroup. Then there is always a closed connected subgroup $L \subset G$ acting properly on $X = G/H$.*

Proof. We prove this by induction on $\dim X$. If $\dim X = 1$, then H is normal in G. Selecting any vector $Y \in \mathfrak{g} \setminus \mathfrak{h}$ and setting $L = \exp \mathbb{R}Y$, we obtain a semidirect product $G = L \ltimes H$. Theorem 2.2 says that L acts properly on X. Now let $\dim X > 1$. We can always choose a codimension 1 (normal) subgroup N between H and $G, H \subset N \subset G$, $\dim G/N = 1$. By the previous case there is an $L \subset G$ which acts properly on G/N. The result follows from the ensuing

2.4 Lemma. *Let X, Y be L-manifolds, $\phi : X \to Y$, a surjective L-equivariant submersion. Then if L acts properly on Y, it also acts properly on X.*

Proof. Let $S \subset X$ be compact. Then $T = \phi(S) \subset Y$ is also compact. Moreover, $L_S \subset L_T$. Indeed, if $s \in S$, $g \cdot s \in S$,, then $t = \phi(s) \in T$ and $g \cdot t = g \cdot \phi(s) = \phi(g \cdot s) \in \phi(S) = T$. The facts that L_S is closed and L_T is compact guarantees that L_S is also compact.

The corollary now follows by applying Lemma 2.4 to the G-equivariant projection $G/H \to G/N$.

2.5 Remarks. (1) Theorem 2.2 and Corollary 2.3 show that in the nilpotent situation, the answers to question I are that one can always find a proper action, but not always one

that has compact quotient.
(2) See [5] for a generalization of Corollary 2.3 in the discrete setting.

3. CI implies Proper when $H = PSL(2,\mathbb{R})$

Now we return to the second of the two categories we investigate in this paper, namely semidirect products. Let $G = H \ltimes V$, where V is a normal vector subgroup. In [4], Kobayashi considers the case $H = GL(2,\mathbb{R})$, $V = \mathbb{R}^2$. It turns out to be surprisingly difficult, but he manages to prove that question II has an affirmative answer in that case. In this section we replace $GL(2,\mathbb{R})$ by $PSL(2,\mathbb{R})$, but we allow V to be arbitrary. We employ the same strategy as Kobayashi—namely, we will classify all the subgroups L of G (up to conjugacy) which are maximal with respect to having the property that they satisfy the CI condition in their action on G/H. Then we shall verify that each of these maximal groups L acts properly on G/H. This is sufficient to prove CI⇒proper by the observation that if $M \subset L$ and L acts properly, then so does M. Here is the main result of this section.

3.1 Theorem. *Let $G = H \ltimes V, H = PSL(2,\mathbb{R})$, (or what amounts to the same thing: suppose we are given a finite-dimensional real representation of $SL(2,\mathbb{R})$ which is trivial on the center). Then any closed connected subgroup of G which satisfies CI on G/H acts properly on G/H.*

Proof. We begin with a simple but essential

3.2 Lemma. *Let $G = H \ltimes V$ and suppose $L \subset G$ has the CI property when acting on G/H. Suppose in addition that L is exponential solvable. Then $\dim L \leq \dim V$.*

Proof. Consider the natural projection $p : G \to V$, $p(g) = p(hv) = v$; it's a smooth and open map (not a homomorphism since H is not normal). Restrict p to L, $p : L \to V$. It is now injective. In fact if $p(\ell_1) = p(\ell_2)$, then there is an $h \in H$ such that $\ell_1 = h\ell_2$. hence $h \in H \cap L$. But the latter, being both compact and exponential solvable, must be trivial. Thus $h = e$; and hence $p : L \to V$ is a diffeomorphism into, implying $\dim L \leq \dim V$.

3.3 Remarks. (1) Lemma 3.2 holds for any reductive group H, not just $H = PSL(2,\mathbb{R})$. The L's to which the lemma will apply are groups conjugate to subgroups of ANV where $H = KAN$ is an Iwasawa decomposition.
(2) Life would be relatively simple if every $L \subset G = H \ltimes V$ which was exponential solvable and maximal with the CI property satisfied $\dim L = \dim V$. However, we shall describe an example in the next section of a maximal CI action in which $\dim L < \dim V$.

Now we return to the proof of Theorem 3.1. Assume first that the action of $H = PSL(2,\mathbb{R})$ on V is irreducible. We use classical facts about the representation theory of $PSL(2,\mathbb{R})$ on a finite-dimensional vector space V. Every such V must have odd dimension. So take $\dim V = 2n+1$, $n \geq 0$. We select a basis $\{X, Y, T\}$ of $\mathfrak{h} = \mathrm{Lie}(H)$ satisfying $[T,X] = 2X$, $[T,Y] = -2Y$, $[X,Y] = T$. We write $\mathfrak{n} = \mathbb{R}X$, $\mathfrak{a} = \mathbb{R}T$, $\mathfrak{k} = \mathbb{R}(X - Y)$ and $N = \exp\mathfrak{n}$, $A = \exp\mathfrak{a}$, $K = \exp\mathfrak{k}$ to connote Iwasawa components. There are vectors $v_{-2n}, v_{-2n+2}, \ldots, v_{2n-2}, v_{2n}$ which satisfy: $X \cdot v_j = v_{j+2}$, $-2n \leq j \leq 2n-2$, $X \cdot v_{2n} = 0$; $Y \cdot v_j \in \mathbb{R}^{\times} v_{j-2}$, $-2n+2 \leq j \leq 2n$, $Y \cdot v_{-2n} = 0$; and $T \cdot v_j = jv_j$, $-2n \leq j \leq 2n$. We use this structure to classify the one-dimensional connected subgroups of G that enjoy the CI property. First, let's classify the adjoint conjugacy classes of $\mathfrak{g} = \mathrm{Lie}(G)$.

3.4 Lemma. *A complete set of representatives for the adjoint orbits in $\mathfrak{g} = \mathfrak{h} + V$ is given by the following list: $tT + yv_0$, $y \in \mathbb{R}, t > 0$; $r(X - Y) + yv_{2n}$, $r \neq 0, y \in \mathbb{R}$;*

$\epsilon X + yv_{-2n}$, $\epsilon = \pm 1, y \in \mathbb{R}$; and $v \in \mathcal{S}$ where $\mathcal{S} \sim V/H$ is a cross-section in V for the action of H.

Proof. The adjoint classes in $\mathfrak{h}$ are specified by the list: tT, $t > 0$; $r(X-Y)$, $r \in \mathbb{R}$; ϵX, $\epsilon = \pm 1$. Then, for $W \in \mathfrak{h}$, $u, v \in V$, the equation $\mathrm{Ad}u(W+v) = W + v - \mathrm{ad}_V W(u)$ shows that the classes in $\mathfrak{g}$ not lying in V are determined by the range of $\mathrm{ad}_V W$ for W in the preceding list. In every case that $W \neq 0$, the range of $\mathrm{ad}_V W$ is of codimension 1. Therefore, if V_1 is a complement of that range, then $W + v_1$, $v_1 \in V_1$ yields distinct classes. The lemma follows from the elementary computations of the ranges and a complement in each case.

Let us write $L^1_{t,y} = \exp \mathbb{R}(tT + yv_0)$, $L^2_{r,y} = \exp \mathbb{R}(r(X-Y) + yv_{2n})$, $L^3_{\epsilon,y} = \exp \mathbb{R}(\epsilon X + yv_{-2n})$, and $L^4_v = \exp \mathbb{R}v$, $v \in \mathcal{S}$. Next we specify which of these enjoy the CI property.

3.5 Lemma. *Up to conjugacy, the one-dimensional connected subgroups L of G which have the CI property are:*

$$\begin{aligned} L^1_y :=& L^1_{1,y}, \quad y \neq 0 \\ L^2_y :=& L^2_{1,y}, \quad y \in \mathbb{R} \\ L^3_y :=& L^3_{1,y}, \quad y \neq 0 \\ & L^4_v, \quad v \in \mathcal{S}_p, \end{aligned}$$

where $\mathcal{S}_p$ is a cross-section in the projective space of V for the action of H.

Proof. For $L \subset H$, the CI property can hold only when L is compact, thereby explaining the exclusion of $y = 0$ in cases 1 and 3. All the other L's in the list satisfy CI. In fact, we just need to check that, for any of the other list entries $\exp \mathbb{R}(W + v)$, $v \neq 0$, we have $\exp \mathbb{R}(W+v) \cap uHu^{-1}$ is compact $\forall u \in V$. This is obvious in the second and fourth cases. As for the other two cases, we reason as follows. In the third case, if $\exp s(X + yv_{-2n}) = uhu^{-1} = h(h^{-1} \cdot u)u^{-1}$, then we must have $\exp sX \exp(syv_{-2n} - s^2 yv_{-2n+2} + - \dots) = h(h^{-1} \cdot u)u^{-1}$. Therefore, $h = \exp sX$. But, for any u, we have $(h^{-1} \cdot u)u^{-1} \in \exp \sum_{j \neq -2n} \mathbb{R}v_j$. Therefore $s = 0$ and the intersection is a singleton (thus compact). In the first case, we reason similarly; namely, if $\exp s(T + yv_0) = h(h^{-1} \cdot u)u^{-1}$, then since T and v_0 commute, we find that $h = \exp sT$ and $(h^{-1} \cdot u)u^{-1} = \exp syv_0$. But if $u = \sum u_j v_j$, then $h^{-1} \cdot u - u = \sum (e^{stj} - 1)u_j v_j$, therefore once again $s = 0$. $\square$

Now, continuing with the proof of Theorem 3.1, we have to see how these 1-dimensional groups could be extended to larger L's satisfying CI. By [4,Prop. A.1.2], any L satisfying CI is a co-compact extension of a solvable group. But its maximal compact subgroup is at most 1-dimensional, and so abelian. Therefore, the Lie algebra $\mathfrak{l}$ is solvable. Then there is a Jordan-Hölder sequence $(0) = \mathfrak{l}_0 \triangleleft \mathfrak{l}_1 \triangleleft \cdots \triangleleft \mathfrak{l}_k = \mathfrak{l}$ with each $\dim \mathfrak{l}_j/\mathfrak{l}_{j-1} = 1$, $1 \leq j \leq k$. Moreover, $\mathfrak{l}_1$ must be one of the 1-dim algebras classified in Lemma 3.5. So let's see if any of the algebras in Lemma 3.5 are maximal (with the CI property); and if not, we have to compute their normalizers to see what the possibilities for $\mathfrak{l}_2$ are. We shall continue up the Jordan-Hölder sequence in this way. So let's consider the four types of algebras in Lemma 3.5.

If hu normalizes $\mathbb{R}(X + yv_{-2n})$, $y \neq 0$, then h must normalize $\mathfrak{n}$. Thus $h \in AN$. Write $h = \exp tT \exp xX$ Then $\exp xX$ fixes X and moves v_{-2n} to $v_{-2n} + xv_{-2n+2} + \dots$; while $\exp tT$ moves X to $e^{2t}X$ and sends v_{-2n} to $e^{-2nt}v_{-2n}$; and of course u commutes with v_{-2n} and moves X to $X - X \cdot u$ This leads instantly to the equation $e^{2t} = e^{-2nt}$, so $t = 0$. The element $hu = (\exp X)u$ could normalize $\mathbb{R}(X + yv_{-2n})$; for example, if $u = xv_{-2n+2} + \frac{1}{2}x^2 v_{-2n+4} + \cdots$, or if $u \in \mathbb{R}v_{2n}$. It is routine to verify that any 2-dimensional group thus generated satisfies CI. A similar argument shows that we may continue in this way to find that the maximal L's satisfying CI and which have L_y^3 as the first element in a Jordan-Hölder sequence are

$$_mL_y^3 = \exp\left(\mathbb{R}(X + yv_{-2n}) + \sum_{j>-2n} \mathbb{R}v_j\right), \ y \neq 0.$$

Now what about L_y^1? If hu normalizes $T + yv_0$ then h normalizes $\mathfrak{a}$, therefore $h \in A$. Such an h in fact commutes with $T + yv_0$. But then the action of u could not normalize unless $u \in \mathbb{R}v_0$. Thus the normalizer is two dimensional and equals $A \exp \mathbb{R}v_0$. The latter does not satisfy the CI condition since its intersection with H is A, which is not compact. So the groups $\exp \mathbb{R}(T + yv_0)$, $y \neq 0$, are maximal CI having their own Lie algebras as first elements in a Jordan-Hölder sequence. But they are *not* maximal L's with the CI property in general. It is clear that that role is played by the n-dimensional groups

$$_mL_y^1 = \exp\left(\mathbb{R}(T + yv_0) + \sum_{j\neq 0} \mathbb{R}v_j\right), \ y \neq 0.$$

Next we look at L_y^2. If hu normalizes it, then h normalizes $X - Y$, so $h \in K$. But then h moves v_{2n} into $-v_{2n-2}$ and u moves $X - Y$ into $-X \cdot u + Y \cdot u$ and commutes with v_{2n}. Clearly it is impossible for hu to normalize if it is not trivial. Thus again we see that the 1- dimensional group $\exp \mathbb{R}(r(X - Y) + yv_{2n})$ cannot occur as the first element in a Jordan-Hölder sequence of an L unless it is L itself. But as before, such an L is not maximal CI. That role is obviously played by the group KV. Moreover, the latter will clearly account for the maximal L's containing any L_v^4.

Summarizing the preceding discussion, we have found three families of groups L, maximal with respect to satisfying the CI property. They are:

$$\begin{aligned}
_mL_y^1 &= \exp\left(\mathbb{R}(T + yv_0) + \sum_{j\neq 0} \mathbb{R}v_j\right), \ y \neq 0 \\
_mL^2 &= KV \\
_mL_y^3 &= \exp\left(\mathbb{R}(X + yv_{-2n}) + \sum_{j>-2n} \mathbb{R}v_j\right), \ y \neq 0.
\end{aligned}$$

The first are exponential solvable groups of dimension n; the second is a solvable group of dimension $n + 1$; and the third are nilpotent groups of dimension n.

The proof of Theorem 3.1 is concluded with

3.6 Proposition. *Each of the maximal groups $_mL_y^1, {_mL^2}, {_mL_y^3}$ satisfying the CI condition acts properly on G/H.*

Proof. We need to show that for any L of the three types, and for any $S \subset V$ a compact set, the intersection $L \cap SHS^{-1}$ is compact. This is absolutely clear in the second case because of the compactness of K. Now in the third case if we have $\exp(s(X + yv_{-2n}) + \sum_{j>-2n} y_j v_j)) = \exp sX \exp(syv_{-2n} + \dots) = uhw^{-1} = h(h^{-1} \cdot u - w)$, then $h = \exp sX$ and $h^{-1} \cdot u - w = (u_{-2n} - w_{-2n})v_{-2n} + \dots$. This implies that s can only vary over a bounded interval and the intersection is compact.

We execute a similar argument in the first case. If we have $\exp(s(T+yv_0)+\sum_{j\neq 0} u_j v_j) = \exp sT \exp(syv_0 + \dots) = uhw^{-1} = h(h^{-1} \cdot u - w)$, then $h = \exp sT$ and $h^{-1} \cdot u - w = (u_0 - w_0)v_0 + \dots$. This implies again that s can only vary over a bounded interval and the intersection is compact. □

This completes the proof of Theorem 3.1 in the case that the action of H on V is irreducible. The general case follows relatively easily from that. The details are straightforward and virtually identical to the preceding, so I omit them. I give only the classification. The maximal CI L's, which are all verified to be proper, again fall into three classes: $L^2 = KV$; $L^3_v = \exp(\mathbb{R}(X + v) + V_*)$, where v is a vector not in $\mathrm{Range}(\mathrm{ad}_V X)$ and V_* is an X-invariant complement (these are $\dim V$-dimensional nilpotent groups); and finally $L^1_v = \exp(\mathbb{R}(T + v) + V_*)$, where v is a vector that commutes with $\mathrm{ad}_V T$ and V_* is a T-invariant complement (these are $\dim V$-dimensional exponential solvable groups).

4. CI Implies Proper when $H = N_3(\mathbb{R})$

Now we turn our attention to a combination of the nilpotent and semidirect product situations, *i.e.* $G = H \ltimes V$ where $H = N$ is itself nilpotent. We shall see in the next section that, to a remarkable degree, the general semidirect product situation reduces to this special case. On the other hand, as is already implicit in [4], there is a great deal of difficulty in answering question II in this case. For the record I will state a formal conjecture (in two generalities), although the evidence for it is not conclusive. I will prove a special case in this section.

4.1 Conjecture. (a) *If $G = N \ltimes V$, a semidirect product of a simply connected nilpotent Lie group N acting unipotently on a normal vector subgroup V, then any connected Lie group $L \subset G$ which acts on G/N with the CI property acts properly.*
(b) *More generally, if $H \subset G$ are simply connected nilpotent Lie groups, the same implication is true for the action of a connected Lie subgroup L of G on G/H.*

In order to establish (4.1a) it is enough to handle the case $V = \mathbb{R}^r$, $N = N_r(\mathbb{R})$= the upper triangular real $r \times r$ matrices (see Lemma 5.5 in the next section). The case $r = 2$ follows from [4]. We do $r = 3$ here. We will use the same strategy as in [4] or the argument in the last section. We note that in this case $\mathfrak{g}$ is nilpotent, so by Lemma 3.2 any L has dimension at most 3. Moreover in its Jordan-Hölder sequence, $(0) \triangleleft \mathfrak{l}_1 \triangleleft \mathfrak{l}_2 \triangleleft \mathfrak{l}_3 = \mathfrak{l}$, we may assume each $\mathfrak{l}_j$ is actually an ideal in all of $\mathfrak{l}$.

Let us choose coordinates. We take $\mathfrak{g}$ to be the six-dimensional Lie algebra with generators $\{A, B, C, X, Y, Z\}$ satisfying the bracket relations

$$[A, B] = C,\ [A, Y] = X,\ [B, Z] = Y,\ [C, Z] = X.$$

We set $\mathfrak{n} = \mathrm{span}\{A, B, C\}$, $N = \exp \mathfrak{n}$, $V = \exp(\mathbb{R}X + \mathbb{R}Y + \mathbb{R}Z)$.

Now in the nilpotent situation, the CI property simplifies, namely $L \cap gHg^{-1}$ will be compact iff it's trivial. So (as observed in section 1) the CI condition becomes $L \cap gHg^{-1} = \{e\}$, which has a Lie algebra formulation, namely $\mathfrak{l} \cap \mathrm{Ad}g(\mathfrak{h}) = \{0\}$.

4.2 Lemma. *The variety* $\mathcal{V} = \cup_{x \in V} Adx(\mathfrak{n}) = \{W - [W, v] : W \in \mathfrak{n},\, v \in V\} = \{aA + bB + cC - (ay + cz)X - bzY : a, b, c, y, z \in \mathbb{R}\}$.

We leave it to the reader to do the simple calculation. The consequence is that L has the property CI iff $\mathfrak{l} \cap \mathcal{V} = \{0\}$. Also, if $W + u \in \mathfrak{g}$, $W \in \mathfrak{n}, u \in V$, then $W + u \notin \mathcal{V}$ iff $u \notin \mathrm{Range}(\mathrm{ad}_V W)$. We will use that repeatedly as we deploy the following strategy for classifying the maximal CI L's.

4.3 Proposition. *Let* $\mathfrak{l}_1$ *be CI.*

(i) *If* $\dim \mathfrak{l}_1 = 3$*, it is proper.*
(ii) *If* $\dim \mathfrak{l}_1 = 1$*, then* $\mathfrak{l}_1$ *can be extended to a 3-dimensional CI algebra* $\mathfrak{l}$ *with* $\dim \mathfrak{l} \cap V = 2$.
(iii) *If* $\dim \mathfrak{l}_1 = 2$ *and* $\dim \mathfrak{l}_1 \cap V \geq 1$*, then it can be extended to a 3-dimensional CI algebra* $\mathfrak{l}$ *with* $\dim \mathfrak{l} \cap V \geq 2$.
(iv) *If* $\dim \mathfrak{l}_1 = 2$ *and* $\dim \mathfrak{l}_1 \cap V = 0$*, then either* $\mathfrak{l}_1$ *is maximal or it can be extended to a 3-dimensional CI algebra* $\mathfrak{l}$ *with* $\dim \mathfrak{l} \cap V = 1$.

We note that once this proposition is proven, the proof of Theorem 4.1 comes down to showing that any group corresponding to a 2-dimensional maximal CI algebra must also act properly. That will come after the

Proof of Proposition 4.3. (i) Let $\mathfrak{l}$ be a subalgebra with $\dim \mathfrak{l} = 3$. By the CI property we have $\mathfrak{l} \cap \mathfrak{n} = \{0\}$, therefore, $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{n}$ simply by counting dimensions. But then $G = L \times N$ as manifolds and so L acts simply transitively on G/N. The result follows from Proposition 2.1.

(ii) Choose a non-zero element $W + u \in \mathfrak{l}_1$, $W \in \mathfrak{n}$, $u \in V$, $u \notin \mathrm{Range}(\mathrm{ad}_V W)$. We will find two vectors $u_1, u_2 \in V$ such that $\mathfrak{l} = \mathbb{R}(W + u) + \mathbb{R}u_1 + \mathbb{R}u_2$ is a 3-dimensional CI algebra. If $W = 0$, that this can be done is obvious, so assume $W \neq 0$. Then $\mathrm{Range}(\mathrm{ad}_V W)$ has dimension 1 or 2. Let $u_1 \in \mathrm{Range}(\mathrm{ad}_V W)$, $u_1 \neq 0$. Set $u_2 = W \cdot u_1$. If $u_2 \neq 0$, note that since $\mathfrak{g}$ is 3-step, $W \cdot u_2 = 0$. In either case, we have produced a non-zero element $\tilde{u}$ (namely u_1 if $u_2 = 0$ and u_2 otherwise) so that $\tilde{u} \in \mathrm{Range}(\mathrm{ad}_V W)$ and $W \cdot \tilde{u} = 0$. Note that the 2-dimensional algebra $\mathfrak{l}_2 = \mathbb{R}(W + u) + \mathbb{R}\tilde{u}$ has property CI because if $a(W + u) + b\tilde{u} \in \mathcal{V}$, then $a(W + u) + b\tilde{u} = W_1 - W_1 \cdot v_1$, for some $W_1 \in \mathfrak{n}$, $v_1 \in V \Rightarrow W_1 = aW$ and $au + b\tilde{u} = -W_1 \cdot v_1$. But $a \neq 0$, therefore $u + \frac{b}{a}\tilde{u} \in \mathrm{Range}(\mathrm{ad}_V W)$ which, since $\tilde{u} \in \mathrm{Range}(\mathrm{ad}_V W)$, implies $u \in \mathrm{Range}(\mathrm{ad}_V W)$, a contradiction.

Now change notation and write $u_1 = \tilde{u}$. Then $\mathfrak{l}_2 = \mathbb{R}(W + u) + \mathbb{R}u_1$ is a 2-dimensional CI algebra. Next we add a third vector. There are two cases. If $\mathrm{Range}(\mathrm{ad}_V W)$ has dimension 2, there is a u_2 in that space, not linearly dependent on u_1. Then $[W + u, u_2] = [W, u_2] = au_1 + bu_2$ for some $a, b \in \mathbb{R}$. But nilpotence guarantees that $b = 0$ and so $\mathrm{span}(W + u, u_1, u_2)$ is either abelian or a Heisenberg Lie algebra. Reasoning exactly as above, we see that it is CI since $u + \alpha u_1 + \beta u_2 \notin \mathrm{Range}(\mathrm{ad}_V W)$, for any real α, β.

It remains to deal with the case $\dim \mathrm{Range}(\mathrm{ad}_V W) = 1$. We still have $W + u$, $u \notin \mathrm{Range}(\mathrm{ad}_V W)$, $u_1 \in \mathrm{Range}(\mathrm{ad}_V W)$, $W \cdot u_1 = 0$. Choose any non-zero vector $u_2 \in V$, not in the 2-dimensional space spanned by u and u_1. Then $\mathfrak{l} = \mathbb{R}(W + u) + \mathbb{R}u_1 + \mathbb{R}u_2$ is clearly 3-dimensional. To see that it is CI we only need to check that $u + \alpha u_1 + \beta u_2 \notin \mathrm{Range}(\mathrm{ad}_V W)$ for any α, β. But if not, since $\mathrm{Range}(\mathrm{ad}_V W) = \mathbb{R}u_1$, then $u + \alpha u_1 + \beta u_2 \in \mathbb{R}u_1$, which is impossible. This finishes the proof of (ii).

(iii) Now we suppose that $\mathfrak{l}_1$ is CI, $\dim \mathfrak{l}_1 = 2$, $\dim \mathfrak{l}_1 \cap V \geq 1$. If the latter is 2, then $\mathfrak{l}_1 \subset V$ and we can take $\mathfrak{l} = V$ for the extension. Otherwise, $\dim \mathfrak{l}_1 \cap V = 1$ so that

$\mathfrak{l}_1 = \mathbb{R}(W+u) + \mathbb{R}u_1$ and $u + \alpha u_1 \notin \mathrm{Range}(\mathrm{ad}_V W)$, $\forall \alpha \in \mathbb{R}$. This puts us back in case (ii)—we may reason as in that case.

(iv) Now finally suppose that $\dim \mathfrak{l}_1 = 2$, $\mathfrak{l}_1 \cap V = \{0\}$. Then either $\mathfrak{l}_1$ is maximal with property CI or it extends to a 3-dimensional subalgebra with the CI property. Could the latter instance occur simultaneously with the property of trivial intersection with V. The answer is no because of the following reasoning. The linear projection $q : \mathfrak{l} \to \mathfrak{n}$ has kernel V. Hence by diagonalizing the $\mathfrak{n}$ component we can assume that a basis of $\mathfrak{l}$ looks like: $T_j = W_j + x_j X + y_j Y + z_j Z$, $j = 1,2,3$, where $W_1 = A$, $W_2 = B$, $W_3 = C$. But then it must be that $[T_1, T_2] = T_3$ and $[T_1, T_3] = [T_2, T_3] = 0$. Expanding those relations, we find that $z_1 = z_2 = z_3 = 0$. But this contradicts (see Lemma 3.2) the fact that the projection $p : \mathfrak{l} \to V$ is injective. Thus if $\mathfrak{l}_1$ has a 3-dimensional CI extension $\mathfrak{l}$, that extension must satisfy $\dim \mathfrak{l} \cap V = 1$.

Now we are ready for the main result of this section.

4.4 Theorem. *If L acts on G/N with property CI, then it is proper.*

Proof. It remains only to dispose of the case of a maximal CI group L satisfying $\dim L = 2$, $\dim L \cap V = 0$, namely to show that such an L must act properly. This turns out to be not so easy. No abstract argument could be found, but rather one needs to classify the possible L's and then verify specifically that each acts properly. So suppose we have such an L. Suppose it is the case that $\forall T \in \mathfrak{l}$, $T = W + u$, $W \in \mathfrak{n}, u \in V$, we have $X \in \mathrm{Range}(\mathrm{ad}_V W)$. Then $\mathfrak{l} + \mathbb{R}X$ satisfies CI. The reason for that is as follows: $\forall \alpha \in \mathbb{R}$, $u + \alpha X \notin \mathrm{Range}(\mathrm{ad}_V W)$ (since $u \notin \mathrm{Range}(\mathrm{ad}_V W)$ and $X \in \mathrm{Range}(\mathrm{ad}_V W)$) But the existence of such a 3-dimensional algebra would contradict maximality. So there must exist $W + u \in \mathfrak{l}$ such that $X \notin \mathrm{Range}(\mathrm{ad}_V W)$. Now given $W \in \mathfrak{n}$, when is $X \notin \mathrm{Range}(\mathrm{ad}_V W)$? Writing $W = aA + bB + cC$ and $u = xX + yY + zZ$, we have $W \cdot u = (ay + cz)X + bzY$. The only way $X \notin \mathrm{Range}(\mathrm{ad}_V W)$ can happen is if $b \neq 0$ and $a = 0$. Hence we conclude that there is a pair of basis vectors for $\mathfrak{l}$:

$$T_1 = \beta_1 B + \gamma_1 C + x_1 X + y_1 Y + z_1 Z,\ \beta_1 \neq 0$$
$$T_2 = \alpha_2 A + \beta_2 B + \gamma_2 C + x_2 X + y_2 Y + z_2 Z.$$

In fact α_2 must be zero. This is because $\mathfrak{l}$ is abelian so

$$0 = [T_1, T_2] \equiv -\beta_1 \alpha_2 C \pmod{V} \Rightarrow \alpha_2 = 0.$$

So now let $W_j = \beta_j B + \gamma_j C$, $u_j = x_j X + y_j Y + z_j Z$, $j = 1,2$. Then $\lambda_1 u_1 + \lambda_2 u_2 \notin \mathrm{Range}(\mathrm{ad}_V \lambda_1 W_1 + \lambda_2 W_2)$, $\forall \lambda_j \in \mathbb{R}$. Once again $\mathfrak{l}$ is abelian, so $[T_1, T_2] = 0 \Rightarrow \beta_1 z_2 = \beta_2 z_1$ and $\gamma_1 z_2 = \gamma_2 z_1$ But then $\beta_1 \gamma_2 z_1 z_2 = \beta_2 \gamma_1 z_1 z_2 \Rightarrow (\beta_1\gamma_2 - \beta_2\gamma_1) z_1 z_2 = 0$. Since $\dim \mathfrak{l} \cap V = 0$, we have $(\beta_1\gamma_2 - \beta_2\gamma_1) \neq 0 \Rightarrow z_1 z_2 = 0$. If $z_1 = 0$, then $\beta_1 z_2 = 0 \Rightarrow z_2 = 0$. Similarly $z_2 = 0 \Rightarrow z_1 = 0$ (since β_2 and γ_2 cannot both vanish). So

$$T_j = \beta_j B + \gamma_j C + x_j X + y_j Y,\ j = 1,2.$$

Next we show there is no loss of generality to assume $x_1 = y_2 = 1, x_2 = y_1 = 0$ In fact if $y_1 = 0$, then $x_1 \neq 0$ (since $T_1 \notin \mathfrak{n}$). Divide T_1 by x_1. Then if $y_2 \neq 0$ divide T_2 by it and add $-x_2 T_1$ to T_2. If $y_2 = 0$, reverse the roles of y_1 and y_2. A similar argument applies if one of x_1 or x_2 vanishes. If all four x_1, x_2, y_1, y_2 are non-zero, then divide T_2 by y_2, add

$-y_1T_2$ to T_1, divide T_1 by x_1 and add $-x_2T_1$ to T_2. All of these computations preserve the property $\beta_1\gamma_2 - \beta_2\gamma_1 \neq 0$. Thus, without loss of generality, we may assume

$$\begin{aligned} T_1 &= \beta_1 B + \gamma_1 C + X \\ T_2 &= \beta_2 B + \gamma_2 C + Y, \end{aligned}$$

with $\beta_1\gamma_2 - \beta_2\gamma_1 \neq 0$ and the algebra $\mathfrak{l} = \operatorname{span}\{T_1, T_2\}$ maximal with the CI property. We shall show that in this case, $L = \exp \mathfrak{l}$ is proper.

Now $\lambda X + \mu Y \notin \operatorname{Range}(\lambda W_1 + \mu W_2), \forall \lambda, \mu \in \mathbb{R}$. This translates into

$$\lambda X + \mu Y \notin \operatorname{Range}(\operatorname{ad}_V((\lambda\beta_1 + \mu\beta_2)B + (\lambda\gamma_1 + \mu\gamma_2)C)),\ \forall \lambda, \mu \in \mathbb{R}.$$

But for $u = xX + yY + zZ$, we have

$$(\lambda W_1 + \mu W_2)\cdot u = (\lambda\beta_1 + \mu\beta_2)zY + (\lambda\gamma_1 + \mu\gamma_2)zX.$$

Thus for every $z \in \mathbb{R}$, the system of equations

$$\begin{aligned} \lambda &= (\lambda\gamma_1 + \mu\gamma_2)z \\ \mu &= (\lambda\beta_1 + \mu\beta_2)z \end{aligned}$$

has no solutions in λ and μ. Hence

$$\gamma_1 + \frac{\mu}{\gamma}\gamma_2 \neq \frac{\lambda}{\mu}\beta_1 + \beta_2,$$

or setting $\omega = \mu/\gamma$ and clearing fractions, we have

$$\gamma_2\omega^2 + \omega(\gamma_1 - \beta_2) - \beta_1 \neq 0, \forall \omega \in \mathbb{R}.$$

This can only happen if the discriminant D is negative

$$D = (\gamma_1 - \beta_2)^2 + 4\gamma_2\beta_1 < 0. \tag{4.5}$$

We show that it is precisely the condition (4.5) which insures the properness of the action. We must prove that $L \cap SNS^{-1}$ is compact for any compact set $S \subset V$. First of all we have

$$L = \{\ell = \exp(\lambda T_1 + \mu T_2) = \exp(\lambda\beta_1 + \mu\beta_2)B \exp(\lambda\gamma_1 + \mu\gamma_2)C \exp \lambda X \exp \mu Y : \lambda, \mu \in \mathbb{R}\}.$$

If we write $S = \{\exp xX \exp yY \exp zZ : x, y, z \text{ in some fixed bounded subsets of } \mathbb{R}\}$, then for $s_1, s_2^{-1} \in S$ we have

$$\begin{aligned} s_1 n s_2 &= \exp x_1 X \exp y_1 Y \exp z_1 Z \exp cC \exp bB \exp aA \exp x_2 X \exp y_2 Y \exp z_2 Z \\ &= \exp cC \exp bB \exp aA \exp(x_1 + x_2 - ay_1 - cz_1)X \exp(y_1 + y_2 - bz_1)Y \exp(z_1 + z_2)Z. \end{aligned}$$

Therefore $\ell = s_1 n s_2$ can happen only when $a = 0$, $z_1 + z_2 = 0$ and

$$\begin{aligned} b &= \lambda\beta_1 + \mu\beta_2 \qquad & \lambda &= x_1 + x_2 - cz_1 \\ c &= \lambda\gamma_1 + \mu\gamma_2 \qquad & \mu &= y_1 + y_2 - bz_1. \end{aligned}$$

This can be reformulated as a matrix equation

$$A\begin{pmatrix}\lambda\\ \mu\end{pmatrix} = \begin{pmatrix}x_1+x_2\\ y_1+y_2\end{pmatrix},$$

where

$$A = \begin{pmatrix}1+\gamma_1 z_1 & \gamma_2 z_1\\ \beta_1 z_1 & 1+\beta_2 z_1\end{pmatrix}.$$

Note that the determinant of A is $z_1^2(\gamma_1\beta_2 - \beta_1\gamma_2) + z_1(\gamma_1+\beta_2)+1$, a quadratic in z_1 whose discriminant is exactly D. Thus A is non-singular and the set of values that

$$\begin{pmatrix}\lambda\\ \mu\end{pmatrix} = A^{-1}\begin{pmatrix}x_1+x_2\\ y_1+y_2\end{pmatrix},$$

can assume is bounded. That is $L\cap SNS^{-1}$ is compact. □

5. An Improvement of Kobayashi's Continuous Analogue of Auslander's Conjecture

Now we relate this work to Auslander's Conjecture. We will recall Kobayashi's continuous analogue—which is a theorem, not just a conjecture, then strengthen Kobayashi's result, and finally see that the two categories we have been studying in §§3,4 are different sides of the same coin. The context for Auslander's Conjecture is $G = GL(n,\mathbb{R})\ltimes\mathbb{R}^n$, $H = GL(n,\mathbb{R}^n)$, $V=\mathbb{R}^n$. Suppose that $\Gamma\subset G$ acts properly discontinuously and freely on G/H and $\Gamma\backslash G/H$ is compact. Auslander's Conjecture asserts that Γ is virtually a solvable group (*i.e.* it contains a solvable subgroup of finite index). The conjecture is still unsettled, but it has received quite a lot of attention (see*e.g.* [6],[2], and especially [3] for a nice discussion and bibliography). In [4] Kobayashi proves a strong analogue of this conjecture in the continuous case.

5.1 Theorem. *(Kobayashi) Let G be a Lie group, H a closed subgroup so that $\mathfrak{h}$ contains a maximal semisimple subalgebra of $\mathfrak{g}$ (a vacuous condition if $\mathfrak{g}$ is solvable). Then any $L\subset G$ which satisfies the CI condition (in its action on G/H) must be amenable (meaning, in particular, if G is connected, that it contains a co-compact solvable subgroup, i.e. is virtually solvable in the context of continuous groups).*

5.2 Remarks. (1) Not to belabor the obvious, but Kobayashi's theorem is not just a continuous analogue, it also drops the compactness assumption from Auslander's Conjecture. (2) Given this theorem, one speculates as to whether Auslander's Conjecture should be formulated in the same context, that is: if G is a connected Lie group, H a closed subgroup such that $\mathfrak{h}$ contains a maximal semisimple Lie subalgebra of $\mathfrak{g}$, then any $\Gamma\subset G$ acting freely and properly discontinuously with compact quotient on G/H is virtually solvable. For discrete groups, examples are known where the result is false if one drops the compact quotient.

Here is our strengthening of Kobayashi's theorem in the algebraic situation.

5.3 Theorem. *Let G be an algebraic group, H an algebraic subgroup containing a Levi factor of G. Suppose that the connected algebraic subgroup $L\subset G$ acts with the CI property on G/H. Then L is a compact extension of a unipotent group.*

Proof. By Theorem 5.1, L is amenable. Now by the CI property, $L\cap gHg^{-1}$ is compact $\forall g\in G$. Let M be any Levi component of L. Then $\exists g\in G$ such that $g^{-1}Mg\subset H$.

Hence M must be compact. Thus to prove the theorem, it is enough to assume L has no compact factors. But then $L = D \ltimes U$, D split abelian and U unipotent. We take A (as in section 3) to be a split abelian component of an Iwasawa decomposition of H. Then there must exist $g_0 \in G$ such that $g_0^{-1} D g_0 \subset A$. Then $D \subset g_0 A g_0^{-1} \Rightarrow D \subset g_0 H g_0^{-1} \cap L$ which is compact. But D has no non-trivial compact subgroups. So $D = \{e\}$ and $L = U$ is unipotent.

Now we tie together the two categories studied in §§3,4.

5.4 Theorem. *Let $G = H \ltimes V$ be algebraic, H reductive, V a vector group, and suppose H acts effectively on V. Let $L \subset G$ be an algebraic subgroup. Then to prove that CI⇒proper for the action of L on G/H, it suffices to prove the result when both L and H are unipotent. Furthermore it is no loss of generality to assume $H = N_r(\mathbb{R})$, $V = \mathbb{R}^r$, for some $r \geq 1$.*

Proof. We begin with a relatively simple

5.5 Lemma. *Let $G = H \ltimes V$ be a semidirect product, $H_1 \subset H$ a closed subgroup. Set $G_1 = H_1 \ltimes V$. Suppose $L \subset G_1$. Then the action of L on G/H is CI (resp. proper) iff the action of L on G_1/H_1 is CI (resp. proper).*

Proof. Clearly both G/H and G_1/H_1 are diffeomorphic to V, and the L action in either case is the same when transferred to V. The result is clear.

Continuing with the proof of Theorem 5.4, we suppose it has been proven that it is no loss of generality to assume L is unipotent. The next step does not require H to be reductive, only that it act effectively on V. Choose a basis in V so that $H \subset GL(r, \mathbb{R})$, $V = \mathbb{R}^r$. Moreover there is a choice of the basis so that $L \subset N_r(\mathbb{R}) \ltimes V$. Set $H_1 = H \cap N_r(\mathbb{R})$, $G_1 = H_1 \ltimes V$, and set $G_r = N_r(\mathbb{R}) \ltimes V$. Then $L \subset G_1 \subset G_r$. The assumption is that the action of L on G/H is CI. By Lemma 5.5, the action of L on G_1/H_1 is also CI. Yet another application of Lemma 5.5 says that L on $G_r/N_r(\mathbb{R})$ is CI. By the presumed step, we can conclude that L acts properly on $G_r/N_r(\mathbb{R})$. Now apply Lemma 5.5 twice again to conclude that L acts properly on G/H. Thus we have come down to proving: to show CI⇒proper for $L \subset H \ltimes V$, it is enough to assume L is unipotent.

Now let $L \subset G = H \ltimes V$ act with the property CI. By Theorem 5.3, L is a compact extension of a unipotent group $L = T \ltimes U$, T compact, U unipotent. The co-compact subgroup U of L has the property CI in its action on G/H. Therefore by the reduction assumption, it acts properly on G/H. We show finally that L itself acts properly on G/H. Let $S \subset V$ be compact. Then $U \cap SHS^{-1}$ is compact. Now replace S by $S_T = T \cdot S$, which is still compact, but also T-invariant. We must still have $U \cap S_T H S_T^{-1}$ compact. Therefore $T(U \cap S_T H S_T^{-1}) = TU \cap S_T H S_T^{-1}$ is compact. But then $TU \cap SHS^{-1} \subset TU \cap S_T H S_T^{-1}$ is compact and we are done.

Thus to prove CI⇒proper in the context of semidirect products, it is enough to assume that the group acting and the "Levi component" are unipotent. Thus the issue comes down to the truth of Conjecture 4.1a. §4 supplies some strong positive evidence. Of course I feel that the main unproven result of interest is CI⇒proper for arbitrary nilpotent groups (*i.e.* Conjecture 4.1b); but its status remains uncertain. What is certain is that its dispensation is as important for general semidirect products of reductive groups as it is for nilpotent groups themselves.

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