Optimal Stochastic Approximations and Encoding Schemes using Weyl-Heisenberg Sets

Radu Balan and Ingrid Daubechies

ABSTRACT In this chapter we study two classes of optimization problems concerning the interaction between stochastic processes and coherent Weyl-Heisenberg sets. One class involves approximation of stochastic signals, the other class refers to signal encoding for transmission in noisy channels. Both problems are studied in continuous and discrete time setting. Explicit solutions are found in Zak transform domain. The optimizers turn out to be generically ill-localized similar to the no-go Balian-Low theorem.

Keywords: stochastic signals, approximation, encoding, amalgam space

1 Introduction

Let $(g; b, a) = \{g_{mn;b,a}\ ;\ m, n \in Z\}$ denote a Weyl-Heisenberg (WH) set with window g and parameters $b > 0$ (frequency modulation) and $a > 0$ (time translation), where

$$g_{mn;b,a}(x) = e^{2\pi imbx} g(x - na). \tag{1.1}$$

When there is no danger of confusion, we denote $g_{mn;b,a}$ by g_{mn}. Our normalization of the Fourier transform is:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx\ ; \tag{1.2}$$

and of the Zak transform:

$$F(t, s) = \sqrt{a} \sum_{k \in Z} e^{2\pi ikt} f(a(s + k)). \tag{1.3}$$

For two WH sets, $(g^1; b, a)$ and $(g^2; b, a)$, we use the shorthand notation $(g^1, g^2; b, a)$. Thus $(g^1, g^2; b, a) = ((g^1; b, a), (g^2; b, a))$ is the *WH pair* composed of $(g^1; b, a)$ and $(g^2; b, a)$. Throughout this paper we implicitly assume that all the WH sets under consideration are at least Bessel sequences. Furthermore, in particular instances, we shall require the WH set is either a Riesz basis for its span (shorthanded by *s-Riesz basis*), or a frame. We formally define the *analysis operator* $T_{g^1;b,a} : f \mapsto \{\langle f, g^1_{m,n;b,a}\rangle\}_{(m,n)}$, the *synthesis operator* $T^*_{g^2;b,a} : c \mapsto \sum_{m,n} c_{m,n} g^2_{m,n;b,a}$, the *frame operator* $S_{g^1,g^2;b,a} = T^*_{g^2;b,a} T_{g^1;b,a}$ and the *Gram operator* $G_{g^1,g^2;b,a} = T_{g^1;b,a} T^*_{g^2;b,a}$. The Bessel sequence property guarantees these operators are well-defined in $L^2(\mathrm{R})$ and $l^2(\mathrm{Z}^2)$. We call $(g^1, g^2; b, a)$ a *dual pair of WH frames* when both WH sets are frames and $S_{g^1,g^2;b,a} = 1$ (the identity in $L^2(\mathrm{R})$), and we call $(g^1, g^2; b, a)$ a *biorthogonal pair of WH s-Riesz bases* when both are s-Riesz bases and $G_{g^1,g^2;b,a} = 1$ (the identity in $l^2(\mathrm{Z}^2)$). When $g^1 = g^2 = g$, we call $(g, g; b, a)$ a *WH iso-pair.*

For $(g^1, g^2; b, a)$ a WH pair of Bessel sequences and for $f \in L^2(\mathrm{R})$, the following series:

$$h = \sum_{m,n\in \mathrm{Z}} c_{mn} g^2_{mn} \tag{1.4}$$

$$c_{mn} = \langle f, g^1_{mn}\rangle \tag{1.5}$$

converges strongly in $L^2(\mathrm{R})$-sense to an element h of $L^2(\mathrm{R})$.

We shall consider two sets of situations. In the first case, the *signal encoding problem*, we work with dual pair of WH frames; we assume $f \in L^2(\mathrm{R})$ is fixed, and the c_{mn}'s coefficients are perturbed by noise, i.e. (1.5) is replaced by $c_{mn} = \langle f, g^1_{mn}\rangle + \nu_{mn}$; we are then interested in studying how close (in mean-square sense) h, defined by (1.4), is to f for optimally chosen dual pairs. In the second case, the *signal approximation problem*, we work with pairs of WH s-Riesz bases and consider f itself is a realization of a stochastic process; we are again interested in studying how well h, defined by (1.4), approximates in mean-square sense the original signal f.

To make sense of the whole setting, we have to define the right functional spaces where the stochastic processes can be realized, and also to make sure expansions of the type (1.4) converge in those functional spaces. Some non-stationary processes can be realized on spaces of finite energy: either $L^2(\mathrm{R})$, in the approximation problem case, or $l^2(\mathrm{Z}^2)$, in the encoding problem case. Yet, stationary processes on the whole line R, or index space Z^2 cannot be realized in spaces of finite energy. Instead, the Wiener amalgam space $W(L^2, l^\infty)$ is best suited for such stochastic processes. Hence we need to check convergence properties of the series (1.4) in $W(L^2, l^\infty)$. Once all these objects are well-defined in the corresponding functional spaces, we explicitly compute the mean-square error $e = \mathrm{E}[\|h - f\|^2]$, where the norm $\|\cdot\|$ is context-dependent. Our goal is to optimize this error, first over g^2 for a given g^1 (the so called *semi-optimization problem*), and then over both g^2 and g^1 (the *optimization problem*). In section 4, we show the optimizers are ill-localized in a sense similar to the well-known Balian-Low (BL), or amalgam BL theorems (see [BeHeWa95]). To obtain explicit closed-form solutions, we use the Zak transform; this requires that we restrict our attention to the case when ba is a rational number. We assume $ba = \frac{p}{q}$, where p, q are relatively prime integers.

The statements presented so far apply to continuous-time signals. They have similar counterparts in the discrete-time signal setting. More specifically, consider $(g^1, g^2; b, a)$ a WH pair of Bessel sequences and $c \in l^2(\mathrm{Z}^2)$. Then the following sequence:

$$d = \langle f, g^2_{mn} \rangle , \quad (1.6)$$

$$f = \sum_{m,n \in \mathrm{Z}} c_{mn} g^1_{mn} \quad (1.7)$$

is well-defined in $l^2(\mathrm{Z}^2)$. Now consider the case where $(g^1; b, a)$, $(g^2; b, a)$ are frames and some disturbance perturbs additively f. Such a context corresponds to a transmission channel affected by noise, where the carrier waves are g^1_{mn}'s, and the symbols are c_{mn}'s. The noise signal is modeled as a stochastic process over the space of finite power signals, $W(L^2, l^\infty)$. By using a convenient measure of the reconstructed coefficient error $||d - c||$, we evaluate the average distortion of the *discrete-time signal encoding scheme* just described, $e_{de} = \mathrm{E}_n[||c - d||^2]$. The semi-optimal and optimal problems for this case are stated in section 2.2. The second case of interest covers the case when the input signal itself, $c = (c_{mn})$, is stochastic, and $(g^1; b, a)$, $(g^2; b, a)$ are s-Riesz bases. The natural input signal representation space is $l^{\infty,\infty}(\mathrm{Z}^2)$. With a proper reconstruction error measure $||c - d||$ in $l^{\infty,\infty}(\mathrm{Z}^2)$, the approximation distortion $e_{da} = \mathrm{E}_s[||c - d||^2]$ is used in the optimization problems associated to the *discrete-time signal approximation scheme*.

Several papers in the literature have dealt with the interaction between stochastic signals and Weyl-Heisenberg coherent sets. In [Munch92], the continuous-time signal encoding problem was considered. The author studied only integral redundant frames (namely $p = 1$ and $q > 1$) and his numerical examples exhibit a discontinuous behavior of the optimal windows; he did not consider this issue further, nor did he look to obtain sub-optimal but better localized solutions. In [BaDaVa00], the continuous-time signal approximation problem was considered. There, the authors applied the solution to a multiple description encoding scheme. The approximation analysis revealed the non-localization phenomenon of the optimal solution. A context-dependent method was proposed to design a sub-optimal solution that is well-localized at the expense of a slight increase in distortion.

In the context of signal modulation analysis, the author of [Koz98] looked at the effect of noise on the discrete-time signal encoding scheme. Since the main issue was the unknown channel transfer function, the analysis was mainly restricted to the case of white noise. The design procedure to select a desirable solution follows a trial-and-error type approach.

An ill-localization phenomenon of some optimal dual windows has been remarked in [Strohmer98]. There, the author notices that for minimal support analysis windows, the minimal supported dual window exhibits a Balian-Low type effect. To avoid this problem, the author proposes different optimization criteria for the dual window design.

Even though we state and develop a one-dimensional theory, all the results can be easily carried to the higher dimensional case virtually without modification. Since the notations would become slightly more complicated, for the convenience of the exposition we preferred to stick to the simpler one dimensional notations.

2 Stochastic Processes and Statement of the Problems

2.1 Stochastic Processes and Gabor Analysis on $l^{\infty,\infty}(\mathrm{Z}^2)$ and $W(L^2, l^\infty)$

2.1.1 The $W(L^2, l^\infty)$ (continuous-time) case.

Consider a continuous-time stochastic signal f. This assumes the existence of a probability space (Ω, Σ, μ) so that realizations of this process are measurable functions $f_\omega : \mathrm{R} \to \mathrm{C}$. Statistics of f are obtained by integrating over Ω with the probability measure μ. To simplify the notation, the expectation symbol $\mathrm{E}[\cdot]$ is used instead of explicit integration. Thus the average of f is defined through:

$$\mathrm{E}[f(t)] := \int_\Omega f_\omega(t) d\mu(\omega) \tag{1.8}$$

and the autocovariance function:

$$\mathrm{E}[f(t)\overline{f(s)}] := \int_\Omega f_\omega(t)\overline{f_\omega(s)} d\mu(\omega) \tag{1.9}$$

Throughout this paper we assume the stochastic signals are zero-mean, wide-sense-stationary and have known autocovariance function $R(\tau)$. In other words, $\mathrm{E}[f(t)] = 0$ and $\mathrm{E}[f(t)\overline{f(s)}] = R(t-s)$. Inspired by the real-world situation, we assume all realizations have finite power. In this case, a natural representation space is the Wiener amalgam $W(L^2, l^\infty)$ defined by

$$W(L^2, l^\infty) := \{f : \mathrm{R} \to \mathrm{C} \mid \|f\|^2_{W(L^2,l^\infty)} := \sup_{n\in\mathrm{Z}} \int_n^{n+1} |f(x)|^2\, dx < \infty\} \tag{1.10}$$

First we make a remark about the necessity of this amalgam space. Note that (wide sense) stationary processes cannot be realized in the space of finite energy. Indeed, the average energy of such a signal is:

$$\mathrm{E}[\|f\|_2^2] = \int_{\mathrm{R}} R(0,0)dt = \infty$$

On the other hand, every stationary covariance function $R \in \mathrm{L}^1(\mathrm{R}) \cap \mathcal{F}^{-1}(L^1(\mathrm{R}))$ can be found to correspond to a stochastic process in $W(L^2, l^\infty)$ as shown in the following example ([Balan98]):

Example 2.1 Assume R is a covariance function (i.e. $\hat{R} \geq 0$). Then define the following probability space:

$$\begin{aligned} \Omega = \mathrm{R} \times \{-1,1\}, d\mu(\omega,q) &= \frac{1}{2\sqrt{2\pi}R(0)}\hat{R}(q|\omega|)d\omega \\ &= \begin{cases} \frac{1}{2\sqrt{2\pi}R(0)}\hat{R}(|\omega|), & q=+1, \\ \frac{1}{2\sqrt{2\pi}R(0)}\hat{R}(-|\omega|), & q=-1, \end{cases} \end{aligned} \tag{1.11}$$

and the stochastic signal:

$$f : \Omega \to W(L^2, l^\infty) \quad , \quad f_{\omega,q}(x) = R(0)e^{iq|\omega|x + i\frac{\pi}{2}\,\mathrm{sgn}\,(\omega)} \tag{1.12}$$

Then direct computations show that:

$$\begin{aligned} \mathrm{E}[f(x)] &= \sum_{q\in\{-1,1\}} \int_{-\infty}^{\infty} d\mu(\omega,q) f_{\omega,q}(x) = 0 \\ \mathrm{E}[f(t)\overline{f(s)}] &= \sum_{q\in\{-1,1\}} \int_{-\infty}^{\infty} d\mu(\omega,q) f_{\omega,q}(t)\overline{f_{\omega,q}(s)} = R(t-s) \end{aligned}$$

Consider now $(g; b, a)$ a WH set. We want to decompose a stochastic signal f, representable in $W(L^2, l^\infty)$, into a space of coefficients. For this amalgam space, the natural space of coefficients is the "amalgam" (or mixed-norm) space $l^{2,\infty}(\mathrm{Z}^2)$ defined by:

$$l^{2,\infty}(\mathrm{Z}^2) = \{c = (c_{mn})_{m,n\in \mathrm{Z}} \mid \|c\|^2_{l^{2,\infty}(\mathrm{Z}^2)} := \sup_n \sum_m |c_{mn}|^2 < \infty\} \tag{1.13}$$

Thus we want the analysis operator, $T : W(L^2, l^\infty) \to l^{2,\infty}(\mathrm{Z}^2)$, $T(f) = \{\langle f, g_{mn}\rangle\}_{(m,n)\in \mathrm{Z}^2}$ to be bounded and well-defined. The standard Gabor analysis started on $L^2(\mathrm{R})$ and then continued with the modulation spaces theory (see [Feich89],[Gröch00]). The analysis operator on modulation spaces (and implicitly $L^2(\mathrm{R})$) is bounded when the window g belongs to the space $M_{1,1}$ (also called the *Feichtinger algebra* $S_0(\mathrm{R})$) defined by

$$M_{1,1} = \{f \in L^2(\mathrm{R}) \mid \|f\|_{M_{1,1}} := \int d\omega \int dt\, |\langle f, g^0_{t,\omega}\rangle| < \infty\} \tag{1.14}$$

where $g^0_{t,\omega}(x) = e^{2\pi i\omega x}e^{-(x-t)^2/2}$ (see [FeiZimm98], Chapter 3). Using complex interpolation techniques, one can easily derive the boundedness of the analysis operator on amalgam spaces, as desired here. Yet, it has been long observed that on $L^2(\mathrm{R})$, a sufficient and weaker condition of boundedness is that $g \in W(L^\infty, l^1)$ ([Waln92]). In [Balan98] the author shows a similar condition is sufficient for the boundedness of the analysis operator T, between $W(L^2, l^\infty)$ and $l^{2,\infty}(\mathrm{Z}^2)$. Note that extra care has to be paid to define the convergence of the synthesis operator properly. To make things more precise, let us first introduce the definition of Wiener amalgam space $W(L^p, l^q)$

$$W(L^p, l^q) = \{f : \mathrm{R} \to \mathrm{C} \mid \|f\|^q_{W(L^p,l^q)} = \sum_{n\in\mathrm{Z}} (\int_n^{n+1} |f(x)|^p\, dx)^{q/p} < \infty\} \tag{1.15}$$

for $1 \le p, q < \infty$ and the usual change for the limiting case $p = \infty$ or $q = \infty$. Now the following result gives sufficient conditions for boundedness:

Theorem 2.2 *Assume $g^1, g^2 \in W(L^\infty, l^1)$, $b, a > 0$ and let $f \in W(L^2, l^\infty)$. Then,*

1. *The sequence*

$$T_{g^1;b,a} f := \{\langle f, g^1_{mn}\rangle\}_{m,n\in\mathrm{Z}} \tag{1.16}$$

belongs to $l^{2,\infty}(\mathrm{Z}^2)$. Moreover, there is a constant $C^1_{b,a}$ such that

$$\|T_{g^1;b,a} f\|_{l^{2,\infty}(\mathrm{Z}^2)} \le C^1_{b,a} \|g^1\|_{W(L^\infty,l^1)} \|f\|_{W(L^2,l^\infty)} \tag{1.17}$$

2. *The series*

$$S_{g^1,g^2;b,a} f := \sum_{m,n} \langle f, g^1_{mn}\rangle g^2_{mn} \tag{1.18}$$

converge unconditionally in the L^2_{loc} topology, that is for every $\varepsilon > 0$ and compact set K there are $N_\varepsilon, M_\varepsilon > 0$ such that for every finite set $S \subset \mathrm{Z}^2 \setminus ([-M_\varepsilon, M_\varepsilon] \times [-N_\varepsilon, N_\varepsilon])$,

$$\|\sum_{(m,n)\in S} \langle f, g^1_{mn}\rangle g^2_{mn}\|_{L^2(K)} < \varepsilon$$

Moreover, (1.18) converges also in weak- topology of $W(L^2, l^\infty)$, i.e. for every $h \in W(L^2, l^1)$ and $\varepsilon > 0$ there are $M_\varepsilon, N_\varepsilon > 0$ such that for every $N > N_\varepsilon, M > M_\varepsilon$*

$$\left| \langle h, f - \sum_{|m|\le M_\varepsilon} \sum_{|n|\le N_\varepsilon} \langle f, g^1_{mn}\rangle g^2_{mn}\rangle \right| < \varepsilon$$

3. *The function defined in (1.18) is in* $W(L^2, l^\infty)$ *and there is a constant* $C_{b,a} > 0$ *such that:*

$$||S_{g^1,g^2;b,a} f||_{W(L^2,l^\infty)} \leq C_{b,a} ||g^1||_{W(L^\infty,l^1)} ||g^2||_{W(L^\infty,l^1)} ||f||_{W(L^2,l^\infty)}. \tag{1.19}$$

Remark 2.3 Note that both $S_{g^1,g^2;b,a} = T^*_{g^2;b,a} T_{g^1;b,a} : W(L^2, l^\infty) \rightarrow W(L^2, l^\infty)$, the frame operator, and $T_{g^1;b,a} : W(L^2, l^\infty) \rightarrow l^{2,\infty}(\mathrm{Z}^2)$ are well-defined and bounded operators. These results will be useful for the continuous-time signal approximation and discrete-time signal encoding problems. In the former case, the data is modeled as a stochastic signal in $W(L^2, l^\infty)$, whereas in the encoding problem, the channel noise is assumed to be realized in $W(L^2, l^\infty)$.

Remark 2.4 In general the series (1.18) is not strongly convergent in the $W(L^2, l^\infty)$-norm. Indeed consider the following example (from [Balan98]):

Example 2.5 Consider $g^1 = g^2 = 1_{[0,1]}$, the characteristic function of $[0, 1]$, $b = a = 1$ and $f = 1_{\mathrm{R}}$, the constant function 1 on the entire real line. Note that $||f||_{W(L^2,l^\infty)} = 1$. Then, for each $N > 0$,

$$\sum_{|n|\leq N} \sum_{m} \langle f, g^1_{mn} \rangle g^2_{mn} = 1_{[-N,N+1]}.$$

Therefore $||f - \sum_{|n|\leq N} \sum_m \langle f, g^1_{mn}\rangle g^2_{mn}||_{W(L^2,l^\infty)} = 1$ for all N.

Summing first over n and then over m still does not lead to strong convergence of the series as can be checked with $h(x) = \sum_{m\in \mathrm{Z}} e^{2\pi i m x} 1_{[m,m+1]}(x)$.

Remark 2.6 As mentioned before, if $g^1, g^2 \in W(L^\infty, l^1)$, then $S_{g^1,g^2;b,a}$ is also bounded between $L^2(\mathrm{R})$ and $L^2(\mathrm{R})$. However, in general, even if $S_{g^1,g^2;b,a}$ is well-defined and bounded on $L^2(\mathrm{R})$, it does not need to be bounded on $W(L^2, l^\infty)$, as the following example (from [Balan98]) shows:

Example 2.7 Consider $I_n = [\frac{2^n-1}{2^n}, \frac{2^{n+1}-1}{2^{n+1}}]$, for $n \geq 0$. Define the set $E = \cup_{n\geq 0}(n + I_n)$ and the functions $g^1 = 1_E$ (the characteristic function of E) and $g^2 = 1_{[0,1]}$. For $b = a = 1$, one can easily check that $(g^1; b, a)$ and $(g^2; b, a)$ are both orthonormal bases of $L^2(\mathrm{R})$, hence Bessel sequences. Therefore $S_{g^1,g^2;b,a}$ is bounded (in fact unitary) on $L^2(\mathrm{R})$. Consider now $f = \sum_{n\geq 0} 2^{(n+1)/2} 1_{n+I_n}$ and, additionally the function $\tilde{f} = \sum_{n\geq 0} 2^{(n+1)/2} 1_{I_n}$. Note that $f \in W(L^2, l^\infty)$, $||f||_{W(L^2,l^\infty)} = 1$; moreover, for $p < 2$, $f, \tilde{f} \in L^p$; however $f, \tilde{f} \notin L^2(\mathrm{R})$. The coefficients of f with respect to $(g^1; b, a)$ are:

$$c_{mn} = \langle f, g^1_m \rangle = \delta_{n,0} \int_0^1 e^{-2\pi i m x} \tilde{f}(x) dx$$

Therefore

$$\sum_{|m|\leq M}\sum_{|n|\leq N} c_{mn} g^2_{mn} = (\sum_{|m|\leq M} e^{2\pi m y}\int_0^1 e^{-2\pi i m x}\tilde{f}(x)dx)1_{[0,1]}$$

By Plancherel's theorem we have:

$$\begin{aligned} &\|\sum_{|m|\leq M}\sum_{|n|\leq N} c_{mn} g^2_{mn}\|^2_{L^2([0,1])} \\ &\qquad = \sum_{|m|\leq M} |\int_0^1 e^{-2\pi i m x}\tilde{f}(x)dx|^2 \overset{M\to\infty}{\to} \|\tilde{f}\|^2_{L^2([0,1])} = \infty. \end{aligned}$$

Thus $S_{g^1,g^2;b,a}f$ can be defined in distributional sense (note $(c_{mn})_{m\in\mathbb{Z}} \in l^{p'}$, $\forall n$ and $p' = (1-\frac{1}{p})^{-1}$) but will not be in $W(L^2,l^\infty)$ (in fact it is not even in L^2_{loc}).

Remark 2.8 The previous example shows that one can have WH Bessel sequences even if $g^1, g^2 \notin W(L^\infty, l^1)$. In fact, one can even find $g^1, g^2 \in W(L^\infty, l^1)$ for which $S_{g^1,g^2;b,a}$ is a bounded operator on $W(L^2, l^\infty)$, as shown in the example below (from [Balan98]). The condition $g^1, g^2 \in W(L^\infty, l^1)$ in Theorem 2.2 is therefore not necessary.

Example 2.9 Consider the same partitions as before, in Example 2.7. Set

$$g^1 = \sum_{n\geq 0}\frac{1}{(n+1)^{\alpha+\frac{1}{2}}}1_{n+I_n} \qquad g^2 = 1_{[0,1]}$$

where $0 < \alpha \leq \frac{1}{2}$. Note that $g^1 \in W(L^\infty, l^2)$, but $g^1 \notin W(L^\infty, l^p)$ for any $p \leq (\alpha+\frac{1}{2})^{-1} < 2$; in particular $g^1 \notin W(L^\infty, l^1)$. We analyze now $S_{g^1,g^2;b,a}$ for $b = a = 1$. Let us consider an arbitrary $f \in W(L^2, l^\infty)$ and denote by $c_{mn} = \langle f, g^1_{mn}\rangle$. They are finite and bounded by $\|f\|_{W(L^2,l^\infty)}$. On the other hand

$$\|(S_{g^1,g^2;b,a}f)\cdot 1_{[N,N+1]}\|^2_{L^2([N,N+1])} = \sum_{m\in\mathbb{Z}} |c_{mN}|^2$$

But $c_{mn} = \langle f, g^1_{mn}\rangle = \int_0^1 e^{-2\pi i m x}(\sum_l f(x+l+n)\overline{g^1(x+l)})dx$. Therefore $\sum_m |c_{mn}|^2 = \int_0^1 |\sum_l f(x+l+N)\overline{g^1(x+l)}|^2 dx$. Note $|\sum_l f(x+l+N)\overline{g^1(x+l)}|^2 = \sum_{l\geq 0}|f(x+l+N)|^2\frac{1}{(l+1)^{1+2\alpha}}1_{I_l}(x)$, thus

$$\begin{aligned} \sum_m |c_{mn}|^2 &= \sum_{l\geq 0}\frac{1}{(l+1)^{1+2\alpha}}\int_{I_l}|f(x+l+N)|^2 dx \\ &\leq \sum_{l\geq 0}\frac{1}{(l+1)^{1+2\alpha}}\|f\|^2_{W(L^2,l^\infty)}, \end{aligned}$$

so that $\|S_{g^1,g^2;b,a}f\|_{W(L^2,l^\infty)} \leq C_\alpha \|f\|_{W(L^2,l^\infty)}$, which proves that $S_{g^1,g^2;b,a}$ is bounded on $W(L^2, l^\infty)$.

Although the converse of Theorem 2.2 is not true, the following result offers a necessary condition to have a bounded analysis, and frame operator on $W(L^2, l^\infty)$. First we give a definition.

Definition 2.10 *A function $f : \mathrm{R} \to \mathrm{C}$ is said to have* persistency length *a if there is a $\delta > 0$ and a compact set K congruent to $[0, a] \bmod a$, such that for every $x \in K$, $|f(x)| \geq \delta$.*

Theorem 2.11 ([Balan98]) *A. Let $(g; b, a)$ be a WH set such that the analysis operator $T_{g;b,a} : f \mapsto \{\langle f, g_{mn}\rangle\}_{(m,n)\in \mathrm{Z}}$ is well-defined and bounded between $W(L^2, l^\infty)$ and $l^{2,\infty}(\mathrm{Z}^2)$. Then $g \in W(L^\infty, l^2)$.*

B. Let $(g^1, g^2; b, a)$ be a WH pair such that the following hold true:

1. *For every $f \in W(L^2, l^\infty)$, the series $\sum_{mn} \langle f, g^1_{mn}\rangle g^2_{mn}$ converges unconditionally in L^2_{loc};*
2. *The frame operator $S_{g^1,g^2;b,a}$ is bounded on $W(L^2, l^\infty)$;*
3. *g^2 has persistency length $\frac{1}{b}$.*

Then $g^1 \in W(L^\infty, l^2)$.

The proof of Theorem 2.2 is fairly standard, and is based on carefully estimation of the partial sums. First the summation over the frequency index is performed using Parseval identity. Then triangle inequality and Cauchy-Schwarz are used in the second summation over the time index (see [Balan98] for details). Instead, the proof of Theorem 2.11 seems more interesting and therefore we are going to present it here.

Proof of Theorem 2.11

A. We know there exists a constant $C > 0$ such that for every $f \in W(L^2, l^\infty)$, $\sum_m |\langle f, g_{mn}\rangle|^2 \leq C\|f\|^2_{W(L^2,l^\infty)}$. Take $f = e^{-i \arg g}$. Obviously $f \in W(L^2, l^\infty)$ and $\|f\|_{W(L^2,l^\infty)} = 1$. For $m = n = 0$, $\langle f, g_{mn}\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} |g(x)|dx \leq C$. Therefore $g \in L^1(\mathrm{R})$.

Next we show $g \in L^\infty(\mathrm{R})$. Suppose the contrary, that for every $D > 0$ there is a measurable subset J of an interval of the form $[\frac{N_0}{b}, \frac{N_0+1}{b}]$ such that $|J| > 0$ and $|g(x)| > D$ for every $x \in J$. Take $f = \frac{1}{\sqrt{|J|}} e^{-i \arg g} 1_J$. Note that $\|f\|_{W(L^2,l^\infty)} \leq \|f\|_{L^2(\mathrm{R})} = 1$ and for $n = 0$,

$$\langle f, g_{mn}\rangle = \frac{1}{\sqrt{|J|}} \int_J |g(x)| e^{-2\pi i m b x} dx.$$

Then:

$$\sum_{m\in\mathrm{Z}} |\langle f, g_{mn}\rangle|^2 = \frac{1}{b} \|\frac{1}{\sqrt{|J|}} g \cdot 1_J\|^2_{L^2[\frac{N_0}{b}, \frac{N_0+1}{b}]} = \frac{1}{b|J|} \int_J |g(x)|^2 dx > D^2$$

which contradicts $\sum_m |\langle f, g_{mn}\rangle|^2 \leq C\|f\|^2_{W(L^2,l^\infty)}$. Therefore $g \in L^\infty(\mathrm{R})$.

Using Parseval identity we obtain:

$$\sum_m |\langle f, g_{mn}\rangle|^2 = \frac{1}{b}\int_0^{\frac{1}{b}} |\sum_{l\in Z} f(x+na+\frac{l}{b})\overline{g(x+\frac{l}{b})}|^2 dx$$

For $n=0$ we need to check that

$$\int_0^{1/b} |\sum_l f(x+\frac{l}{b})\overline{g(x+\frac{l}{b})}|^2 dx \leq C\|f\|^2_{W(L^2,l^\infty)}.$$

To avoid messy computation, we may take without loss of generality $b=1$. For each $n\in Z$ denote by J_n the measurable subset of $[n,n+1]$ defined by $J_n = \{x\in[n,n+1] \mid |g(x)| \geq \frac{1}{2}\|g\|_{L^\infty[n,n+\frac{1}{2}]}\}$. If $|J_n|\leq\varepsilon$, define $J_{n,\varepsilon}=J_n$; if $|J_n|>\varepsilon$, then take a subset $J_{n,\varepsilon}$ of J_n with $|J_{n,\varepsilon}|=\varepsilon$. Note that, by the definition of J_n, $|J_{n,\varepsilon}|>0$ for all n. Let N_ε be an integer such that for every $|n|<N_\varepsilon$, $|J_{n,\varepsilon}|\geq\frac{\varepsilon}{2}$. Obviously $\lim_{\varepsilon\to 0}N_\varepsilon = \infty$. Take $f=\sum_{|n|\leq N_\varepsilon} 1_{J_{n,\varepsilon}} e^{i\arg g}$. Then $\|f\|^2_{W(L^2,l^\infty)}\leq\varepsilon$ and $|\sum_l f(x+l)\overline{g(x+l)}|^2 \geq \sum_{|n|\leq N_\varepsilon}|g(x+n)|^2 1_{J_{n,\varepsilon}}(x+n)$ which implies $\int_0^1 |\sum_l f(x+l)\overline{g(x+l)}|^2 dx \geq \frac{\varepsilon}{8}\sum_{|n|\leq N_\varepsilon}\|g\|^2_{L^\infty[n,n+1]|}$. Using now the boundedness of the analysis operator $T_{g;b,a}$, we obtain that $\sum_{|n|\leq N_\varepsilon}\|g\|^2_{L^\infty[n,n+1]}\leq 8C$. Since $\lim_{\varepsilon\to 0}N_\varepsilon =$ we get $\sum_{n\in Z}\|g\|^2_{L^\infty[n,n+1]}\leq 8C$ which means $g\in W(L^\infty,l^2)$. □

B. We know that $f\mapsto\sum_{m,n}\langle f,g^1_{mn}\rangle g^2_{mn}$ is bounded on $W(L^2,l^\infty)$ and the series converges unconditionally in L^2_{loc}. We claim that $f\mapsto\sum_m\langle f,g^1_{mn}\rangle g^2_{mn}$ is uniformly bounded on $W(L^2,l^\infty)$ for every n. To see this we prove first for every compact K there is a constant $C(K)$ such that for every n, $\|\sum_m\langle f,g^1_{mn}\rangle g^2_{mn}\|_{L^2(K)}\leq C(K)\|f\|_{W(L^2,l^\infty)}$.

Indeed, for every fixed f, the sequence $\sum_{m=-M}^M\langle f,g^1_{mn}\rangle g^2_{mn}$ converges in L^2_{loc}, for $M\to\infty$. Thus it is bounded. On the other hand the partial sums of operators $S_{M,n} := \sum_{m=-M}^M\langle\cdot,g^1_{mn}\rangle g^2_{mn}$ are bounded operators, therefore by the uniform boundedness principle they are also uniformly bounded, i.e. for every M, $\|S_{M,n}\|_{B(W(L^2,l^\infty),L^2(K))}\leq C_n$ for some $C_n>0$ (here $B(W(L^2,l^\infty),L^2(K))$ denotes the Banach space of bounded operators from $W(L^2,l^\infty)$ to $L^2(K)$, endowed with operator norm). Next, for every $\varepsilon>0$ and for every $f\in W(L^2,l^\infty)$ with $\|f\|_{W(L^2,l^\infty)}=1$, there is a M_0 such that $\|\sum_{|m|>M_0}\langle f,g^1_{mn}\rangle g^2_{mn}\|_{L^2(K)}<\varepsilon$. Hence

$$\begin{aligned}\|\sum_m\langle f,g^1_{mn}\rangle g^2_{mn}\|_{L^2(K)} &\leq \|\sum_{|m|\leq M_0}\langle f,g^1_{mn}\rangle g^2_{mn}\|_{L^2(K)} + \\ &+ \|\sum_{|m|>M_0}\langle f,g^1_{mn}\rangle g^2_{mn}\|_{L^2(K)} < \varepsilon + C_n.\end{aligned}$$

Since ε was arbitrary, we get that $f\mapsto S_n(f) := \sum_m\langle f,g^1_{mn}\rangle g^2_{mn}$ is a bounded operator in $B(W(L^2,l^\infty),L^2(K))$. Next we apply again the uniform boundedness principle to the sequence of operators S_n. Each is bounded from $W(L^2,l^\infty)$ to $L^2(K)$ as we have seen. For every fixed $f \in W(L^2,l^\infty)$, the series $\sum_n S_n(f)$ converges in $L^2(K)$ therefore each term is bounded by the same constant. Thus we obtain a constant $C(K)$ such that $\|S_n\|_{B(W(L^2,l^\infty),L^2(K))} < C(K)$ for every n.

Now we return to the operator $f \mapsto \sum_m \langle f, g^1_{mn}\rangle g^2_{mn}$ on $W(L^2,l^\infty)$. Notice that

$$\|S_n\|_{B(W(L^2,l^\infty),L^2(K+a))} = \|S_{n+1}\|_{B(W(L^2,l^\infty),L^2(K))} < C(K)$$

Thus if we take $K = [0,a]$ we get immediately that

$$\|\sum_m \langle f, g^1_{mn}\rangle g^2_{mn}\|_{W(L^2,l^\infty)} \leq C\|f\|_{W(L^2,l^\infty)} \quad \text{for every } n.$$

Let K_δ and $\delta > 0$ be the compact set, respectively the positive constant from the definition of persistency for g^2; remember that K_δ is congruent to $[0,\frac{1}{b}]$ modulo $\frac{1}{b}$. Then, for every n:

$$\begin{aligned}\|\sum_m \langle f, g^1_{mn}\rangle g^2_{mn}\|_{L^2(K_\delta+na)} &= \|g^2(\cdot)\sum_m \langle f, g^1_{mn}\rangle e^{2\pi imb(\cdot+na)}\|_{L^2(K_\delta)} \\ &\geq \delta(\sum_m |\langle f, g^1_{mn}\rangle|^2)^{1/2}\end{aligned}$$

and thus $(\sum_m |\langle f, g^1_{mn}\rangle|^2)^{1/2} \leq \frac{C}{\delta}\|f\|_{W(L^2,l^\infty)}$ for every $f \in W(L^2,l^\infty)$ and $n \in \mathrm{Z}$. Now we apply the result at point A, and obtain the conclusion. □

Remark 2.12 Similar results have been obtained independently in [GrHeOk01]. There, the authors extend these results to the general (weighted) amalgam space $W(L^p,l^q_\nu)$. Again a sufficient condition for boundedness is that the window belongs to $W(L^\infty,l^1)$. Whereas boundedness on $W(L^p,l^\infty)$ (together with unconditionally convergence and persistency of g^2) implies g^1 is in $W(L^\infty,l^p)$.

Remark 2.13 The norm on $W(L^2,l^\infty)$ is often hard to compute and optimize. Instead we look at weighted-L^2 norms defined by some nonnegative weight w. Specifically we assume $w \geq 0$ has persistency a and is in $W(L^\infty,l^1)$. Typical models for such weights are characteristic functions. With such a slight change of the continuous norm, we denote by $W_a(L^2_w,l^\infty)$ the Wiener amalgam space:

$$W_a(L^2_w,l^\infty) = \{f : \mathrm{R} \to \mathrm{C}\ |\|f\|_{W_a(L^2_w,l^\infty)} := \sup_{n\in\mathrm{Z}} \int_{-\infty}^{\infty} w(x)|f(x-n\beta)|^2 dx\} \tag{1.20}$$

One can easily show (see [Balan98]) that for such weights, $W_a(L^2_w,l^\infty)$ is a Banach space norm-equivalent to $W(L^2,l^\infty)$.

2.1.2 The $l^{\infty,\infty}(\mathbb{Z}^2)$ (discrete-time) case.

Consider now a stochastic process ν over the coefficient index space $\mathbb{Z}^2$. The natural representation space is $l^{\infty,\infty}(\mathbb{Z}^2) = l^\infty(\mathbb{Z}^2)$ which is simply the space of bounded sequences over $\mathbb{Z}^2$. Thus the stochastic process ν corresponds to a map $\nu : \Omega \to l^{\infty,\infty}(\mathbb{Z}^2)$ over the probability space (Ω, Σ, μ) so that the statistics of ν are defined similarly to (1.8) and (1.9). We convene to denote by ν_ω a particular realization, that is a complex-valued sequence over $\mathbb{Z}^2$. The synthesis operator $T^*_{g;b,a}$ maps $l^{\infty,\infty}(\mathbb{Z}^2)$ into a distribution space, in general, unless g is trivial. The right distribution space is $M'_{1,1}$, where the *modulation space* $M_{1,1}$ was defined in (1.14). Indeed this is the case because the analysis operator associated to the Gaussian $g^0(x) = e^{-x^2/2}$ and sufficiently small b, a, maps $M_{1,1}$ into $l^{1,1}(\mathbb{Z}^2) = l^1(\mathbb{Z}^2)$. Then, by duality, the synthesis operator $T^*_{g^0;b,a}$ maps $l^{\infty,\infty}(\mathbb{Z}^2) = l^{1,1}(\mathbb{Z}^2)^*$ into $M^*_{1,1}$. More generally, the Gaussian window g^0 can be replaced by any function of $M_{1,1}$ without changing the space. In this case, one obtains merely an equivalent norm.

We are now interested to know when the synthesis operator $T^*_{g;b,a}$ and Gram operator $G_{g^1,g^2;b,a} := T_{g^1;b,a}T^*_{g^2;b,a}$ are bounded when acting on $l^{\infty,\infty}(\mathbb{Z}^2)$. Unlike the $W(L^2, l^\infty)$ space for continuous-time stochastic signal case, here the following results solve this problem completely:

Theorem 2.14 *A. Let $(g; b, a)$ be a WH set. Then the synthesis operator $T^*_{g;b,a} : l^{\infty,\infty}(\mathbb{Z}^2) \to M'_{1,1}$ is bounded if and only if $g \in M_{1,1}$.*

B. Let $(g^1, g^2; b, a)$ be a WH pair. If $g^1, g^2 \in M_{1,1}$ then the Gram operator $G_{g^1,g^2;b,a} : l^{\infty,\infty}(\mathbb{Z}^2) \to l^{\infty,\infty}(\mathbb{Z}^2)$ is bounded.

Proof A The first part of the statement was proved by Feichtinger (see [FeiZimm98], Theorem 3.3.1 and Corollary 3.3.2). In particular, if $g \in M_{1,1}$, $T_{g;b,a} : M_{1,1} \to l^{1,1}(\mathbb{Z}^2)$ is bounded by $\|T_{g;b,a}\|_{B(M_{1,1},l^{1,1}(\mathbb{Z}^2))} \leq C_{b,a}\|g\|_{M_{1,1}}$, and then, by duality, $T^*_{g;b,a} : l^{\infty,\infty}(\mathbb{Z}^2) \to M'_{1,1}$ is bounded by the same bound as well.

The interesting part is the converse. This seems to be new and is proved by the following argument. First we prove that if $T^*_{g;b,a} : l^{\infty,\infty}(\mathbb{Z}^2) \to M'_{1,1}$ is bounded, then $T^*_{g;b/P,a/Q} : l^{\infty,\infty}(\mathbb{Z}^2) \to M_{1,1}$ is bounded as well. To show this note first that:

$$\begin{aligned} T^*_{g;b/P,a/Q}c &= \sum_{m,n} c_{mn} g_{m,n;b/P,a/Q} = \\ &= \sum_{l=0}^{P-1}\sum_{k=0}^{Q-1}\sum_{m',n'} c_{m'P+l,n'Q+k}\, e^{2\pi i l n' \frac{ba}{P}} g^{l,k}_{m',n';b,a}, \end{aligned} \quad (1.21)$$

where $g^{l,k}(x) = e^{2\pi i l b x/P} g(x - k\frac{a}{Q})$. Then the boundedness of $T^*_{g;b,a}$ on

$l^{\infty,\infty}(\mathrm{Z}^2)$ implies:

$$|\langle T^*_{g;b,a}d, h\rangle| = |\sum_{m,n} d_{m,n}\langle g_{m,n;b,a}, h\rangle| \leq C_g \|h\|_{M_{1,1}} \|d\|_{l^{\infty,\infty}(\mathrm{Z}^2)}$$

for every $d \in l^{\infty,\infty}(\mathrm{Z}^2)$ and $h \in M_{1,1}$. In particular set

$$d_{m,n} = c_{m'P+l,n'Q+k}\, e^{2\pi i(m'+\frac{l}{P})\frac{ba}{Q}}$$

and

$$h(x) = e^{-2\pi il\frac{b}{P}x} f(x + k\frac{a}{Q}).$$

Since the $M_{1,1}$ norm is invariant to time-frequency shifts, the previous inequality turns into:

$$|\sum_{m,n} c_{m'P+l,n'Q+k}\, e^{2\pi iln'\frac{ba}{P}} \langle g^{l,k}_{m',n';b,a}, f\rangle| \leq C_g \|f\|_{M_{1,1}} \|c\|_{l^{\infty,\infty}(\mathrm{Z}^2)}$$

which shows that each of the PQ terms in (1.21) defines a bounded operator from $l^{\infty,\infty}(\mathrm{Z}^2)$ into $M'_{1,1}$.

Consider now the Gaussian window $g^0(x) = e^{-x^2/2}$. There are $b_0, a_0 > 0$ such that for every $0 < b' < b_0$ and $0 < a' < a_0$, $T_{g^0;b',a'} : M_{1,1} \to l^{1,1}(\mathrm{Z}^2)$ is bounded. Moreover, by Theorem 3.2.16 in [FeiZimm98], Chapter 3, if $g \in L^2(\mathrm{R})$ and $T_{g^0;b',a'}g \in l^{1,1}(\mathrm{Z}^2)$, then $g \in M_{1,1}$. Choose $P, Q > 1$ so that $b/P < a_0$ and $a/Q < a_0$. Now, for every $c \in l^{\infty,\infty}(\mathrm{Z}^2)$,

$$C_g\|c\|_{l^{\infty,\infty}(\mathrm{Z}^2)} \geq |\sum_{m,n} c_{m,n}\langle g_{m,n;b/P,a/Q}, g^0\rangle| = |\sum_{m,n} c'_{m,n}\langle g, g^0_{m,n;b/P,a/Q}\rangle|$$

with $c'_{m,n} = c_{m,n}e^{2\pi imn\frac{ba}{PQ}}$. Thus $\{\langle g, g^0_{m,n;b/P,a/Q}\rangle\}_{m,n}$ has to be in $l^{1,1}(\mathrm{Z}^2)$. Hence $T_{g^0;b/P,a/Q}g \in l^{1,1}(\mathrm{Z}^2)$ which shows $g \in M_{1,1}$.

B. The second statement comes immediately from Corollary 3.3.2, i)c), combined with Theorem 3.3.1, i) c) from [FeiZimm98], Chapter 3. These show that $T^*_{g^2;b\,be} : l^{\infty,\infty}(\mathrm{Z}^2) \to M'_{1,1}$ and $T_{g^1;b,a} : M'_{1,1} \to l^{\infty,\infty}(\mathrm{Z}^2)$ are both bounded, hence their composition, $G_{g^1,g^2;b,a} : l^{\infty,\infty}(\mathrm{Z}^2) \to l^{\infty,\infty}(\mathrm{Z}^2)$ is bounded as well.

Remark 2.15 An explicit computation shows that $G_{g^1,g^2;b,a}$ is bounded if and only if:

$$\sum_{m,n} |\langle g^1, g^2_{m,n;b,a}\rangle| < \infty \tag{1.22}$$

The condition $g^1, g^2 \in M_{1,1}$ guarantees just that. Naturally, one can ask whether the converse is true. In general the answer is negative, as the following example shows. Take $g^1 = g^2 = 1_{[0,1]}$ and $b = a = 1$. Then the

Gram operator is identity on $l^2(\mathrm{Z}^2)$, and therefore is identity on $l^{\infty,\infty}(\mathrm{Z}^2)$ as well (hence bounded). Yet, $1_{[0,1]}$ is not in $M_{1,1}$. It fails to be in $M_{1,1}$ because its Fourier transform decays like $\frac{1}{\omega}$, too slowly to be integrable. But it does not fail to be in $M_{1+\varepsilon,1}$, for any $\varepsilon > 0$. Note that $g^1 = g^2 = 1_{[0,1]}$ satisfies (1.22) because of exact cancellations that occur at integer values. Should we take $b < 1$ these cancellation no longer occur, and the Gram operator becomes unbounded on $l^{\infty,\infty}(\mathrm{Z}^2)$.

2.2 Models and Statement of Problems

This rather long introduction of function spaces allows us to present the stochastic optimization problems we study here. To fix the notations, denote by $(\Omega, d\mu)$ a probability space. The expectation operator E replaces the integration operator over Ω with measure $d\mu$. By *continuous-time stochastic signal* we mean a function f of $L^2(\Omega, W(L^2, l^\infty); d\mu)$. We use f also to denote a realization f_ω, when no confusion can arise. By *discrete-time stochastic signal* we mean a function c of $L^2(\Omega, l^{\infty,\infty}(\mathrm{Z}^2); d\mu)$. Again, when there is no danger of confusion, c would also denote a realization c_ω. This choice of definition for stochastic signals implies the autocovariance function for continuous-time signals, $t \mapsto R(t,t) := \mathrm{E}[|f(t)|^2]$, is in $W(L^1, l^\infty)$, and for discrete-time signals, $n \mapsto R_{m,n;m,n} := \mathrm{E}[|c_{m,n}|^2]$ is in $l^{\infty,\infty}(\mathrm{Z}^2)$, because:

$$\begin{aligned} \|R(\cdot,\cdot)\|_{W(L^1,l^\infty)} &:= \sup_n \int_n^{n+1} dx \int_\Omega |f_\omega(x)|^2 d\mu(\omega) \\ \leq \int_\Omega d\mu(\omega)(\sup_n \int_n^{n+1} |f_\omega(x)|^2 dx) &= \|f\|^2_{L^2(\Omega, W(L^2,l^\infty); d\mu)} \end{aligned} \tag{1.23}$$

and

$$\begin{aligned} \|R_{\cdot;\cdot}\|_{l^{\infty,\infty}(\mathrm{Z}^2)} &= \sup_{m,n} \int_\Omega |c_{\omega;m,n}|^2 d\mu(\omega) \\ &\leq \int_\Omega \sup_{m,n} |c_{\omega;m,n}|^2 d\mu(\omega) = \|c\|^2_{L^2(\Omega, l^{\infty,\infty}(\mathrm{Z}^2); d\mu)}. \end{aligned} \tag{1.24}$$

2.2.1 Continuous-Time Signal Approximation (CTSA)

Assume f a stationary continuous-time stochastic signal into $W(L^2, l^\infty)$ of zero average and autocovariance function $R(\cdot)$. Thus:

$$\mathrm{E}[f(t_1)\overline{f(t_2)}] = R(t_1 - t_2) \tag{1.25}$$

$$\mathrm{E}[f(t)] = 0 \tag{1.26}$$

We want to approximate f by a coherent expansion of the form $S_{g^1,g^2;b,a}f$. To distinguish among different approximation solutions, we consider a measure of the approximation error. Obviously this question is trivial when

$(g^1;b,a)$ is a frame and $(g^2;b,a)$ is a dual (in other words, when $(g^1,g^2;b,a)$ is a dual pair of WH frames). In general we are interested in the case when both $(g^1;b,a)$ and $(g^2;b,a)$ are incomplete sets, such as s-Riesz bases. When $g^1,g^2 \in W(L^\infty,l^1)$, f and $S_{g^1,g^2;b,a}f$ are both in $W(L^2,l^\infty)$, by Theorem 2.11. Consider now a nonnegative bounded summable weight $w \geq 0$, $w \in L^1(\mathrm{R}) \cap L^\infty(\mathrm{R})$. Typical such weights are characteristic functions of intervals. Then the weighted $L^2(\mathrm{R})$ norm of the approximation error measures how well $S_{g^1,g^2;b,a}f$ approximates f and its expectation is a measure of the *stochastic approximation of the continuous-time signal f by the WH pair $(g^1,g^2;b,a)$*:

$$J_{ca}(g^1,g^2;b,a,w,R) = \int_{-\infty}^{\infty} \mathrm{E}[|f(x) - S_{g^1,g^2;b,a}f(x)|^2]w(x)\,dx \qquad (1.27)$$

The optimization of J_{ca} concerns the set of problems termed as *continuous-time stochastic signal approximation problems*. These are as follows. Assume $b,a>0$ so that $ba>1$ are given.

1. (Semi-optimization Problems) For a fixed $g^1 \in W(L^\infty,l^1)$ such that $(g^1;b,a)$ is a s-Riesz basis, find the best $g^2 \in W(L^\infty,l^1)$ that minimizes J_{ca} so that $(g^2;b,a)$ is s-Riesz basis:

$$\inf_{\substack{(g^2;b,a)\, s-Riesz\ basis \\ g^2 \in W(L^\infty,l^1)}} J_{ca}(g^1,g^2;b,a,w,R) \quad , \quad \text{given } (g^1;b,a) \qquad (1.28)$$

   Conversely, for a fixed g^2, find the best g^1 that minimizes J_{ca}.

2. (Optimization Problem) Find the best WH pairs of s-Riesz bases $(g^1,g^2;b,a)$:

$$\inf_{\substack{(g^1,g^2;b,a)\, pair\ of\ s-Riesz\ bases \\ g^1,g^2 \in W(L^\infty,l^1)}} J_{ca}(g^1,g^2;b,a,w,R) \qquad (1.29)$$

Remark 2.16 These approximation problems are very much of the same type as the standard Karhunen-Loeve approximation problems. In fact the measure we use is merely an extension of the mean-square measure used in KL decompositions. What is non-trivial is the structure of the approximation. While it is true that in finite dimensional spaces, or for compact domains (and therefore periodic signals), the KL problem turns into an eigenproblem for the covariance operator, it is not a priori clear what constraint the WH set structure imposes on the solution. In fact it is not obvious that any of these problems have minimizers (i.e. solutions that satisfy the constraints).

Remark 2.17 Theorem 2.11 allows us to perform freely the usual algebraic manipulations: permutation of summations symbols and commutation of bounded operators and summations. Moreover, J_{ca} is bounded above by the $W_a(L^2_w, l^\infty)$-norm from (1.20), and in turn, by a constant times $W(L^2, l^\infty)$-norm of f. More specifically, the criterion (1.27) is always bounded by:

$$\begin{aligned} J_{ca}(g^1, g^2; b, a, w, R) \quad \leq \quad & C_a(1 + C_{b,a}\|g^1\|_{W(L^\infty, l^1)}\|g^2\|_{W(L^\infty, l^1)})^2 \cdot \\ & \cdot \|w\|_{L^1}\|f\|^2_{L^2(\Omega, W(L^2, l^\infty); d\mu)}. \end{aligned} \tag{1.30}$$

Remark 2.18 All the derivations we perform here, apply equally well to non-stationary signals as well. For the sake of simplicity of notation we consider only the stationary case. An interesting non-stationary case is when $R(t_1, t_2) = \mathrm{E}[f(t_1)\overline{f(t_2)}]$ is in $L^2(\mathrm{R}^2)$ and $w = 1$.

Remark 2.19 An alternate measure of the approximation error is given by the *average power*:

$$P_{ca} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{E}[|f(x) - S_{g^1, g^2; b, a} f(x)|^2] dx \tag{1.31}$$

As we prove later, this criterion is equivalent to (1.27) for $w = \frac{1}{a}1_{[0,a]}$. The equivalence can be carried over even for non-stationary signals. In that case P_{ca} turns out to be equivalent to J_{ca} associated to a fictitious cyclostationary process whose covariance function is given by $R_c(t_1, t_2) = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{k=-K}^{K} R(t_1 + ka, t_2 + ka)$ and weight $w = 1_{[0,a]}$.

2.2.2 Discrete-Time Signal Approximation (DTSA)

Consider the following scenario: Assume we are given a stochastic signal over $l^{\infty,\infty}(\mathrm{Z}^2)$, c, and we want to approximate it using a synthesis-analysis pair of WH sets, $(g^1, g^2; b, a)$. More specifically we want to approximate it by $G_{g^1, g^2; b, a} c$, the Gram operator associated to the WH pair $(g^1, g^2; b, a)$. The problem is nontrivial for the case when $(g^1; b, a)$ and $(g^2; b, a)$ are overcomplete sets. To fix the notations, assume $b, a > 0$ are given so that $ba < 1$. Assume c is a stationary stochastic signal of zero average and known covariance function:

$$\mathrm{E}[c_{mn}] \quad = \quad 0 \tag{1.32}$$

$$\mathrm{E}[c_{m_1 n_1}\overline{c_{m_2 n_2}}] \quad = \quad R_{m_1 - m_2, n_1 - n_2} \tag{1.33}$$

Choose nonnegative summable weights $w_{mn} \geq 0$, $\sum_{m,n} w_{mn} < \infty$ and define the approximation error as:

$$J_{da}(g^1, g^2; b, a, w, R) = \sum_{m,n\in\mathrm{Z}} \mathrm{E}[|c_{mn} - G_{g^1, g^2; b, a} c_{mn}|^2] w_{mn} \tag{1.34}$$

The set of problems we consider in this paper are the following:

1. (Semi-Optimal Pairs) For a given frame $(g^1;b,a)$, with $g^1 \in M_{1,1}$, find the frame $(g^2;b,a)$ that minimizes (1.34), that is

$$\inf_{\substack{(g^2;b,a)\,frame\\ g^2\in M_{1,1}}} J_{da}(g^1,g^2;b,a,w,R) \quad , \quad given\ (g^1;b,a) \tag{1.35}$$

   and conversely, for a given frame $(g^2;b,a)$, $g^2 \in M_{1,1}$, find the best frame $(g^1;b,a)$, $g^1 \in M_{1,1}$ that minimizes J_{da}.

2. (Optimal Pair) Find the best WH pairs of frames $(g^1,g^2;b,a)$:

$$\inf_{\substack{(g^1,g^2;b,a)\,pair\ of\ frames\\ g^1,g^2\in M_{1,1}}} J_{da}(g^1,g^2;b,a,w,R) \tag{1.36}$$

Remark 2.20 The same remarks as in the continuous-time case, apply here. Again Theorem 2.14 allows us to freely perform the usual algebraic manipulations: permutation of summations symbols, commutation of bounded operators and summations. Moreover, J_{da} is bounded above by:

$$\begin{aligned} J_{da}(g^1,g^2;b,a,w,R) \quad &\leq \quad C_a(1+C_{b,a}\|g^1\|_{M_{1,1}}\|g^2\|_{M_{1,1}})^2\cdot \\ &\cdot\|w\|_{l^{1,1}(\mathrm{Z}^2)}\|c\|^2_{L^2(\Omega,l^{\infty,\infty}(\mathrm{Z}^2);d\mu)}. \end{aligned} \tag{1.37}$$

An alternate measure of the approximation error is given by the *average power*:

$$P_{da} = \lim_{M,N\to\infty}\frac{1}{(2M+1)(2N+1)}\sum_{|n|\leq N}\sum_{|m|\leq M}\mathrm{E}[|c_{mn}-(G_{g^1,g^2;b,a}c)_{mn}|^2] \tag{1.38}$$

As we prove later, this criterion is equivalent to (1.34) for $w_{mn} = 1/q$ for $m = 0$ and $0 \leq n \leq q-1$, and $w_{mn} = 0$ otherwise.

2.2.3 Continuous-Time Signal Encoding (CTSE)

Consider now the following scenario (see Figure 1). A continuous-time finite energy signal $f \in L^2(\mathrm{R})$ is encoded using a WH frame $(g^1;b,a)$. The coefficients $c_{mn} = \langle f, g^1_{mn}\rangle$ are sent through a communication channel and received perturbed by additive noise $\tilde{c}_{mn} = c_{mn} + \nu_{mn}$. For reconstruction, a dual frame $(g^2;b,a)$ is used and the obtained signal is $\tilde{f} = \sum_{mn}\tilde{c}_{mn}g^2_{mn}$. Because $(g^1,g^2;b,a)$ is a dual frame pair, the decoded signal becomes:

$$\tilde{f} = f + \sum_{mn}\nu_{mn}g^2_{mn}$$

and the transmission error is then:

$$\varepsilon := \tilde{f} - f = \sum_{mn} \nu_{mn} g^2_{mn} \tag{1.39}$$

In general, when ν_{mn} is a stationary process, thus in $l^{\infty,\infty}(\mathrm{Z}^2)$, ε is not a function, but a distribution in $M'_{1,1}$. We introduce a convenient measure on the error (1.39). Assume only a finite number of coefficients are perturbed by noise, say those for $|n| \leq N$, $|m| \leq M$. Then we compute the average mean square error per coefficient and then we take the limit for $M, N \to \infty$. More specifically, consider (d_{mn}) a zero average and (wide-sense) stationary discrete signal over Z^2:

$$\mathrm{E}[d_{mn}] = 0 \quad \mathrm{E}[d_{mn}\overline{d_{m'n'}}] = R_{m-m',n-n'} \tag{1.40}$$

Then assume:

$$\nu^{MN}_{mn} = \begin{cases} \frac{1}{\sqrt{(2N+1)(2M+1)}} d_{mn} & for \quad |m| \leq M, \ |n| \leq N \\ 0 & otherwise. \end{cases} \tag{1.41}$$

and define the *transmission error* measure:

$$J_{ce}(g^1, g^2; b, a, R) = \lim_{M,N\to\infty} \mathrm{E}[\| f - \tilde{f} \|^2_{L^2(\mathrm{R})}] \tag{1.42}$$

We are interested in minimizing J_{ce} under a series of constraints:

1. (Semi-Optimization Problem) For a given encoding window g^1, find the best dual frame pair $(g^1, g^2; b, a)$ that minimizes J_{ce}, i.e.

$$\inf_{g^2,(g^1,g^2;b,a) \ dual \ pair} J_{ce}(g^1, g^2; b, a, R) \tag{1.43}$$

2. (Iso-Pairing Problem) Find the best dual frame pair of the form $(g, g; b, a)$, i.e.

$$\inf_{g,(g,g;b,a) \ dual \ pair} J_{ce}(g, g; b, a, R) \tag{1.44}$$

3. (Optimization Problem) Find the best norm-constrained encoder g^1 and its associated optimal decoder, i.e.

$$\inf_{g^1,||g^1||=1} \ \inf_{g^2,(g^1,g^2;b,a) \ dual \ pair} J_{ce}(g^1, g^2; b, a, R) \tag{1.45}$$

Remark 2.21 1. Note that $\sum_{mn} d_{mn} g^2_{mn}$ is in general not a function. It is a distribution in $M'_{1,1}$ because of Theorem 2.14, but the series is not convergent in any other way (pointwise or locally on a compact in L^p-sense). On the other hand, using only a weighted L^2 norm instead of L^2-norm to

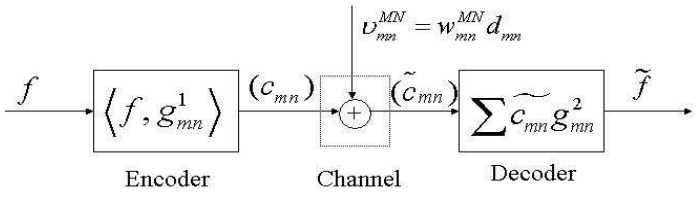

FIGURE 1. Continuous-Time Signal Encoding Scheme

measure the error, would not solve the problem. Hence the need of using finite set of coefficients through ν_{mn} and then taking the average.

2. We consider dual frame pairs because we want unbiased estimators of the original signal f. In general, given more information about the source, one can search the solution over arbitrary pairs of frames.

3. The iso-pairing problem can be equivalently stated as the optimal normalized tight frame problem for the criterion J_{ce}.

4. In the optimal problem a norm constraint is required. Indeed, if no such constraint is posed, the optimal solution would correspond to an infinite energy encoding window g^1, and a zero energy decoding window g^2 which would make $J_{ce} = 0$. But this is not relevant from a practical point of view.

5. Note the covariance operator of $(\nu^{M,N}_{mn})_{mn}$ is trace-class with trace independent of (hence uniformly bounded over) M, N. Hence it is sufficient to assume $(g^2; b, a)$ is Bessel sequence to obtain:

$$J_{ce}(g^1, g^2; b, a, R) \leq C_{b,a} \|S_{g^2;b,a}\|_{L^2(\mathrm{R})} \|d\|^2_{L^2(\Omega, l^{\infty,\infty}(\mathrm{Z}^2); d\mu)} \tag{1.46}$$

2.2.4 Discrete-Time Signal Encoding (DTSE)

Similar to the Continuous-Time Signal Encoding scheme, consider now the discrete version of that problem. The *Discrete-Time Signal Encoding* scheme, pictured in Figure 2, assumes the following scenario. The input data is given by the sequence $(c_{mn})_{mn}$ in $l^2(\mathrm{Z}^2)$; it is encoded using the synthesis operator associated to a s-Riesz basis $(g^1; b, a)$, into the continuous-time signal $f = \sum_{mn} c_{mn} g^1_{mn}$; next f passes through a communication channel where is perturbed additively by the continuous-time noise ν, $\tilde{f} = f + \nu$, and it is decoded using the biorthogonal window g^2, $\tilde{c}_{mn} = \langle \tilde{f}, g^2_{mn} \rangle$. Because $(g^1, g^2; b, a)$ is a biorthogonal pair, the reconstructed signal is:

$$\tilde{c}_{mn} = c_{mn} + \langle \nu, g^2_{mn} \rangle$$

The transmission error is then:

$$\varepsilon_{mn} = \tilde{c}_{mn} - c_{mn} = \langle \nu, g^2_{mn} \rangle \tag{1.47}$$

We consider two measures of the transmission error, and we show later they are equivalent. First we assume a disturbance model similar to the continuous-time case, namely

$$\nu^T(t) = \frac{1}{\sqrt{2T}} 1_{[-T,T]} d(t) \tag{1.48}$$

where $d(\cdot)$ is a continuous-time zero-mean, (wide sense) stationary stochastic signal with autocovariance function R,

$$\mathrm{E}[d(t)] = 0 \quad \mathrm{E}[d(t)\overline{d(t')}] = R(t - t') \tag{1.49}$$

The *average transmission error* is defined by:

$$J_{de}(g^1, g^2; b, a, R) = \lim_{T \to \infty} \sum_{m,n} \mathrm{E}[|\tilde{c}_{mn} - c_{mn}|^2] \tag{1.50}$$

a second measure of the transmission error is given by the *average distortion per coefficient* defined as follows. Assume the channel perturbation is $\nu = d$ and define now the distortion by:

$$P_{de}(g^1, g^2; b, a, R) = \lim_{M,N \to \infty} \frac{1}{2N+1} \sum_{|m| \leq M} \sum_{|n| \leq N} \mathrm{E}[|\tilde{c}_{mn} - c_{mn}|^2] \tag{1.51}$$

Remarkably, the two transmission error measures are identical, as we prove later.

We state now the optimization problems in terms of J_{de}.

1. (Semi-Optimization Problem) For a given encoder $(g^1; b, a)$, find the best decoder g^2, so that $(g^1, g^2; b, a)$ is a biorthogonal s-Riesz basis pair, i.e.

$$\inf_{g^2, (g^1, g^2; b, a) \ \textit{biorthogonal pair}} J_{de}(g^1, g^2; b, a, R) \tag{1.52}$$

2. (Iso-Pair Problem) Find the best biorthogonal pair $(g, g; b, a)$, i.e.

$$\inf_{g, (g, g; b, a) \ \textit{biorthogonal pair}} J_{de}(g, g; b, a, R) \tag{1.53}$$

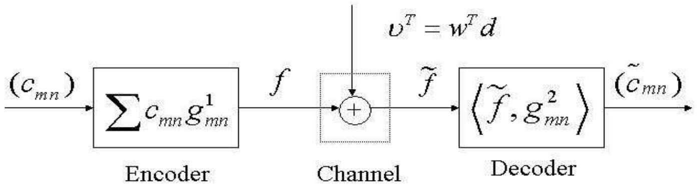

FIGURE 2. Discrete-Time Signal Encoding Scheme

3. (Optimization Problem) Find the best norm-constrained encoder, that is $(g^1, g^2; b, a)$ biorthogonal s-Riesz basis pair and $||g^1|| = 1$ that minimizes J_{de},

$$\inf_{g^1, ||g^1||=1} \; \inf_{g^2, (g^1,g^2;b,a) \; biorthogonal \; pair} J_{de}(g^1, g^2; b, a, R) \tag{1.54}$$

Remark 2.22 1. As discussed in subsection 2.2 we assume the stationary signal d is realized in $W(L^2, l^\infty)$, and for $g^2 \in W(L^\infty, l^1)$ the decoder always produces an output in $l^{2,\infty}(\mathrm{Z}^2)$. In the first case, $\nu = \frac{1}{\sqrt{2T}} 1_{[-T,T]} d$ has finite average energy, $\mathrm{E}[||\nu||^2_{L^2}] = R(0) < \infty$ and thus J_{de} is well defined for every T. In the latter case, for $\nu = d$, the output has finite power and thus the average distortion (1.51) makes perfect sense.

2. The iso-pair obviously corresponds to an orthogonal WH set. Thus, the iso-pair problem asks for the best orthogonal WH set $(g; b, a)$ with respect to the criterion (1.50).

Table 1.1 summarizes the class of problems we have introduced.

3 Semi-optimal and Optimal Solutions

In this section we compute explicitly the criteria (1.27, 1.34,1.42,1.50). To this end we need several important results: a summation result obtained as a variation of Parseval relation, matrix computations involving the Zak transform and solutions of some matrix optimization problems.

Problem	Input Data	Operation Performed	Type of Error	Cause of Error
CTSA	stochastic function	representation by coeffs in s-Riesz basis	approximation error after reconstruction	incompleteness of s-Riesz basis
DTSA	stochastic sequence	representation by function constructed using a frame	"	frame coeffs do not span $l^2(\mathrm{Z}^2)$
CTSE	deterministic function	analysis and synthesis using a frame	reconstruction error after transmission over noisy channel	stochastic channel noise
DTSE	deterministic sequence	synthesis and analysis using a s-Riesz basis	"	"

TABLE 1.1. Classification of the Stochastic Problems.

3.1 The Weak Poisson Summation Formula

The computations we perform use a special form of the Poisson summation formula. Actually, in our framework it is merely a consequence of the Parseval identity. We call it the *weak form of the Poisson summation formula.* It has been proved and used by many authors before (see the proof of Theorem 4.1.5 in [HeWa89], Theorem 2 in [Chui93], or Lemma 3.2 in [DaLaLa96]). The result that follows has been proposed and proved in in [BaDaVa00] (Appendix C):

Theorem 3.1 *Suppose $f_1, f_2 \in W(L^2, l^\infty)$ and $g_1, g_2 \in W(L^\infty, l^1)$. Then*

$$\sum_m \int\int f_1(x)g^1(x)f_2(y)g^2(y)e^{2\pi imb(x-y)} =$$
$$= \frac{1}{b}\sum_m \int dx f_1(x)g^1(x)f_2(x+\frac{m}{b})g^2(x+\frac{m}{b}) \qquad (1.55)$$

and the integrals converge absolutely.

Remark 3.2 The products f_1g^1 and f_2g^2 can be replaced, equivalently, by h_1, h_2 in $W(L^2, l^1)$. Note that $W(L^2, l^\infty) \cdot W(L^\infty, l^1) \equiv W(L^2, l^1)$, where the equivalence should be understood in the following sense: if $f \in W(L^2, l^\infty)$ and $g \in W(L^\infty, l^1)$, then $fg \in W(L^2, l^1)$ and $||fg||_{W(L^2,l^1)} \leq ||f||_{W(L^2,l^\infty)}||g||_{W(L^\infty,l^1)}$; conversely, any function $h \in W(L^2, l^1)$ can be factorized as a product $h = fg$ with $f \in W(L^2, l^\infty)$, $||f||_{W(L^2,l^\infty)} \leq ||h||_{W(L^2,l^1)}$ and $g \in W(L^\infty, l^1)$, $||g|| \leq ||h||_{W(L^2,l^1)}$. For instance, for $x \in [n, n+1)$ define $f(x) = h(x)/||h||_{L^2[n,n+1]}$ and $g(x) = ||h||_{L^2[n,n+1]}$, if $||h||_{L^2[n,n+1]} \neq 0$, and $f(x) = 0, g(x) = 0$ otherwise.

3.2 Certain Matrix Optimization Problems

A. Consider the minimization problem of a functional of the type:

$$I(X,A) = Tr\ \{S(I - XA)R(I - A^*X^*)\} \tag{1.56}$$

with S, R selfadjoint invertible matrices. To fix the notation, let assume X is a $p \times q$ matrix, A is a $q \times p$ matrix and S, R are $p \times p$ strictly positive matrices. To make it nontrivial, assume $rank(A) = q < p$. For such a problem, the optimal X is given by:

$$X_o = RA^*(ARA^*)^{-1} \tag{1.57}$$

and the optimal value is:

$$I(A) := I(X_o, A) = Tr\ \{R^{1/2}SR^{1/2}(I - R^{1/2}A^*(ARA^*)^{-1}AR^{1/2})\} \tag{1.58}$$

Moreover, $I(A)$ can be further optimized over A by noting that $P = R^{1/2}A^*(ARA^*)^{-1}AR^{1/2}$ is an orthogonal projection. Hence the optimal A should correspond to an eigenspace associated to the largest q eigenvalues of $R^{1/2}SR^{1/2}$, say $\lambda_1, \ldots, \lambda_q$. Note the eigenvalues of $R^{1/2}SR^{1/2}$ are the same with the eigenvalues of RS. Denote by P such an orthogonal projection. It is uniquely defined when the q^{th} eigenvalue of $R^{1/2}SR^{1/2}$ is nondegenerate. Let $\{v_1, \ldots, v_q\}$ be an orthonormal basis in $Ran(P)$ (for instance the first q eigenvectors of $R^{1/2}SR^{1/2}$) and denote by V the $p \times q$ matrix whose columns are these vectors. Thus $P = VV^*$, $V^*V = I_q$ and $R^{1/2}SR^{1/2}V = V \cdot diag(\lambda_1, \cdots, \lambda_q)$. Then any optimal pair (A, X) has the form:

$$A_{opt} = LV^*R^{-1/2} \quad X_{opt} = R^{1/2}VL^{-1} \tag{1.59}$$

where L is an arbitrary $q \times q$ invertible matrix, and the optimal value becomes:

$$I_{min} = \sum_{k=q+1}^{p} \lambda_k \tag{1.60}$$

B. Consider now the following functional:

$$I(X) = Tr\ \{XRX^*\} \tag{1.61}$$

where $X \in \mathrm{C}^{p\times q}$ and $R \in \mathrm{C}^{q\times q}$, $R > 0$ and $p < q$, subject to the constraint

$$AX^* = pI_p \tag{1.62}$$

where $A \in \mathrm{C}^{p\times q}$. We want to minimize $I(X)$ subject to (1.62). The Lagrange functional is

$$L(X,\Lambda) = Tr\ \{XRX^* - \Lambda(AX^* - pI_p) - (XA^* - pI_p)\Lambda^*\}$$

and solving for the stationary points we obtain

$$X_o = p(AR^{-1}A^*)^{-1}AR^{-1}, \tag{1.63}$$

and the criterion becomes

$$I_o(X_0) = p^2 Tr\ \{(AR^{-1}A^*)^{-1}\} \tag{1.64}$$

Since we assumed $R > 0$, this is the global minimum point.

Next consider again the functional (1.61) subject to the constraint

$$XX^* = pI_p \tag{1.65}$$

Clearly the minimum of $I(X)$ is achieved when the columns of X^* form an orthonormal basis in the p-dimensional invariant space of R associated to the lowest eigenvalues. Then the solution is

$$X = \sqrt{p}UV^* \tag{1.66}$$

where U is a $C^{p\times p}$ unitary matrix and V is the $C^{q\times p}$ complex matrix whose columns are the p eigenvectors of R corresponding to the lowest eigenvalues $\lambda_{q-p+1}, \dots, \lambda_q$, so that $V^*V = I_p$. The optimal value of the criterion is then

$$I(X) = p \sum_{k=q-p+1}^{q} \lambda_k \tag{1.67}$$

Now consider (1.61,1.62) where X, A, R are matrix-valued functions over a domain D, and we define

$$J(A, X) = \int_D Tr\ \{XRX^*\} \tag{1.68}$$

where X is subject to (1.62) at every point of D, and A is constrained by

$$\int_D Tr\ \{AA^*\} = 1 \tag{1.69}$$

We want now to minimize J over these two constraints. The minimization over X has been already carried out before, and the criterion turned out (1.64), that is

$$J(A, X_o) = p^2 \int_D Tr\ \{(AR^{-1}A^*)^{-1}\}$$

Now we want to optimize further over A, subject to (1.69). The optimization decouples into two steps. First, at each point of D we have to minimize (1.64) subject to $Tr\ \{AA^*\} = c^2$, for some yet unknown real-valued function c defined over D. The solution is that A has to correspond to the invariant space of R associated to the lowest p eigenvalues. Thus

$$A = cUV^*$$

where $V \in \mathbf{C}^{q\times p}$ is the matrix whose columns are the p normalized eigenvectors of R corresponding to the lowest eigenvalues, say $\lambda_{q-p+1}, \ldots, \lambda_q$ so that $V^*V = I_p$, and $U \in \mathbf{C}^{p\times p}$ is a unitary matrix. Now, J turns into:

$$J(c) = p^2 \int_D \frac{1}{c^2} \sum_{k=q-p+1}^{q} \lambda_k \quad , \quad subject\ to \int_D p\cdot c^2 = 1$$

Optimizing further over c, we obtain

$$c_{opt} = \frac{(\sum_{k=q-p+1}^{q} \lambda_k)^{1/4}}{(p\int_D(\sum_{k=q-p+1}^{q} \lambda_k)^{1/2})^{1/2}}, \tag{1.70}$$

The optimizer turns into:

$$A_{opt} = c_{opt}UV^* \tag{1.71}$$

$$X_{opt} = \frac{p}{c_{opt}}UV^* \tag{1.72}$$

and the optimal criterion becomes:

$$J_{opt} = p^3(\int_D(\sum_{k=q-p+1}^{q} \lambda_k)^{1/2})^2 \tag{1.73}$$

3.3 Zak Transform

For a function $g \in L^2(\mathbf{R})$, we use the *Zak transform* as normalized in [BaDaVa00]:

$$G(t,s) = \sqrt{a}\sum_{k\in\mathbf{Z}} e^{2\pi ikt} g(a(s+k)) \tag{1.74}$$

For more information on the Zak transform we refer the reader to [Jans82] and [Jans88]. We recall here the inversion formulae in time and frequency domain:

$$g(x) = \frac{1}{\sqrt{a}}\int_0^1 G(t,\frac{x}{a})dt \quad , \quad \hat{g}(\xi) = \sqrt{\frac{a}{2\pi}}\int_0^1 e^{-ias\xi}G(-\frac{a\xi}{2\pi}, s)ds \tag{1.75}$$

and the quasiperiodicity relations:

$$G(t+1,s) = G(t,s) \quad , \quad G(t,s+1) = e^{-2\pi it}G(t,s) \tag{1.76}$$

Assume $ba = \frac{p}{q}$ with p,q relatively prime. Then we denote by $\Gamma(t,s)$ the $p\times q$ matrix whose (j,k) entry is $G(t+\frac{k}{q}, s+j\frac{q}{p})$, $j = 0,1,\ldots,p-1$,

$k = 0, 1, \ldots, q-1$:

$$\Gamma(t,s) = \begin{bmatrix} G(t,s) & G(t+\frac{1}{q},s) & \cdots & G(t+\frac{q-1}{q},s) \\ G(t,s+\frac{q}{p}) & G(t+\frac{1}{q},s+\frac{q}{p}) & \cdots & G(t+\frac{q-1}{q},s+\frac{q}{p}) \\ \vdots & \vdots & & \vdots \\ G(t,s+(p-1)\frac{q}{p}) & G(t+\frac{1}{q},s+(p-1)\frac{q}{p}) & \cdots & G(t+\frac{q-1}{q},s+(p-1)\frac{q}{p}) \end{bmatrix} \tag{1.77}$$

In general we denote the time domain functions by lower case letters $(f, g, \ldots)$, the Zak domain functions by upper case letters $(F, G, \ldots)$, and the matrix representation (1.77) by Greek upper case letters $(\Phi, \Gamma, \ldots)$. Note $g \mapsto \Gamma$ is a unitary operator mapping $L^2(\mathrm{R})$ into $L^2(\Box, \mathrm{C}^{p\times q})$, where $L^2(\Box, \mathrm{C}^{p\times q})$ is the Hilbert space of $p \times q$ complex valued functions defined over rectangle $\Box = [0, \frac{1}{q}] \times [0, \frac{1}{p}]$ endowed with inner product, $\Phi, \Gamma \in L^2(\Box, \mathrm{C}^{p\times q})$,

$$\langle \Phi, \Gamma \rangle = \int\int_{\Box} Tr\ \{\Phi\Gamma^*\} dt\, ds \tag{1.78}$$

where $*$ stands for hermitian conjugation ($M^* = \bar{M}^T$). Let $E(t) \in \mathrm{C}^{p\times p}$, $Q, D \in \mathrm{C}^{q\times q}$ be defined by:

$$E(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ e^{-2\pi i q t} & 0 & 0 & \cdots & 0 \end{bmatrix} \ ,\ Q = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{1.79}$$

$$D = \begin{bmatrix} 1 & & & & \\ & e^{-2\pi i \frac{1}{q}} & & & \\ & & e^{-2\pi i \frac{2}{q}} & & \\ & & & \ddots & \\ & & & & e^{-2\pi i \frac{q-1}{q}} \end{bmatrix}. \tag{1.80}$$

Then the quasiperiodicity relations (1.76) turn into:

$$\Gamma(t+\frac{1}{q}, s) = \Gamma(t,s)\cdot Q \ \ ,\ \ \Gamma(t, s+\frac{1}{p}) = e^{-2\pi i n_0 t} E(t)^{r_0} \cdot \Gamma(t,s) \cdot D^{n_0} \tag{1.81}$$

where (r_0, n_0) are coprime factors of (q, p), i.e. $r_0 q + n_0 p = 1$. In particular, (1.80) and (1.81) are useful in checking the consistency of design procedure results.

We recall the following result (see [ZiZe97, BaDaVa00] for details).

Theorem 3.3 *1. $(g^1, g^2; b, a)$ is dual pair of frames if and only if*

$$\Gamma^1 \Gamma^{2*} = pI_p \tag{1.82}$$

for a.e. $(t,s) \in \square = [0, \frac{1}{q}] \times [0, \frac{1}{p}]$.

2. $(g^1; b, a)$ is a biorthogonal pair of S-Riesz bases if and only if

$$\Gamma^{1*} \Gamma^2 = pI_q \tag{1.83}$$

for a.e. $(t,s) \in \square$. $\square$

Now, without further ado, we start analyzing the stochastic criteria.

3.4 Continuous Time Signal Approximation Problem

Consider J_{ca} defined in (1.27). First expand the square:

$$J_{ca} = \mathrm{E} \int |f(x)|^2 w(x) dx - \mathrm{E} \sum_{m,n} \langle f, g^2_{mn} \rangle_w \langle g^1_{mn}, f \rangle$$
$$-\mathrm{E} \sum_{m,n} \langle f, g^1_{mn} \rangle \langle g^2_{mn}, f \rangle_w + \mathrm{E} \sum_{m,n,m',n'} \langle f, g^1_{mn} \rangle \langle g^2_{mn}, g^2_{m'n'} \rangle_w \langle g^1_{m'n'}, f \rangle$$

and apply Theorem 3.1 to summations over frequency indices m and m',

$$\begin{aligned} J_{ca} \;=\; & \mathrm{E} \int |f(x)|^2 w(x) dx - \\ & \frac{1}{b} \mathrm{E} \sum_{m,n} \int dx\, f(x) \overline{g^2(x-na)} g^1(x + \frac{m}{b} - na) \overline{f(x + \frac{m}{b})} w(x) - c.c. \\ & + \frac{1}{b^2} \mathrm{E} \sum_{m,n,m',n'} \int dx\, f(x + \frac{m}{b}) \overline{g^1(x + \frac{m}{b} - na)} g^2(x-na) \cdot \\ & \qquad \cdot \overline{g^2(x - n'a)} g^1(x + \frac{m'}{b} - n'a) \overline{f(x + \frac{m'}{b})} w(x), \end{aligned}$$

where *c.c.* stands for *complex-conjugate* of the previous term. Next we compute the expectations using the autocovariance function R. Let us consider the general non-stationary case, that is $\mathrm{E}[f(x)\overline{f(y)}] = R(x,y)$. Then the above expression turns into:

$$\begin{aligned} J_{ca} \;=\; & \int dx\, R(x,x) w(x) - \\ & \frac{1}{b} \sum_m \int dx\, R(x, x + \frac{m}{b}) w(x) (\sum_n g^1(x + \frac{m}{b} - na) \overline{g^2(x-na)}) - c.c. \\ + \; & \frac{1}{b^2} \sum_{m,m'} \int dx\, R(x + \frac{m}{b}, x + \frac{m'}{b}) w(x) (\sum_{n,n'} \overline{g^1(x + \frac{m}{b} - na)} g^2(x-na) \cdot \\ & \qquad \cdot \overline{g^2(x - n'a)} g^1(x + \frac{m'}{b} - n'a)). \end{aligned} \tag{1.84}$$

An entire similar derivation can be made for the average power criterion (1.31). One obtains an expression similar to (1.84) where w is replaced by 1 and $\int$ is replaced by $\lim_{t\to\infty}\frac{1}{2T}\int_{-T}^{T}$. Note the $\sum_n$ and $\sum_{n,n'}$ terms are a-periodic. Thus, denoting formally $R_c(t_1,t_2)=\lim_{K\to\infty}\frac{1}{2K+1}\sum_{k=-K}^{K}R(t_1+ka,t_2,k+a)$, one has

$$\begin{aligned}
P_{ca} &= \int_0^a dx\, R_c(x) \\
&= \frac{1}{b}\sum_m \int_0^a dx\, R_c(x,x+\frac{m}{b})(\sum_n g^1(x+\frac{m}{b}-na)\overline{g^2(x-na)}) - c.c. \\
&\quad +\frac{1}{b^2}\sum_{m,m'}\int_0^a dx\, R_c(x+\frac{m}{b},x+\frac{m'}{b})\,\cdot \\
&\quad \cdot\Big(\sum_{n,n'}\overline{g^1(x+\frac{m}{b}-na)}g^2(x-na)\overline{g^2(x-n'a)}g^1(x+\frac{m'}{b}-n'a)\Big),
\end{aligned}$$

which is formally equal to (1.84) for $R(t_1,t_2)=R_c(t_1,t_2)$ and $w=1_{[0,a]}$, that is for the cyclostationary process whose autocovariance function is R_c.

Now let us continue with (1.84). Denote by

$$M^i_{m_1,m_2}(s)=\sum_{k\in\mathbb{Z}}R(a(s+k+\frac{m_1}{ba}),a(s+k+\frac{m_2}{ba})w(a(s+k)) \qquad (1.85)$$

Clearly $M^1_{m_1,m_2}$ is 1-periodic in s. Using

$$|R(x_1,x_2)|\le\sqrt{R(x_1,x_2)}\sqrt{R(x_2,x_2)}$$

and (1.23) we deduce that $M^i_{m_1,m_2}$ is in $L^1[0,a]$. Then (1.84) turns into:

$$\begin{aligned}
J_{ca} &= a\int_0^a M^i_{0,0}(s)ds \\
&\quad -\frac{a}{b}\sum_m\int_0^1 M^i_{0,m}(s)\sum_n g^1(a(s+\frac{m}{ba}-n))\overline{g^2(a(s-n))}ds - c.c. \\
&\quad +\frac{a}{b^2}\sum_{m,m'}\int_0^1 M^i_{m,m'}(s)\sum_{n,n'}\overline{g^1(a(s+\frac{m}{ba}-n))}g^2(a(s-n))\,\cdot \\
&\qquad\qquad \cdot\overline{g^2(a(s-n'))}g^1(a(s+\frac{m'}{ba}-n'))\,ds. \qquad (1.86)
\end{aligned}$$

At this point we use the stationarity of f, and then the rationality assumption $ba=\frac{p}{q}$. Thus $M^i_{m_1,m_2}$ becomes

$$M^i_{m_1,m_2}(s)=R(\frac{m_1-m_2}{b})\sum_k w(a(s+k)) \qquad (1.87)$$

Let us denote by

$$\omega(s) = \sum_k w(a(s+k)) \tag{1.88}$$

so that $M^i_{m_1,m_2}(s) = R(\frac{m_1-m_2}{b})\omega(s)$. Let $m = m_1 p + r$, $m' = m_1^{'} p + r'$ with $0 \leq r, r' \leq p-1$. Then, using the Zak transforms G^1, G^2 of g^1, respectively g^2, the quasiperiodicity relations (1.76) and again the weak Poisson summation formula (1.55) we obtain:

$$\begin{aligned}
J_{ca} &= aq \int_0^1 ds \int_0^{1/q} dt\, \rho_0(t)\omega(s) \\
&- \frac{aq}{p} \sum_{r=0}^{p-1} \int_0^1 ds \int_0^{1/q} dt \rho_{-r}(t)\omega(s) G^1(t+\frac{l}{q}, s + r\frac{q}{p}) \times \overline{G^2(t+\frac{l}{q}, s)} \\
&- c.c. + \frac{aq}{p^2} \int_0^1 ds \int_0^{1/q} dt \sum_{r,r'=0}^{p-1} \rho_{r-r'}(t)\omega(s) \sum_{l_1,l_2=0}^{q-1} \overline{G^1(t+\frac{l_1}{q}, s+r\frac{q}{p})} \\
&\quad \times G^2(t+\frac{l_1}{q}, s)\overline{G^2(t+\frac{l_2}{q}, s)} G^1(t+\frac{l_2}{q}, s+r'\frac{q}{p}),
\end{aligned}$$

where

$$\rho_r(t) = \sum_m e^{2\pi i m q t} R(\frac{mp+r}{b}) \tag{1.89}$$

Because the integrand is 1-periodic, the integral over s can be split into p integrals over intervals of length $\frac{1}{p}$ of the form $\int_0^1 ds = \sum_{r_1=0}^{p-1} \int_{r_1 q/p}^{r_1 q/p + 1/p} ds$. Let us denote by $M(t)$ the $p \times p$ matrix whose (r_1, r_2) element is $\rho_{r_1 - r_2}(t)$, $0 \leq r_1, r_2 \leq p-1$, and by $W(s)$ the $p \times p$ diagonal matrix whose (r,r) element is $\omega(s + r\frac{q}{p})$, $0 \leq r \leq p-1$:

$$M(t) = \begin{bmatrix} \rho_0(t) & \rho_{-1}(t) & \cdots & \rho_{-(p-1)}(t) \\ \rho_1(t) & \rho_0(t) & \ddots & \rho_{-(p-2)}(t) \\ \cdots & \ddots & \ddots & \vdots \\ \rho_{p-1}(t) & \rho_{p-2}(t) & \cdots & \rho_0(t) \end{bmatrix}, \tag{1.90}$$

$$W(s) = \begin{bmatrix} \omega(s) & 0 & \cdots & 0 \\ 0 & \omega(s+\frac{q}{p}) & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \omega(s+(p-1)\frac{q}{p}) \end{bmatrix}. \tag{1.91}$$

Note the following properties of $\rho_r(t)$:

$$\rho_{r+p}(t) = e^{-2\pi iqt}\rho_r(t) \tag{1.92}$$

$$\rho_r(t+\frac{1}{q}) = \rho_r(t) \tag{1.93}$$

$$\rho_{-r}(t) = \overline{\rho_r(t)} \tag{1.94}$$

Therefore $M^*(t) = M(t)$ and $M(t+\frac{1}{q}) = M(t)$, i.e. M is self-adjoint Toeplitz and $\frac{1}{q}$-periodic. Note also the following properties:

$$W(s+\frac{1}{p}) = E(t)^{r_0}W(s)E(t)^{*r_0} \tag{1.95}$$

$$M(t) = E(t)M(t)E(t)^* \tag{1.96}$$

Using these propierties, the expression for J_{ca} can be simplified to

$$\begin{aligned} J_{ca} &= aq\int\int_{\square} Tr\ \{W(s)\big(I-\frac{1}{p}\Gamma^2(t,s)\Gamma^{1*}(t,s)\big)\ \cdot \\ &\quad \cdot M(t)(I-\frac{1}{p}\Gamma^1(t,s)\Gamma^{2*}(t,s))\}. \end{aligned} \tag{1.97}$$

Semi-Optimization Problems

The optimization problems decouple fiberwise, i.e. for every (t,s). Thus, given g^1, that is Γ^1, the best Γ^2 is obtained as a least square problem by minimizing a functional of type (1.56). The solution is then:

$$\Gamma^2_{semi-opt} = pM\Gamma^1(\Gamma^{1*}M\Gamma^1)^{-1} \tag{1.98}$$

and the criterion becomes:

$$J_{ca}(\Gamma^1) = aq\int\int_{\square} Tr\ \{MW(I-\Gamma^2_{semi-opt}\Gamma^{1*})\}\,dt\,ds \tag{1.99}$$

On the other hand, given g^2, that is Γ^2, the best Γ^1 is obtained similarly:

$$\Gamma^1_{semi-opt} = pW\Gamma^2(\Gamma^{2*}W\Gamma^2)^{-1} \tag{1.100}$$

and the criterion becomes:

$$J_{ca}(\Gamma^2) = aq\int\int_{\square} Tr\ \{MW(I-\Gamma^2\Gamma^{1*}_{semi-opt})\}\,dt\,ds \tag{1.101}$$

Remark 3.4 When $(g^1;b,a)$ is a s-Riesz basis, then the necessary and sufficient condition for $(g^2;b,a)$ defined in (1.98) to be a s-Riesz basis is simply the eigenvalues of $M(t)$ to be bounded above and below away from zero. Similarly, when $(g^2;b,a)$ is a s-Riesz basis, $(g^1;b,a)$ defined by (1.100) is also a s-Riesz basis whenever the eigenvalues of $W(s)$ are bounded above and away from zero. This happens when $w\in W(L^\infty,l^1)$ and has persistency length a.

The Optimization Problem

We can further optimize (1.99) over Γ^1. Using (1.59) and (1.60) first we have to solve the following eigenproblem:

$$M^{1/2}(t)W(s)M^{1/2}(t)V(t,s) = V(t,s)\cdot diag(\lambda_j^{ord}(t,s))_{0\leq j\leq q-1} \quad (1.102)$$

where $\lambda_0^{ord}\geq\lambda_1^{ord}\geq\ldots\geq\lambda_{p-1}^{ord}$ are the ordered eigenvalues of $M^{1/2}WM^{1/2}$ and V is $\mathbb{C}^{p\times q}$ normalized so that $V^*V=I_q$ and VV^* is the orthogonal projection onto the invariant space associated to the first q eigenvalues. Then we obtain

$$\Gamma^1_{opt} = M^{-1/2}VL \quad (1.103)$$
$$\Gamma^2_{opt} = pM^{1/2}VL^{-*} \quad (1.104)$$

with $L^{-*}=(L^{-1})^*$, and the optimal value of the criterion:

$$J_{ca} = aq\int\int_{\Box}\sum_{j=q}^{p-1}\lambda_j^{ord}(t,s)\,dt\,ds \quad (1.105)$$

where L is an arbitrary $\mathbb{C}^{q\times q}$ valued measurable function over $\Box$, invertible with bounded inverse, i.e. $\sup_{(t,s)\in\Box}\parallel L\parallel\leq\infty$ and $\sup_{(t,s)\in\Box}\parallel L^{-1}\parallel\leq\infty$.

3.5 Discrete Time Signal Approximation Problem

Consider the criterion (1.34). Let us compute first $\varepsilon_{mn}=\mathrm{E}[|c_{mn}-\tilde{c}_{mn}|^2]$. Using $\tilde{c}_{mn}=\langle\sum_{m',n'}c_{m'n'}g^1_{m'n'},g^2_{mn}\rangle$ and $\mathrm{E}[c_{m'n'}\overline{c_{mn}}]=R_{m'-m,n'-n}$ we obtain:

$$\begin{aligned}\varepsilon_{mn} = {} & R_{0,0}-\sum_{m',n'}R_{m'-m,n'-n}\int e^{2\pi i(m'-m)bx}g^1(x-n'a)\overline{g^2(x-na)}\,dx\\ & -\sum_{m',n'}R_{m-m',n-n'}\int e^{2\pi i(m-m')bx}g^2(x-na)\overline{g^1(x-n'a)}\,dx\\ & +\sum_{m',n',m'',n''}R_{m'-m'',n'-n''}\int dx\int dy\, e^{2\pi i(m'-m)bx}e^{2\pi i(m-m'')bx}\\ & g^1(x-n'a)\overline{g^2(x-na)}g^2(y-na)\overline{g^1(y-n''a)}\end{aligned}$$

Now replace g^1 and g^2 by their Zak transforms via (1.75). Using the pseudoperiodicity relations (1.76) and the summation formula (1.55) we obtain:

$$\begin{aligned}\varepsilon_{mn} = {} & p\int_0^{1/p}ds\int_0^{1/q}dt\sum_{l,l_1,l_2=0}^{q-1}e^{2\pi inl\frac{p}{q}}\rho_{l_1,l}(t,s)[\delta_{l_1,l_2}-\frac{1}{p}(\Gamma^{2*}\Gamma^1)_{l_2-lp,l_1}\delta_{l_1,l_2}\\ & -\frac{1}{p}(\Gamma^{1*}\Gamma^2)_{l_1,l_2+lp}\delta_{l_1,l_2}+\frac{1}{p^2}(\Gamma^{2*}\Gamma^1)_{l_2,l_1}(\Gamma^{1*}\Gamma^2)_{l_1,l_2+lp}],\end{aligned}$$

where:

$$\rho_{l_1,l}(t,s) = \sum_{k,j\in Z} e^{2\pi ik(t+\frac{l_1}{q})} e^{2\pi i(jq+l)\frac{p}{q}s} R_{jq+l,k} \tag{1.106}$$

and Γ^1, Γ^2 are the matrix representations (1.77). Now let us compute J_{da} from (1.34). It is given by

$$J_{da} = \sum_{mn} w_{mn}\varepsilon_{mn} = p\int_0^{1/p} ds \int_0^{1/q} dt \sum_{l,l_1,l_2=0}^{q-1} \sigma_l \rho_{l_1,l}(t,s)[\delta_{l_1,l_2}$$
$$-\frac{1}{p}(\Gamma^{2*}\Gamma^1)_{l_2-lp,l_1}\delta_{l_1,l_2} - \frac{1}{p}(\Gamma^{1*}\Gamma^2)_{l_1,l_2+lp}\delta_{l_1,l_2} + \frac{1}{p^2}(\Gamma^{2*}\Gamma^1)_{l_2,l_1}(\Gamma^{1*}\Gamma^2)_{l_1,l_2+lp}] \tag{1.107}$$

where

$$\sigma_l = \sum_{m,n} w_{mn} e^{2\pi inl\frac{p}{q}} = \sum_{l'=0}^{q-1}(\sum_{m,n'} w_{m,n'q+l'})e^{2\pi il'l\frac{p}{q}}. \tag{1.108}$$

This expression can be rewritten more compactly if we use the following $q\times q$ matrices:

$$D^l = \sigma_l \cdot \mathrm{diag}(\rho_{l_1,l}(t,s))_{0\le l_1\le q-1}, \tag{1.109}$$
$$(U^l)_{l_1,l_2} = \delta_{l_1,l_2+lp}, \tag{1.110}$$

that is D^l is $q\times q$ diagonal matrix and U^l is a $q\times q$ permutation matrix. Then (1.107) becomes:

$$J_{da} = p\int\int_{\square} \sum_{l=0}^{q-1} Tr\ \{(I - \frac{1}{p}\Gamma^{2*}\Gamma^1)D^l(I - \frac{1}{p}\Gamma^{1*}\Gamma^2)U^l\}, \tag{1.111}$$

with $\square = [0,\frac{1}{q}]\times[0,\frac{1}{p}]$. Note the following properties of the two sets of matrices D^l and U^l:

$$D^l(t+\frac{1}{q},s) = Q^* D^l(t,s)Q, \tag{1.112}$$
$$D^l(t,s+\frac{1}{p}) = e^{2\pi i\frac{l}{q}}D^l(t,s), \tag{1.113}$$
$$(D^l)^* = D^{(q-l)\,mod\,q}, \tag{1.114}$$
$$U^l = Q^{lp}, \tag{1.115}$$
$$(U^l)^* = U^{(q-l)\,mod\,q}. \tag{1.116}$$

Remark 3.5 Since $\varepsilon_{m+1,n} = \varepsilon_{mn}$ and $\varepsilon_{m,n+q} = \varepsilon_{mn}$ the average power (1.38) turns into:

$$P_{da} = \frac{1}{q}\sum_{n=0}^{q-1}\varepsilon_{0,n}$$

This is equivalent to J_{da} for the particular choice of weights $w_{mn} = \frac{1}{q}$, $m = 0$ and $0 \leq n \leq q-1$, and $w_{mn} = 0$ otherwise. Note this corresponds to choosing $\sigma_0 = 1$ and $\sigma_l = 0$, for $0 < l \leq q-1$.

Semi-Optimization Problems

Since (1.111) is an integral over independent fibers, the optimization problems decouple into finite optimization problems, fiberwise:

$$J(\Gamma^1, \Gamma^2) = \sum_{l=0}^{q-1} Tr\ \{(I - \frac{1}{p}\Gamma^{2*}\Gamma^1)D^l(I - \frac{1}{p}\Gamma^{1*}\Gamma^2)U^l\} \tag{1.117}$$

Recall we consider the case when both $(g^1; b, a)$ and $(g^2; b, a)$ are frame sets, that is $\frac{p}{q} < 1$. Thus $\Gamma^{2*}\Gamma^1$ has always rank $p < q$ and cannot be I.

First consider the case when g^1 is fixed and we look for the best g^2. The solution is given by the following linear system:

$$\sum_{l=0}^{q-1} \Gamma^1 D^l U^l = \frac{1}{p}\sum_{l=0}^{q-1} \Gamma^1 D^l \Gamma^{1*}\Gamma^2 U^l \tag{1.118}$$

where the unknown is the $p \times q$ matrix Γ^2.

Next consider the dual case, when g^2 is fixed and we search for the optimal g^1. Similarly, its Zak representation matrix Γ^1 has to satisfy the following linear system:

$$\sum_{l=0}^{q-1} \Gamma^2 U^l D^l = \frac{1}{p}\sum_{l=0}^{q-1} \Gamma^2 U^l \Gamma^{2*}\Gamma^1 D^l \tag{1.119}$$

For either case, the criterion takes the following form:

$$J_{da} = p \int\int_{\square} Tr\ \{(\sum_{l=0}^{q-1} U^l D^l)(I - \frac{1}{p}\Gamma^{1*}\Gamma^2)\}\, dt\, ds \tag{1.120}$$

where (Γ^1, Γ^2) are related to one another via (1.118), or (1.119).

The Optimization Problem

The optimization problem continues by further minimizing (1.117) over both (Γ^1, Γ^2). One can easily show that it is equivalent to require that (Γ^1, Γ^2) satisfy simultaneously (1.118) and (1.119). Denote $R = \sum_{l=0}^{q-1} U^l D^l$ and $X = \frac{1}{p}\Gamma^{1*}\Gamma^2$. Note that $R^* = \sum_{l=0}^{q-1} D^l U^l$. Then the system (1.118,1.119) turns equivalently into:

$$X[R - \sum_{l=0}^{q-1} U^l X^* D^l] \quad = \quad [R - \sum_{l=0}^{q-1} U^l X^* D^l]X = 0 \tag{1.121}$$

The optimizer corresponds to a rank p, $q \times q$ complex matrix X that satisfies (1.121). Note there may be more solutions of (1.121) (and in general this is the case). These correspond to other critical points of J_{da}.

In general we cannot obtain a closed form solution. However, the interesting practical problem is for the average power P_{da}, i.e. for uniform weights $w_{mn} = \frac{1}{q}$ for $m = 0$ and $0 \leq n \leq q-1$, and 0 otherwise. In this case we obtain $D^l = 0$ for $l > 0$ and since $U^0 = I$, $R = D^0 = R^*$ and (1.121) turns into:

$$X(I - X^*)R = (I - X^*)RX = 0$$

In general R is invertible, hence from the first equation we obtain $X = XX^*$ and thus X has to be a (selfadjoint) orthogonal projection. The second equation turns into $RX = XRX$, that means X is associated to an invariant space of R. Then J_{da} is given by the remaining eigenvalues of R. Since we want to minimize J_{da}, X has to correspond to the largest p eigenvectors of R. But $R = D^0$ is already diagonal. Thus X is immediate. Let us return to Γ^1 and Γ^2. For a general solution X, Γ^1 and Γ^2 are obtained by factorizing X (which is of rank p) into a product of a $q \times p$ complex matrix with another $p \times q$ complex matrix. The set of solutions will be parameterized by arbitrary $p \times p$ invertible with bounded inverse (over $\square$) complex matrices. For the practical case we consider, Γ^1 and Γ^2 are obtained from the ordered eigenvectors of R. Since R is diagonal, the eigenvectors are simply the columns of the identity matrix. Let us denote by e_j the q-vector whose j^{th} component is 1, and the rest is zero, i.e. the j^{th} column of the identity matrix I_q. Denote by $\pi_{t,s} : \{0, 1, \ldots, q-1\} \to \{0, 1, \ldots, q-1\}$ the ordering permutation of the diagonal elements of D^0, that is

$$\rho_{\pi_{t,s}(j),0}(t,s) \geq \rho_{\pi_{t,s}(j+1),0}(t,s) \quad , \quad 0 \leq j \leq q-2$$

Then:

$$\begin{aligned} \Gamma^1(t,s) &= \sqrt{p} L^* V^* && (1.122) \\ \Gamma^2(t,s) &= \sqrt{p} L^{-1} V^* && (1.123) \end{aligned}$$

where:

$$V(t,s) = [e_{\pi_{t,s}(0)} \;\; e_{\pi_{t,s}(1)} \;\; \cdots \;\; e_{\pi_{t,s}(p-1)}] \tag{1.124}$$

and L is an invertible with bounded inverse $C^{p \times p}$ matrix valued function over $\square$. The criterion becomes:

$$J_{da} = p \int\int_{\square} \sum_{k=p}^{q-1} \rho_{\pi_{t,s}(k)}(t,s)\, dt\, ds \tag{1.125}$$

3.6 Continuous Time Signal Encoding Problem

Consider now the *Continuous Time Signal Encoding* problem stated in subsection 2.2.3. Recall the criterion J_{ce} given by (1.42). Since $(g^1, g^2; b, a)$ is a dual pair of frames, the criterion is:

$$J_{ce}(g^1, g^2; b, a, R) = \lim_{M,N\to\infty} \frac{1}{(2N+1)(2M+1)} \mathrm{E}\,[\int |\sum_{|m|\le M}\sum_{|n|\le N} d_{mn} g^2_{mn}|^2 dx].$$

We assume g^1, g^2 are sufficiently well localized so that we can commute summation with integrals and expectations, and apply the weak Poisson summation formula (Theorem 3.1). By expanding the sum and replacing the expectation with the covariance function $R_{\cdot,\cdot}$ the finite sum turns into

$$E_{M,N} = \frac{1}{(2N+1)(2M+1)} \sum_{|m_1|,|m_2|\le M}\sum_{|n_1|,|n_2|\le N} R_{m_1-m_2,n_1-n_2} e^{2\pi i(m_2-m_1)n_1 ba} \times \int e^{2\pi i(m_2-m_1)bx} g^2(x-(n_2-n_1)a)\overline{g^2(x)}\,dx$$

Denote $m = m_1 - m_2$ and $n = n_1 - n_2$. For fixed m, n, m_1, n_1 run over the sets I_m, respectively I_n where

$$I_m = \begin{cases} \{-M, -M+1, \ldots, M+m\} & if \quad m \le 0 \\ \{-M+m, -M+1, \ldots, M\} & if \quad m > 0 \end{cases},$$

of cardinality

$$|I_m| = \begin{cases} 2M+m+1 & if \quad m \le 0 \\ 2M-m+1 & if \quad m > 0 \end{cases},$$

and similarly for I_n. Then $E_{M,N}$ turns into

$$E_{M,N} = \sum_{m=-2M}^{2M} (1 - \frac{|m|}{2M+1}) \sum_{n=-2N}^{2N} (\frac{1}{2N+1} \sum_{n_1\in I_n} e^{2\pi i m n_1 ba}) R_{mn}\gamma_{mn}$$

where

$$\gamma_{mn} = \int e^{2\pi i mbx} g^2(x-na)\overline{g^2(x)}dx$$

Now, for I_n as defined before and $z = e^{2\pi imba}$ we obtain

$$\sum_{n_1\in I_n} e^{2\pi i m n_1 ba} = \begin{cases} \frac{z^{-N} - z^{N+n+1}}{1-z} & if \quad m \le 0 \\ \frac{z^{-N+n} - z^{N+1}}{1-z} & if \quad m > 0 \end{cases}$$

when $z \neq 1$, and

$$\sum_{n_1 \in I_n} e^{2\pi i m n_1 b a} = |I_n|$$

otherwise. Assume $(R_{mn}\gamma_{mn}) \in l^{1,1}(Z^2)$ (which is no constraint on the stochastic process when $g \in M_{1,1}$ for instance, because then $(\gamma_{mn}) \in l^{1,1}(Z^2)$). Then apply Lebesgue's dominated convergence theorem to compute $\lim_{M,N\to\infty} E_{M,N}$. We obtain

$$J_{ce} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta_{mba\, mod\, 1,0} R_{mn}\gamma_{mn}$$

Thus, for $ba \notin Q$ we obtain

$$J_{ce} = \sum_n R_{0,n}\gamma_{0,n} \tag{1.126}$$

But we are interested in the case $ba = \frac{p}{q}$. Thus J_{ce} turns into

$$J_{ce} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R_{mq,n}\gamma_{mq,n} \tag{1.127}$$

Next we use the Zak transform of g^2 and write the criterion in terms of the matrix representation Γ^2. Simple algebra and application of the weak form of Poisson summation formula (1.55) gives

$$J_{ce} = \int\int_{\Box} Tr\ \{\Gamma^2 R \Gamma^{2*}\}\, dt\, ds \tag{1.128}$$

where $\Box = [0, \frac{1}{q}] \times [0, \frac{1}{p}]$, $R(t,s)$ is the $q \times q$ diagonal matrix

$$R(t,s) = diag(\mu(t + \frac{l}{q}, s))_{0 \leq l \leq q-1}, \tag{1.129}$$

and $\mu(t,s)$ are $(1, \frac{1}{p})$ periodic functions defined by

$$\mu(t,s) = \sum_{m,n} e^{2\pi i m p s} e^{2\pi i n t} R_{mq,n} \tag{1.130}$$

Now we are ready to analyze the optimization problems (1.43– 1.45).

The Semi-Optimization Problem

Given g^1, that is Γ^1, we look for Γ^2 that minimizes (1.130) and satisfies (1.82). This problem has been solved in subsection 3.2, as Problem B. The solution (1.63) reads:

$$\Gamma^2 = p(\Gamma^1 R^{-1} \Gamma^{1*})^{-1} \Gamma^1 R^{-1}, \tag{1.131}$$

and the average distortion becomes

$$J_{ce} = p^2 \int\int_{\square} Tr\ \{(\Gamma^1 R^{-1} \Gamma^{1*})^{-1}\}\, dt\, ds \tag{1.132}$$

Remark 3.6 When $(g^1; b, a)$ is a frame, then the necessary and sufficient condition for $(g^2; b, a)$ defined in (1.131) to be a frame is simply the eigenvalues of $R(t,s)$ to be bounded above and below away from zero, that is:

$$0 < A_0 \leq \mu(t + \frac{l}{q}, s) \leq B_0 < \infty \ \ , \ \ \forall t, s\ ,\ 0 \leq l \leq q-1 \tag{1.133}$$

for some A_0, B_0.

The Iso-Pair Problem

This case has been solved in (1.66,1.67). The solution reads:

$$\Gamma^1 = \Gamma^2 = \sqrt{p} U V^* \tag{1.134}$$

where $U : (t,s) \mapsto U(t,s) \in C^{p\times p}$ is a unitary valued measurable map over $\square$, and $V : (t,s) \mapsto V(t,s) \in C^{q\times p}$ is a measurable function so that the columns of $V(t,s)$ are the p eigenvectors of $R(t,s)$ corresponding to the lowest eigenvalues. Since R is already diagonal, these columns are a subset of the canonical basis $\{e_0, \ldots, e_{q-1}\}$ of C^q. Assume $\pi_{t,s}$ is the q-permutation so that

$$\mu(t + \frac{\pi_{t,s}(l)}{q}, s) \geq \mu(t + \frac{\pi_{t,s}(l+1)}{q}, s) \ \ , \ \ 0 \leq l \leq q-2$$

Then

$$V(t,s) = \left[\ e_{\pi_{t,s}(q-p)} \quad e_{\pi_{t,s}(q-p+1)} \quad \cdots \quad e_{\pi_{t,s}(q-1)}\ \right] \tag{1.135}$$

The distortion is then

$$J_{ce} = p \int\int_{\square} \sum_{l=q-p}^{q-1} \mu(t + \frac{\pi_{t,s}(l)}{q}, s)\, dt\, ds \tag{1.136}$$

The Optimization Problem

Since the Zak transform is unitary, the norm constraint $\|g^1\| = 1$ is equivalent to $\int\int_{\square} Tr\ \{\Gamma^1\Gamma^{1*}\}\, dt\, ds = 1$. Then, the norm-constraint optimizers of J_{ce} are parameterized as in (1.70–1.72). With the current notations, this becomes

$$\Gamma^1_{opt} = cUV^* \tag{1.137}$$

$$\Gamma^2_{opt} = \frac{p}{c} UV^* \tag{1.138}$$

where

$$c(t,s) = \frac{[\sum_{l=q-p}^{q-1} \mu(t+\frac{\pi_{t,s}(l)}{q},s)]^{1/4}}{\{p\int\int_{\Box}[\sum_{l=q-p}^{q-1} \mu(t'+\frac{\pi_{t',s'}(l)}{q},s')]^{1/2}\,dt'\,ds'\}^{1/2}} \tag{1.139}$$

and U, V as in (1.134). The optimal distortion becomes:

$$J_{ce} = p^3 \left(\int\int_{\Box} \left(\sum_{l=q-p}^{q-1} \mu(t+\frac{\pi_{t,s}(l)}{q},s) \right)^{1/2} dt\,ds \right)^2 \tag{1.140}$$

3.7 Discrete Time Signal Encoding Problem

In the Discrete-Time Signal Encoding problem, the average transmission error is defined by (1.50):

$$J_{de}(g^1,g^2;b,a,R) = \lim_{T\to\infty} \sum_{m,n} \mathrm{E}[|\tilde{c}_{mn} - c_{mn}|^2],$$

where T denotes the width of the time window during which noise is added; see (1.48). However one can consider an alternative error measure, namely the average distortion per coefficient defined by (1.51),

$$P_{de}(g^1,g^2;b,a,R) = \lim_{M,N\to\infty} \frac{1}{2N+1} \sum_{|m|\le M} \sum_{|n|\le N} \mathrm{E}[|\tilde{c}_{mn} - c_{mn}|^2]$$

We are going to show these two functionals are identical. Using (1.47),(1.48) and (1.49) we obtain

$$\begin{aligned} J_{de} &= \lim_{T\to\infty} \sum_{m,n} \mathrm{E}[|\langle \nu^T, g^2_{mn}\rangle|^2] \\ &= \lim_{T\to\infty} \frac{1}{2T} \sum_n \int_{-T}^{T} dx \int_{-T}^{T} dy \sum_m e^{2\pi i m b(y-x)} \overline{g^2(x-na)} g^2(y-na) \end{aligned}$$

Now apply the weak Poisson summation formula (1.55) and then periodize the integrand with $\sum_n$. Denote

$$G_m(x) = \sum_n \overline{g^2(x-na)} g^2(x - \frac{m}{b} - na),$$

We obtain:

$$J_{de} = \frac{1}{b} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_m R(\frac{m}{b}) G_m(x) dx$$

Since the integrand is a-periodic, the average is simply the integral over a period, that is

$$J_{de} = \frac{1}{b}\sum_{m,n} R(\frac{m}{b}) \int_0^a \overline{g^2(x-na)} g^2(x-\frac{m}{b}-na)dx \tag{1.141}$$

Consider now P_{de}. Expanding the expectation operator and performing the summation over m from $-\infty$ to $+\infty$ via the weak Poisson summation formula, we obtain

$$P_{de} = \frac{1}{b}\lim_{N\to\infty}\frac{1}{2N+1}\sum_{|n|\leq N}\sum_m R(\frac{m}{b})\int_{-\infty}^{\infty}\overline{g^2(x-na)}g^2(x-\frac{m}{b}-na)dx$$

Note that by a change of variable, the integral does not depend on n. Therefore the average over n does not change the outcome and we remain with

$$P_{de} = \frac{1}{b}\sum_m R(\frac{m}{b})\int \overline{g^2(x)}g^2(x-\frac{m}{b})dx$$

Now by a-periodizing the integrand we obtain again (1.141). Thus we proved $P_{de} = J_{de}$. Note, everywhere we assume the integrals converge absolutely and we can freely commute summation symbols with expectation and integration symbols. That is true as long as g^2 is sufficiently well localized, for instance $g^2 \in W(L^\infty, l^1)$.

Next we compute (1.141) in terms of Zak transform. By replacing g^2 with its Zak transform, then using pseudoperiodicity relation (1.76), and then the summation formula (1.55), we obtain

$$J_{de} = \int_0^1 ds\int_0^1 dt\frac{1}{b}\sum_{r=0}^{p-1}\sum_{m'} e^{2\pi i m' qt} R(\frac{m'p-r}{b})\overline{G^2(t,s)}G^2(t,s+r\frac{q}{p})$$

Now we $\frac{1}{q}$-periodize the integrand over t, and $\frac{1}{p}$ over s. Using again the pseudoperiodicity we obtain:

$$J_{de} = \int\int_{\Box}\sum_{r_1,r_2=0}^{p-1}\sum_{l=0}^{q-1}\sigma_{r_1-r_2}(t)\overline{G^2(t+\frac{l}{q},s+r_2\frac{q}{p})}G^2(t+\frac{l}{q},s+r_2\frac{q}{p})\,dt\,ds$$

where

$$\sigma_r(t) = \frac{1}{b}\sum_{m'} e^{2\pi i m' qt} R(\frac{m'p+r}{b}) \tag{1.142}$$

This can be compactly rewritten as:

$$J_{de} = \int\int_{\Box} Tr\ \{\Gamma^{2*}S\Gamma^2\}\,dt\,ds \tag{1.143}$$

where $S : \square \to C^{p\times p}$ is the Toeplitz self-adjoint matrix

$$S = (\sigma_{r_1-r_2})_{0\le r_1,r_2\le p-1},$$

that is

$$S(t) = \begin{bmatrix} \sigma_0(t) & \sigma_{-1}(t) & \sigma_{-2}(t) & \cdots & \sigma_{-(p-1)}(t) \\ \sigma_1(t) & \sigma_0(t) & \sigma_{-1}(t) & \ddots & \sigma_{-(p-2)}(t) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \sigma_{p-1}(t) & \sigma_{p-2}(t) & \cdots & \sigma_1(t) & \sigma_0(t) \end{bmatrix}. \quad (1.144)$$

Note the following property of $\sigma(t)$

$$\sigma_{-r}(t) = e^{2\pi iqt}\sigma_{p-r}(t) \quad (1.145)$$

and then

$$S(t) = E(t)S(t)E(t)^* \quad (1.146)$$

where $E(t)$ was defined in (1.79). This commutativity relation allows us to compute the eigenvectors of $S(t)$. The eigenvectors of $E(t)$, hence of $S(t)$ as well, are given by:

$$f_k = \frac{1}{\sqrt{p}}\begin{bmatrix} 1 & \varepsilon_k & \varepsilon_k^2 & \cdots & \varepsilon_k^{p-1} \end{bmatrix}^T \quad , \quad 0 \le k \le p-1 \quad (1.147)$$

where ε_k is one of the complex p^{th} root of $e^{-2\pi iqt}$,

$$\varepsilon_k = e^{-2\pi i\frac{q}{p}(t+\frac{k}{q})} \quad , \quad 0 \le k \le p-1 \quad (1.148)$$

The eigenvalues of $S(t)$ are then

$$m_k = \langle f_k, Sf_k\rangle = \hat{R}(\frac{qt+k}{p}) \quad , \quad 0 \le k \le p-1 \quad (1.149)$$

where

$$\hat{R}(\omega) = \frac{1}{b}\sum_m e^{2\pi im\omega}R(\frac{m}{b}) \quad (1.150)$$

The Semi-Optimization Problem

Given g^1, that is Γ^1, we look for Γ^2 that minimizes (1.143) and satisfies (1.83). The correspondence to the problem B in subsection 3.2 is given by $A = \Gamma^{1*}$, $X = \Gamma^{2*}$, $R = S$ and $I_p = I_q$. The solution (1.63) becomes:

$$\Gamma^2 = pS^{-1}\Gamma^1(\Gamma^{1*}S^{-1}\Gamma^1)^{-1}, \quad (1.151)$$

and the average distortion becomes

$$J_{de} = p^2 \int\int_{\square} Tr\ \{(\Gamma^{1*}S^{-1}\Gamma^1)^{-1}\}\,dt\,ds \quad (1.152)$$

Remark 3.7 When $(g^1; b, a)$ is a s-Riesz basis, then the necessary and sufficient condition for $(g^2; b, a)$ defined in (1.151) to be a s-Riesz basis is simply the eigenvalues of $S(t)$ to be bounded above and below away from zero, that is:

$$0 < A_0 \leq m_k(t) \leq B_0 < \infty \quad , \quad \forall t \ , \ 0 \leq k \leq p-1 \tag{1.153}$$

for some A_0, B_0.

The Iso-Pair Problem

We optimize now J_{de} over the constraint $\Gamma^*\Gamma = pI_q$. Again the solution is given by the problem B mentioned before. Equations (1.66,1.67) and the aforementioned correspondence give

$$\Gamma^1 = \Gamma^2 = \sqrt{p} V U \tag{1.154}$$

where $U : \Box \to \mathbf{C}^{q\times q}$ is a measurable unitary valued map, and $V : \Box \to \mathbf{C}^{p\times q}$ is a measurable function whose columns are normalized eigenvectors of $S(t)$ corresponding to the lowest q eigenvalues so that $V^*V = I_q$. Denote by π_t the permutation that orders the p eigenvalues m_k introduced in (1.149),

$$m_{\pi_t(0)}(t) \geq m_{\pi_t(1)}(t) \geq \ldots \geq m_{\pi_t(p-1)}(t)$$

Then

$$V = \left[\begin{array}{cccc} f_{\pi_t(p-q)}(t) & f_{\pi_t(p-q+1)}(t) & \cdots & f_{\pi_t(p-1)}(t) \end{array}\right] \tag{1.155}$$

and the distortion becomes

$$J_{de} = p \int \int_{\Box} \sum_{k=p-q}^{p-1} m_{\pi_t(k)}(t)\, dt\, ds \tag{1.156}$$

The Optimization Problem

Using again the analysis in subsection 3.2, we obtain the optimal solution of J_{de} subject to the norm constraint $||g^1|| = 1$ and biorthogonality $\Gamma^{1*}\Gamma^2 = pI_q$,

$$\Gamma^1 = cVU \tag{1.157}$$

$$\Gamma^2 = \frac{p}{c} VU \tag{1.158}$$

where U, V are as in (1.154) and $c = c(t)$ is given by

$$c(t) = \frac{[\sum_{k=p-q}^{p-1} m_{\pi_t(k)}(t)]^{1/4}}{\{\frac{q}{p}\int_0^{1/q} [\sum_{k'=p-q}^{p-1} m_{\pi_{t'}(k')}(t')]^{1/2}\, dt'\}^{1/2}} \tag{1.159}$$

The optimal distortion becomes:

$$J_{de} = q \left(\int_0^{1/q} \Big(\sum_{k=p-q}^{p-1} m_{\pi_t(k)}(t) \Big)^{1/2} dt \right)^2 \tag{1.160}$$

4 Non-Localization Results

In the previous section we obtained explicit solutions to several optimization problems. The optimizers share a common property: they are all obtained from eigenspaces of some selfadjoint operator. In this section we study how well the optimal windows can be localized in the time-frequency domain. We prove in general the optimizers are not well-localized in the sense the windows cannot belong to the following Banach spaces:

$$C(L^\infty, l^1) = \{f : \mathrm{R} \to \mathrm{C} \mid f \text{ continuous}, \ \|f\|_{W(L^\infty,l^1)} < \infty\}, \tag{1.161}$$

$$H^{1,1} = \{f \in L^2(\mathrm{R}) \mid \|f\|^2_{H^{1,1}} := \int (\frac{1}{2} + x^2)(|f(x)|^2 + |\hat{f}(x)|^2)\, dx \ < \infty\}. \tag{1.162}$$

4.1 CTSA

Recall the parametrization of the optimal solution (1.103, 1.104):

$$\Gamma^1 = M^{-1/2} V L \tag{1.163}$$

$$\Gamma^2 = M^{1/2} V L^{-*} \tag{1.164}$$

where L is a bounded invertible $\mathrm{C}^{q\times q}$-valued function defined over $\square$, $V^*V = I_q$ and the columns of V span an invariant subspace of $M^{1/2}WM^{1/2}$ associated to the largest q eigenvalues:

$$M^{1/2}WM^{1/2}V = V\Lambda. \tag{1.165}$$

Next we consider the case when $w = 1_{[0,a]}$, i.e. a uniform weight over an interval of the length of a translation step. Then $W(s) = I$, for all s, and (1.165) turns into an eigenproblem for $M(t)$. Since $M(t)$ commutes with $E(t)$ (see (1.96)), the eigenvectors of $M(t)$ coincide with the eigenvectors of $E(t)$ which are given by (1.147). Thus the columns of V form a subset of $\{f_0, f_1, \ldots, f_{p-1}\}$ given by the ordering of the associated eigenvalues. The eigenvalues are then:

$$\lambda_k(t) = \langle f_k, Mf_k\rangle = \hat{R}(\frac{qt+k}{p}) \tag{1.166}$$

where

$$\hat{R}(\omega) = \sum_m e^{2\pi i m\omega} R(\frac{m}{b}) \tag{1.167}$$

Let us denote by π_t the ordering permutation, i.e.

$$\lambda_{\pi_t(0)}(t) \geq \lambda_{\pi_t(1)}(t) \geq \ldots \geq \lambda_{\pi_t(p-1)}(t) \tag{1.168}$$

and by D_q the set of points t where the q^{th} eigenvalue is degenerate as follows:

$$D_q = \{t \in [0, \frac{1}{q}) \ , \ \lambda_{\pi_t(q-1)}(t) = \lambda_{\pi_t(q)}(t)\} \tag{1.169}$$

Assume the entries of $M(t)$ are continuous functions (for instance when $R \in W(L^\infty, l^1)$). Then $\lambda_k(t)$ and $\lambda_{\pi_t(k)}(t)$ are continuous functions, for every $0 \leq k \leq p-1$. Next we show D_q is a nonempty set. Using (1.166) note the following:

$$\lambda_k(t + \frac{1}{q}) = \lambda_{k+1}(t) \ , \ 0 \leq k \leq q-2 \tag{1.170}$$

$$\lambda_{p-1}(t + \frac{1}{q}) = \lambda_0(t) \tag{1.171}$$

Denote by π^0 the circular shift, $\pi^0(k) = k+1$, $0 \leq k \leq p-2$ and $\pi^0(p-1) = 0$. Then the above relations on λ imply that

$$\pi^0 \pi_{t+\frac{1}{q}} = \pi_t \tag{1.172}$$

Now assume $S = \{\pi_t(0), \pi_t(1), \ldots, \pi_t(q-1)\}$ is independent of t. It follows that S has to be invariant to π^0 as well, and this is impossible. Hence, for at least one t, $\lambda_{\pi_t(q-1)}(t) = \lambda_{\pi_t(q)}(t)$ and therefore D_q cannot be empty. Assume now that D_q contains isolated points. Note this is a structurally stable property with respect to the $W(L^\infty, l^1)$-norm on R, i.e. for ε-perturbations of R in $W(L^\infty, l^1)$-norm, D_q would still contain isolated points. Moreover, this is a generic property with respect to the same topolgy, namely, assume R is such that D_q does not contain isolated points, then one can choose an arbitrarily small perturbation so that only isolated points are left in D_q. Let t_0 be such an isolated point. Then for some $\varepsilon > 0$, $V(t)$ is uniquely determined on both $(t_0 - \varepsilon, t_0)$ and $(t_0, t_0 + \varepsilon)$. However, one column of $\lim_{t \searrow t_0} V(t)$ is orthogonal to all columns of $\lim_{t \nearrow t_0} V(t)$. Hence the two limits cannot coincide and Γ^1, Γ^2 cannot be continuous as functions of t. Similarly to the amalgam version of the Balian-Low theorem (see [BeHeWa95]), this implies the corresponding optimal windows g^1, g^2 cannot be in $C(L^\infty, l^1)$. On the other hand, since the discontinuity occurs along a segment of the form $\{(t_0, s) \ ; \ 0 \leq s \leq \frac{1}{p}\}$, the first derivatives $\frac{\partial G^1}{\partial t}$ and $\frac{\partial G^2}{\partial t}$ cannot be in $L^2(\Box)$. Therefore g^1, g^2 cannot belong to $H^{1,1}$. In effect we proved:

Theorem 4.1 *Assume $R \in W(L^\infty, l^1)$, $w = 1_{[0,a]}$ and D_q contains an isolated point. Then any pair of windows (g^1, g^2) optimal with respect to J_{ca} cannot be well-localized in the sense that*

$$g^1, g^2, \hat{g}^1, \hat{g}^2 \notin C(L^\infty, l^1) \tag{1.173}$$

and

$$g^1, g^2 \notin H^{1,1} \tag{1.174}$$

Moreover, the existence of isolated points in D_q is a generically and structurally stable property with respect to the $W(L^\infty, l^1)$-norm.

4.2 DTSA

Consider the discrete-time signal approximation problem in the case of uniform weights $w_{mn} = \frac{1}{q}$, for $m = 0$ and $0 \leq n \leq q-1$, and $w_{mn} = 0$, otherwise. The optimal pair of windows with respect to J_{da} criterion is parametrized by (1.122) and (1.123),

$$\Gamma^1 = \frac{1}{\sqrt{p}} L^* V^* \tag{1.175}$$

$$\Gamma^2 = \frac{1}{\sqrt{p}} L^{-1} V^* \tag{1.176}$$

where L is a bounded, invertible $C^{p\times p}$ matrix valued function over $\square$, and the columns of V form a subset of the canonical basis of C^q (i.e. the clumns of the identity matrix I_q). The choice of columns of V is based on the ordering of the diagonal elements $(\rho_{l,0}(t,s))_{0\leq l \leq q-1}$ of D^0 (see 1.109). Assume $R \in l^{1,1}(Z^2)$. Then $\rho_{l,k}(t,s)$ are continuous functions in (t,s). Using (1.106) we obtain

$$\rho_{l,0}(t+\frac{1}{q}, s) = \rho_{l+1,0}(t,s) \quad , \quad 0 \leq l \leq q-2 \tag{1.177}$$

$$\rho_{q-1,0}(t+\frac{1}{q}, s) = \rho_{0,0}(t,s) \tag{1.178}$$

Let us denote by $\pi_{t,s}$ the ordering permutation

$$\rho_{\pi_{t,s}(0),0}(t,s) \geq \rho_{\pi_{t,s}(1),0}(t,s) \geq \ldots \geq \rho_{\pi_{t,s}(q-1),0}(t,s) \tag{1.179}$$

and E_p the set of points where the p^{th} eigenvalue is degenerate as follows:

$$E_p = \{(t,s) \in \square \mid \rho_{\pi_{t,s}(p-1),0}(t,s) = \rho_{\pi_{t,s}(p),0}(t,s)\} \tag{1.180}$$

Using a similar argument as in the CTSA case, we obtain E_p as well as $E_p \cap ([0, \frac{1}{q}] \times \{s_0\})$ are non empty sets, for every $s_0 \in [0, \frac{1}{p}]$. Note E_p is a 2D subset of $\square$. Its topology may be complicated, in general. However, the generic E_p is made of continuous curves. We say E_q contains an *isolated curve* if there is an open set U of $\square$ such that $E_p \cap U$ has empty interior and separates U, i.e. $U = U_- \cup (E_p \cap U) \cup U_+$, where U_-, U_+ are open, and together with $E_p \cap U$ disjoint subsets of U. When E_p contains an isolated curve, $V(t,s)$ cannot be chosen continuously. If, moreover, $E_p \cap U$ is diffeomorphically equivalent to a straight line, i.e. there is a differentable $\phi : (\bar{a}, \bar{b}) \to U$ so that $E_p \cap U = \phi((\bar{a}, \bar{b}))$, then we say E_p contains an *isolated smooth curve.* In such a case, one can easily show there is a coordinate system over U so that $E_p \cap U$ becomes a straight line, and the first derivatives of G^1, G^2 cannot be square integrable. Thus $g^1, g^2 \notin H^{1,1}$. All these can be summarized into:

Theorem 4.2 *Assume $R \in l^{1,1}(\mathbb{Z}^2)$ and E_p contains isolated curves. Then any optimizer (g^1, g^2) of J_{da} cannot be well-localized in the sense:*

$$g^1, g^2, \hat{g}^1, \hat{g}^2 \notin C(L^\infty, l^1) \tag{1.181}$$

Furthermore, if E_p contains an isolated smooth curve, then g^1, g^2 cannot belong to $H^{1,1}$. The existence of isolated curves in E_q is a generically and structurally stable property with respect to the $l^{1,1}(\mathbb{Z}^2)$-norm.

Remark 4.3 Note this theorem proves that, generically, the optimum value of J_{da} is not achieved in the class $M_{1,1}$, where the optimization problem (1.36) is formulated.

4.3 CTSE

The optimal solutions of continuous-time signal encoding problem are similar to those of the discrete-time signal approximation problem. The optimal iso-pairs and dual frame pairs (g^1, g^2) are parametrized via a product of the form UV^*, with U a $\mathbb{C}^{p\times p}$ unitary valued map on $\Box$, and V a $\mathbb{C}^{q\times p}$ matrix valued map over $\Box$ whose columns form a subset of the canonical basis of $\mathbb{C}^q$; see (1.134, 1.137,1.138). The choice of the columns of V is made based on the ordering of the elements $(\mu_l(t,s) = \mu(t+\frac{l}{q}, s))_{0\leq l\leq q-1}$ from (1.129). Note

$$\begin{aligned} \mu_l(t+\frac{1}{q}, s) &= \mu_{l+1}(t,s) \quad , \quad 0 \leq l \leq q-2 && (1.182)\\ \mu_{q-1}(t+\frac{1}{q}, s) &= \mu_0(t,s) && (1.183)\end{aligned}$$

Denote by $\pi_{t,s}$ the ordering permutation of these elements:

$$\mu(t+\frac{\pi_{t,s}(0)}{q}, s) \geq \mu(t+\frac{\pi_{t,s}(1)}{q}, s) \geq \ldots \geq \mu(t+\frac{\pi_{t,s}(q-1)}{q}, s) \tag{1.184}$$

and

$$E_{q-p} = \{(t,s) \in \Box \mid \mu(t+\frac{\pi_{t,s}(q-p-1)}{q}, s) = \mu(t+\frac{\pi_{t,s}(q-p)}{q}, s)\} \tag{1.185}$$

Using the same arguments as in the previous subsection, when $R \in l^{1,1}(\mathbb{Z}^2)$, $E_{q-p} \cap ([0, \frac{1}{q}] \times \{s_0\})$ is not empty, for every s_0. Moreover, when E_{q-p}

contains an isolated curve, Γ^1, Γ^2 cannot be continuous, whereas when E_{q-p} contains an isolated smooth curve, the first order derivatives of G^1, G^2 cannot be in $L^2(\square)$. All these are summarized in the following

Theorem 4.4 *Assume $R \in l^{1,1}(\mathbb{Z}^2)$ and E_{q-p} contains isolated curves. Then any optimizer (g^1, g^2) of J_{ce} cannot be well-localized in the sense:*

$$g^1, g^2, \hat{g}^1, \hat{g}^2 \notin C(L^\infty, l^1) \tag{1.186}$$

Furthermore, if E_{q-p} contains an isolated smooth curve, then g^1, g^2 cannot belong to $H^{1,1}$. The existence of isolated curves in E_{q-p} is a generically and structurally stable property with respect to the $l^{1,1}(\mathbb{Z}^2)$-norm.

4.4 DTSE

The optimizers of the discrete-time signal encoding problem are similar to the optimizers of the continuous-time signal approximation problem. The optimal iso-pairs (given by (1.154)) and optimal biorthogonal pairs (given by (1.157,1.158)) are parametrized by matrix products of the form VU for some $U = U(t,s)$, a $\mathbb{C}^{q\times q}$ unitary matrix valued function over $\square$, and $V = V(t)$ a $\mathbb{C}^{p\times q}$ matrix valued function over $\square$ whose columns form a subset of the orthonormal basis $\{f_0, f_1, \ldots, f_{p-1}\}$ of $\mathbb{C}^p$ introduced by (1.147). The choice of the columns of $V(t)$ is dictated by the ordering of the eigenvalues of $S(t)$ of (1.144). These eigenvalues, denoted by $m_k(t)$, were computed in (1.149). Let π_t be the ordering permutation

$$m_{\pi_t(0)}(t) \geq m_{\pi_t(1)}(t) \geq \ldots \geq m_{\pi_t(p-1)}(t) \tag{1.187}$$

and let D_{p-q} denote the set of points where the $p-q^{th}$ eigenvalue is degenerate as follows:

$$D_{p-q} = \{t \in [0, \frac{1}{q}) \mid m_{\pi_t(p-q-1)}(t) = m_{\pi_t(p-q)}(t)\} \tag{1.188}$$

When $R \in W(L^\infty, l^1)$, m_k's are continuous and by a similar argument as in subsection 4.1 we obtain D_{p-q} is not empty. We also obtain, when D_{p-q} contains isolated points, $V(t)$ cannot be continuous and therefore the windows g^1, g^2 are not well-localized. All these results are summarized in the following:

Theorem 4.5 *Assume $R \in W(L^\infty, l^1)$ and D_{p-q} contains an isolated point. Then any optimal iso-pair or biorthogonal pair (g^1, g^2) of J_{de} cannot be well-localized in the following sense:*

$$g^1, g^2, \hat{g}^1, \hat{g}^2 \notin C(L^\infty, l^1) \tag{1.189}$$

and

$$g^1, g^2 \notin H^{1,1} \tag{1.190}$$

Moreover, the existence of isolated points in D_{p-q} is a generically and structurally stable property with respect to the $W(L^\infty, l^1)$-norm.

5 Numerical Examples

In this section we present a set of examples of the solutions presented before.

The semi-optimal problems start with a fixed analysis window, and design the optimal synthesis window, for a given weight and signal autocovariance functions. We choose the gaussian window

$$g^1(x) = e^{-8x^2} \tag{1.191}$$

as analysis window, and uniform weights, i.e.

$$w(x) = 1_{[0,a](x)} \tag{1.192}$$

for the CTSA problem, respectively

$$w_{mn} = \frac{1}{q}\delta_{m,0}1_{[0,q]}(n) \tag{1.193}$$

for the DTSA problem. The time step is set to one unit, $a = 1$. The covariance function is chosen to decrease exponentially fast:

$$R(x) = e^{-|x|} \tag{1.194}$$

in the CTSA case, and

$$R_{mn} = e^{-(|m|+|n|)} \tag{1.195}$$

Note the Fourier transform of $R(x)$ is $\hat{R}(\omega) = \frac{2}{\sqrt{2\pi}}\frac{1}{1+\omega^2} \geq 0$, and the discrete Fourier transform of R_{mn} is $\hat{R}(\omega_1, \omega_2) := \sum_{mn} R_{mn}e^{-im\omega_1 - in\omega_2} = \frac{(1-e^{-2})^2}{|1-e^{-1+i\omega_1}|^2|1-e^{-1+i\omega_2}|^2} \geq 0$. These prove the choice above corresponds indeed to two stationary stochastic processes.

Once this data is set, one can straightforwardly apply (1.98), (1.131), or (1.151) to obtain respectively the CTSA, CTSE, or DTSE semi-optimal solution. For the uniform weights (1.193), the equation (1.118) turns into:

$$\Gamma^1 D^0 = \frac{1}{p}\Gamma^1 D^0 \Gamma^{1*}\Gamma^2$$

whose solution is

$$\Gamma^2 = p(\Gamma^1 D^0 \Gamma^{1*})^{-1}\Gamma^1 D^0 \tag{1.196}$$

This gives the semi-optimal solution of the DTSA problem.

The optimal windows are given by the solution to an eigenvalue problem associated to $M(t)$ of (1.90), $D^0(t,s)$ of (1.110), $R(t,s)$ of (1.129), and $S(t)$ of (1.144). The optimal solution is not completely determined by the solution of the corresponding eigenproblem. There is a second term, namely the L-factor, that parametrizes all equivalent solutions. We are going to choose L in a particular way, so that the optimal windows become real valued functions. To this end we use the following result whose proof is straightforward:

Proposition 5.1 *Denote by $G(t,s)$ the Zak transform of an $L^2(\mathrm{R})$-function g, and by $\Gamma(t,s)$ the matrix representation (1.77). Then the following are equivalent:*

1. *g is real-valued;*
2. *$\overline{G(t,s)} = G(1-t,s)$;*
3. *$\overline{\Gamma(t,s)} = \Gamma(\frac{1}{q}-t,s)T$ where T is the $q \times q$ matrix with one on the antidiagonal, and zero in the rest, i.e. $T_{l_1,l_2} = \delta_{l_1+l_2,q-1}$ for $0 \leq l_1, l_2 \leq q-1$.*

Let us apply this result to the CTSA problem. When R is real valued, the matrix $M(t)$ has the following property:

$$\overline{M(t)} = M(\frac{1}{q}-t) \tag{1.197}$$

and therefore the eigenvectors of $M(\frac{1}{q}-t)$ can be chosen so that $V(\frac{1}{q}-t,s) = \overline{V(t,s)}$. Thus we solve the eigenproblem $M(t)V(t) = V(t)\Lambda(t)$ for $0 \leq t \leq \frac{1}{2q}$, and we set:

$$\Gamma^1(t,s) = \begin{cases} V(t) & , \text{ for } 0 \leq t \leq \frac{1}{2q} \\ \overline{V(t)}T & , \ otherwise \end{cases} \tag{1.198}$$

Similar argument goes for $S(t)$ and $c(t)$ associated to the DTSE problem.

For the DTSA problem, when R_{mn} are real, the diagonal matrix $D^0(t,s)$ is real, its diagonal elements $\rho_{l,0}(t,s)$ have the following property:

$$\rho_{l,0}(\frac{1}{q}-t,s) = \rho_{q-1-l,0}(t,s) \tag{1.199}$$

Hence D^0 has the following invariance property:

$$D^0(\frac{1}{q}-t,s) = TD^0(t,s)T^* \tag{1.200}$$

and therefore the matrix $V(t,s)$ in (1.124) can be chosen so that $V(\frac{1}{q}-t,s)=T\,V(t,s)$. In turn, this implies we can choose Γ^1 and Γ^2 via (1.122) and (1.123) to satisfy:

$$\Gamma^1(\frac{1}{q}-t,s)=\overline{\Gamma^1(t,s)}T \quad \Gamma^2(\frac{1}{q}-t,s)=\overline{\Gamma^2(t,s)}T.$$

and thus, optimal real-valued windows g^1, g^2. Similar argument applies to $R(t,s)$ and $c(t,s)$ assocuated to the CTSE problem.

Now we present the numerical results. In the left-hand side of the Figure 3 we plot the semi-optimal solutions of the CTSA problem, for several values of p and q. For comparison, the right-hand side of the same figure renders the optimal solution obtained as explained before. In Table 1.2 we summarize the values of the criterion J_{ca} obtained for the semi-optimal solution, respectively optimal solution. Similarly, the left-hand side of Figure 4 contains the semi-optimal solutions of the DTSA problem, whereas the right-hand side of the same figure contains the optimal window. Numerical values of the criterion J_{da} are presented in Table 1.3. For the CTSE problem we contrast the semi-optimal and optimal solutions in Figure 5, and the channel distortion is summarized in Table 1.4. Finally, the Figure 6 contains the same type of results for the DTSE problem, whereas Table 1.5 contains the channel distortion numbers. The typical behaviour of the eigenvalues is presented in Figure 7. There, the t-dependence over the interval $[0,\frac{1}{q}]$ is plotted. The abscisis is discretized in 50 points. Note the self-crossing of the eigenvalue maps which are responsible for the ill time-frequency localization of the optimal windows. Instead, the semi-optimal windows are better localized, but with a price in the approximation error.

p	4	3	2	5	3	4	5	6
q	1	1	1	3	2	3	4	5
Semi-opt.	0.206	0.199	0.179	0.162	0.145	0.116	0.095	0.080
Optimum	0.185	0.177	0.152	0.132	0.116	0.093	0.078	0.067

TABLE 1.2. Table of values of J_{ca}.

q	4	3	2	5	3	4	5	6
p	1	1	1	3	2	3	4	5
Semi-opt.	0.658	0.550	0.365	0.280	0.225	0.161	0.122	0.098
Optimum	0.534	0.429	0.275	0.206	0.166	0.120	0.094	0.078

TABLE 1.3. Table of values of J_{da}.

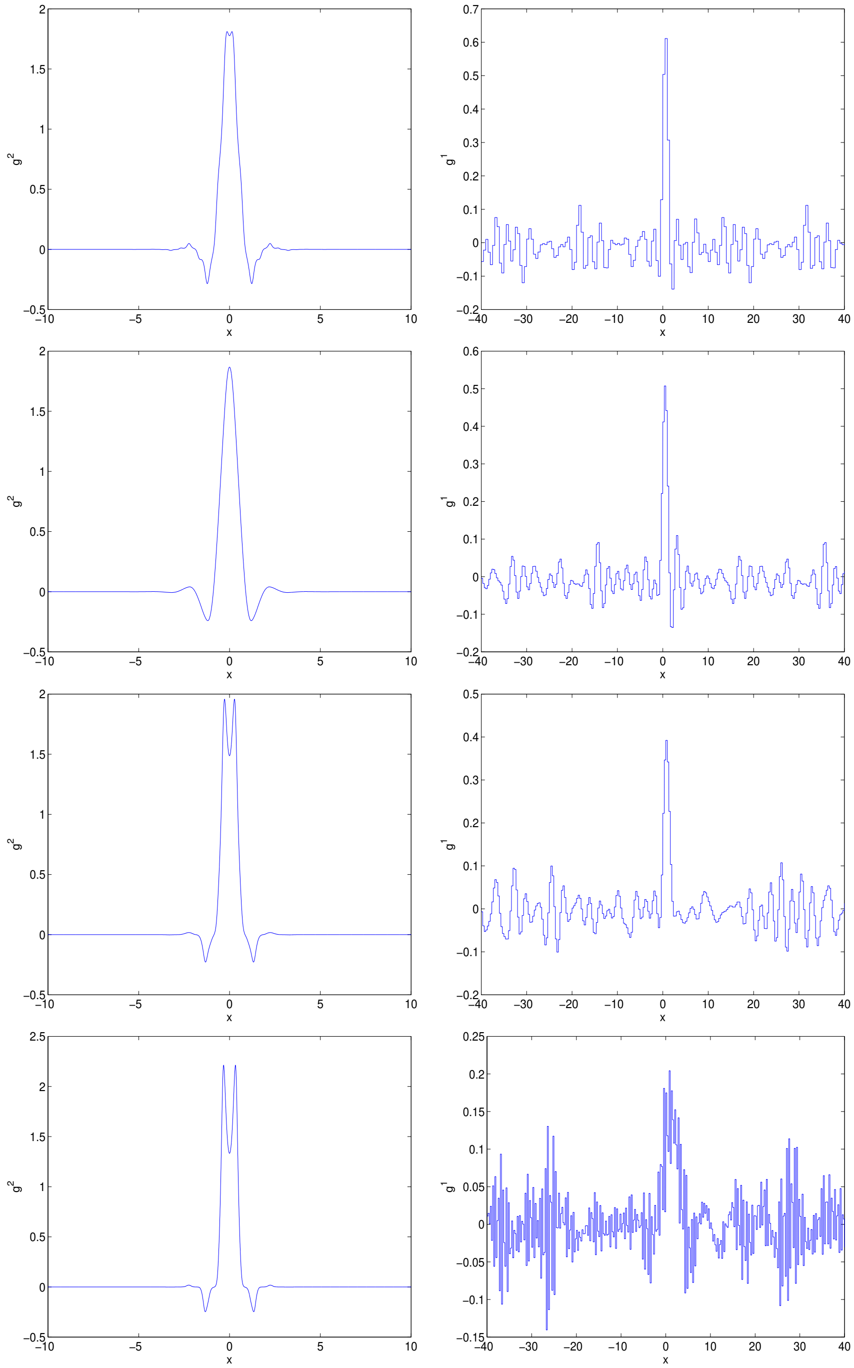

FIGURE 3. The semi-optimal and optimal solutions for the CTSA problem. Left plots contain the semi-optimal windows g^2 when the gaussian (1.191) is used for analysis; right plots contain the optimal solutions; the first row is for $p=2$, $q=1$; the second row is for $p=3$, $q=1$; the third row is for $p=3$, $q=2$; the fourth row is for $p=4$, $q=3$.

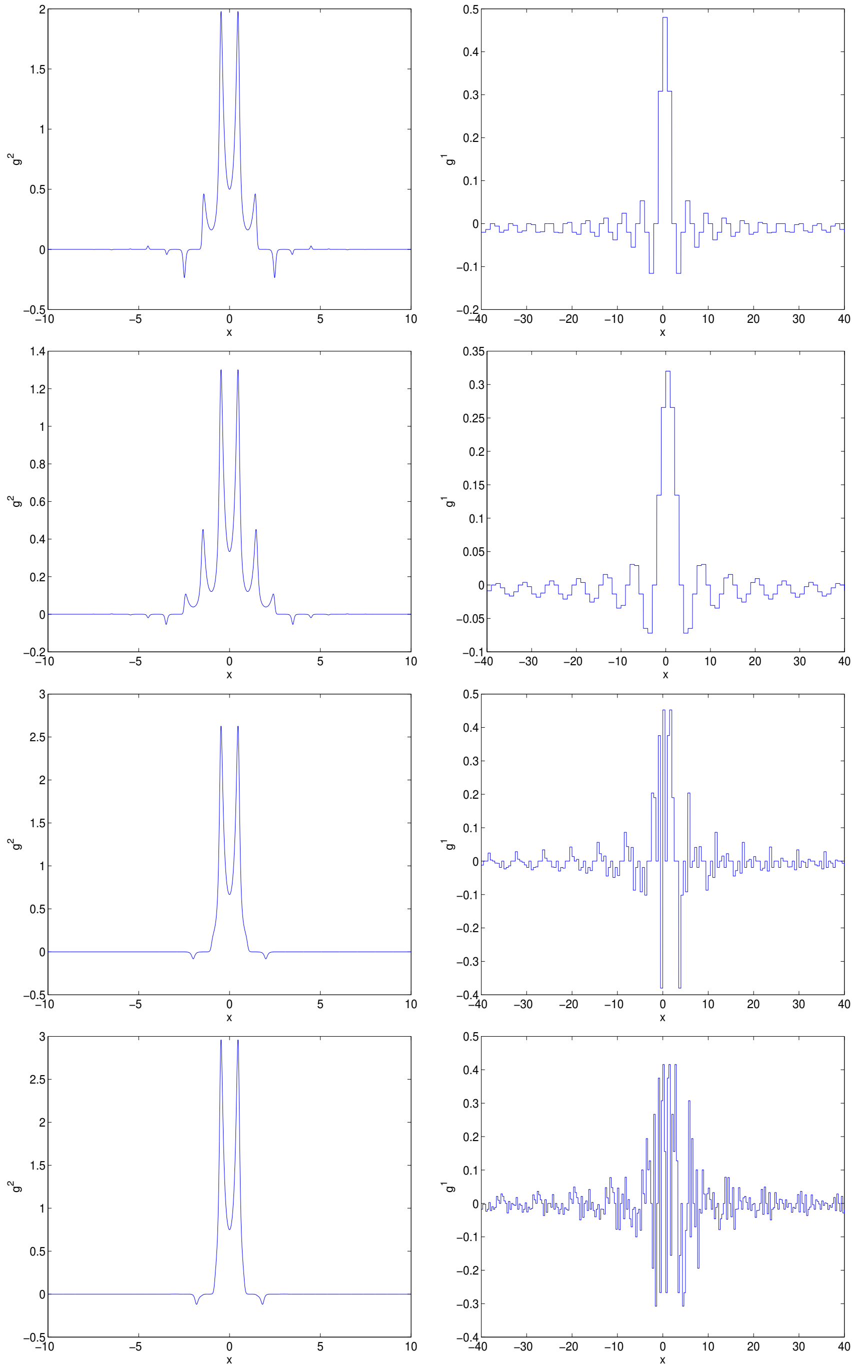

FIGURE 4. The semi-optimal and optimal solutions for the DTSA problem. Left plots contain the semi-optimal windows g^2 when the gaussian (1.191) is used for analysis; right plots contain the optimal solutions; the first row is for $p = 1$, $q = 2$; the second row is for $p = 1$, $q = 3$; the third row is for $p = 2$, $q = 3$; the fourth row is for $p = 3$, $q = 4$.

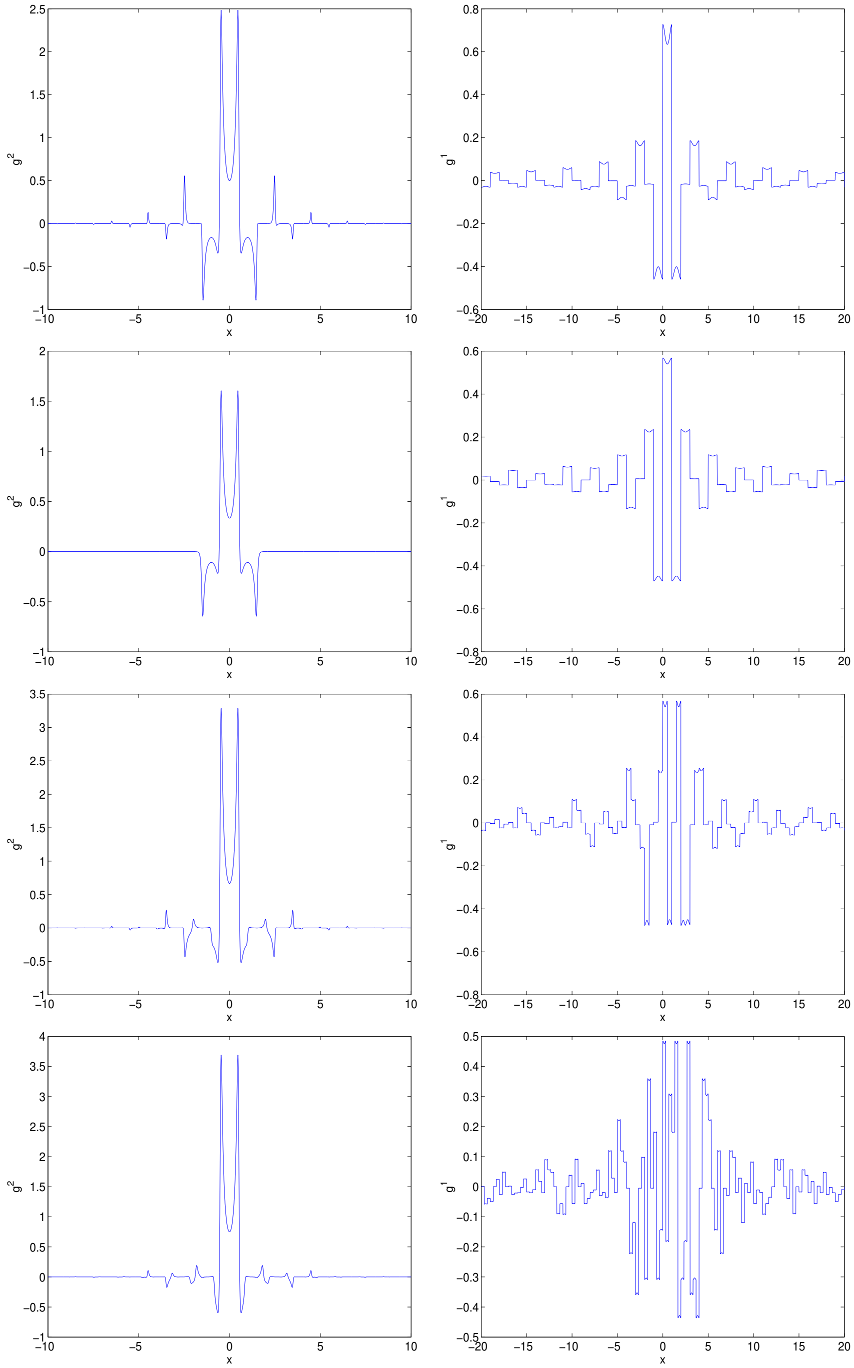

FIGURE 5. The semi-optimal and optimal solutions for the DTSE problem. Left plots contain the semi-optimal windows g^2 when the gaussian (1.191) is used for analysis; right plots contain the optimal solutions; the first row is for $p = 1$, $q = 2$; the second row is for $p = 1$, $q = 3$; the third row is for $p = 2$, $q = 3$; the fourth row is for $p = 3$, $q = 4$.

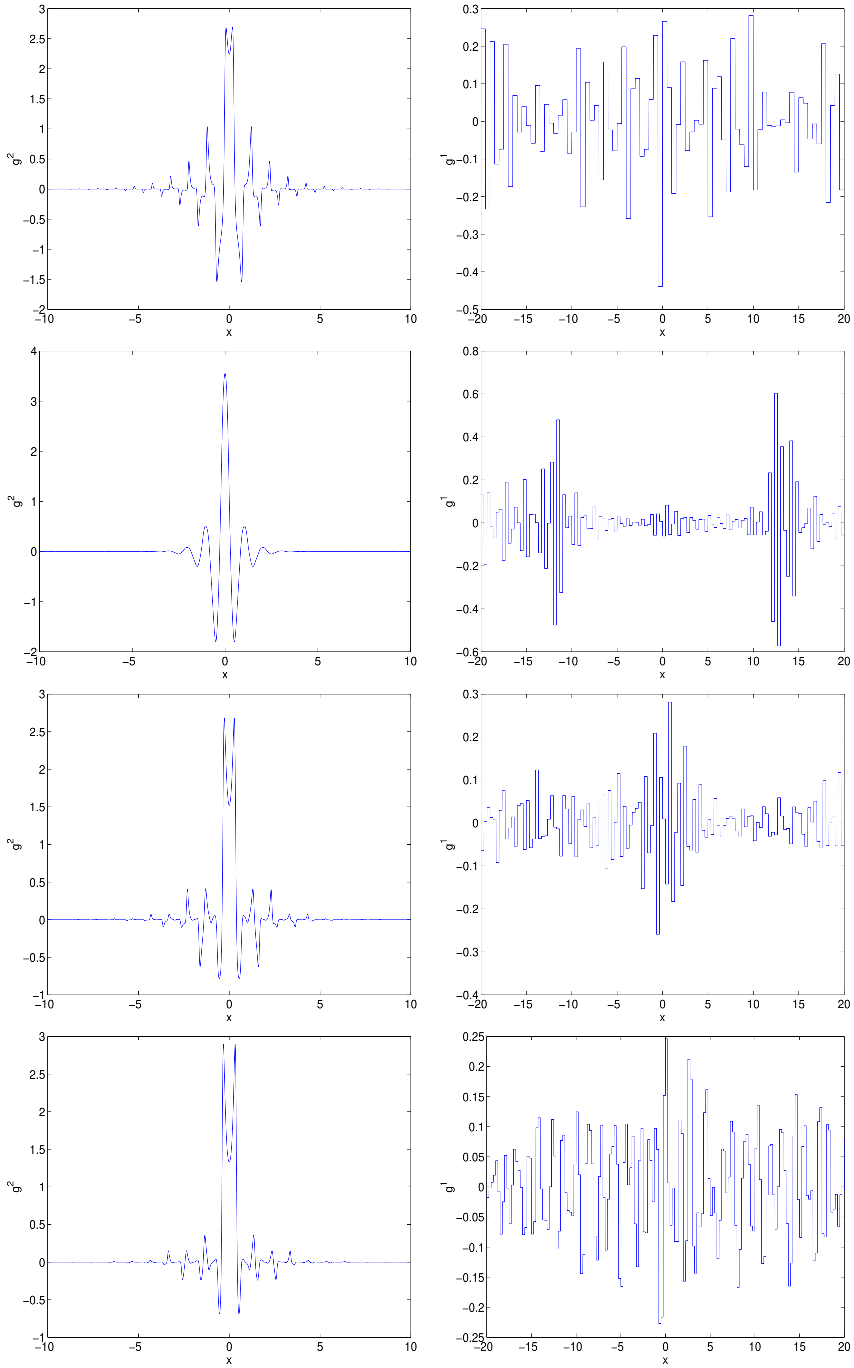

FIGURE 6. The semi-optimal and optimal solutions for the DTSE problem. Left plots contain the semi-optimal windows g^2 when the gaussian (1.191) is used for analysis; right plots contain the optimal solutions; the first row is for $p = 1$, $q = 2$; the second row is for $p = 1$, $q = 3$; the third row is for $p = 2$, $q = 3$; the fourth row is for $p = 3$, $q = 4$.

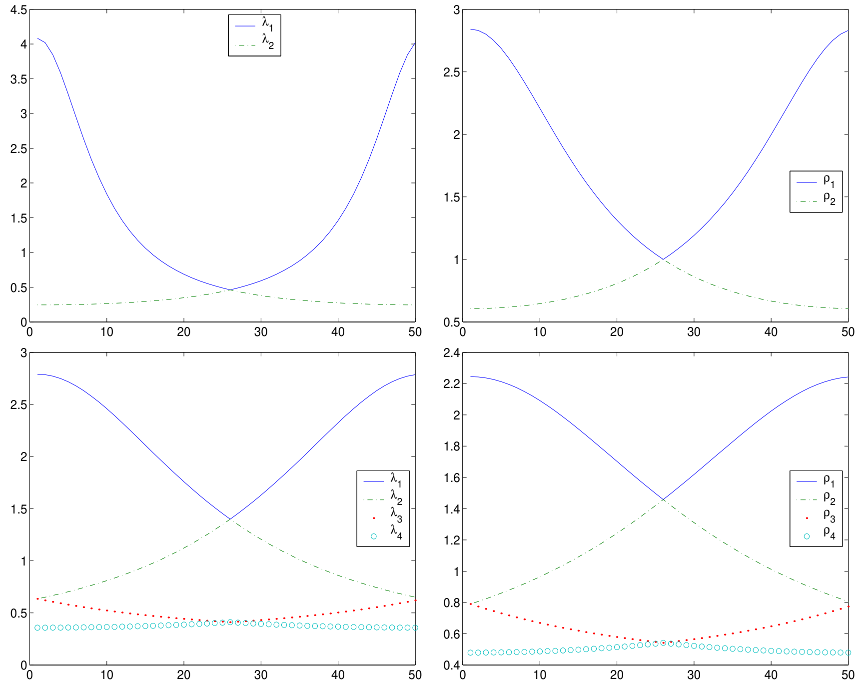

FIGURE 7. The eigenvalue maps. Left plots correspond to the CTSA problem, right plots to the DTSA problem; the top row is for $p = 2$, $q = 1$ (left) and $p = 1$, $q = 2$ (right); the bottom row is for $p = 4$, $q = 3$ (left), $p = 3$, $q = 4$ (right).

q	4	3	2	5	3	4	5	6
p	1	1	1	3	2	3	4	5
Semi-opt.	0.385	0.653	1.361	2.609	3.425	4.161	4.821	5.212
Optimum	0.030	0.055	0.135	0.216	0.285	0.400	0.487	0.554

TABLE 1.4. Table of values of J_{ce}.

p	4	3	2	5	3	4	5	6
q	1	1	1	3	2	3	4	5
Semi-opt.	0.587	0.649	1.018	1.491	1.83	2.469	2.959	3.350
Optimum	0.131	0.181	0.303	0.404	0.483	0.604	0.686	0.743

TABLE 1.5. Table of values of J_{de}.

6 Conclusions

In this chapter we presented an application of the Gabor analysis to four classes of stochastic signal problems. First we established a rigouros framework for stochastic signal analysis, namely through the use of amalgam spaces. In particular, continuous-time stationary signals (i.e. functions) are realized as elements of $W(L^2, l^\infty)$, whereas discrete-time stationary signals (i.e. sequences) are realized in $l^{\infty,\infty}(\mathbb{Z}^2)$. Thus we carried the Gabor analysis onto these two Banach spaces. We obtained necessary and sufficient conditions on the analysis and synthesis windows to have bounded operators. The amalgam space $W(L^2, l^\infty)$ is naturally mapped to the mixed-norm space $l^{2,\infty}(\mathbb{Z}^2)$, whereas $l^{\infty,\infty}(\mathbb{Z}^2)$ is mapped into the space of distributions $M'_{1,1}$. We proved the analysis window is in $W(L^\infty, l^1)$ is a sufficient condition for the analysis operator of the former case to be bounded. Surprisingly, it turned out the necessary decay of the window is given by $W(L^\infty, l^2)$, i.e. when the analysis operator is bounded then necessarily the window has to be in $W(L^\infty, l^2)$. Moreover, in some sense this seems to be optimal. For the latter case, the synthesis operator is bounded if and only if the synthesis window is in $M_{1,1}$.

These results helped us to formally define and represent the stochastic signals. It also constituted the formal argument for the algebraic manipulations we performed next. We considered two classes of situations. The first class concerns the approximation problem. Given a signal, whether continuous-time or discrete-time, we want to approximate it by an element in a linear (and smaller in the sense of trace) subspace constrained to be time-frequency shift invariant. We stated and analyzed two optimization problems and obtained explicit solutions involving only linear, quadratic or eigenvalue matrix equations. The second class of situations concerns the signal transmission problem. The signal is encoded either with a frame, when the data is a continuous-time signal, or with a Riesz basis for its span, when the data is a discrete-time signal. The received data is perturbed by the channel noise. Thus the decoder will use some channel information, more specific its second order statistics, and replace the standard dual frame, or biorthogonal decoding window, by another dual, or biorthogonal, optimally adapted to the channel. With this scenario in mind, we stated and analyzed three optimization problems, and obtained explicit solutions similar to the class of approximation problems. Interestingly, there seems to exist a duality between the solutions of these two classes of problems.

Next we analyzed the time-frequency localization of the optimizers of the four problems stated before. We proved that, in general, they are ill-localized in a sense similar to the Balian-Low non-localization phenomenon.

We concluded our study by giving a numerical example in designing optimal and semi-optimal analysis/synthesis windows.

7 References

[Balan98] R. Balan, *A Study of Weyl-Heisenberg and Wavelet Frames*, Ph.D. thesis, Princeton University 1998.

[BaDaVa00] R. Balan, I. Daubechies, V. Vaishampayan, "The Analysis and Design of Windowed Fourier Frame based Multiple Description Encoding Schemes", *IEEE Trans.Inform.Theory*, vol. 46 no.7 , pp.2491–2537 , 2000.

[Balian81] R. Balian, "Un principe d'incertitude fort en théorie du signal on en mécanique quantique", *C.R.Acad.Sci. Paris* , vol. 292, pp.1357–1362, 1981.

[Batt88] G. Battle, "Heisenberg Proof of the Balian-Low Theorem", *Letters in Mathematical Physics* , vol. 15, pp. 175–177, 1988.

[BeHeWa95] J. J. Benedetto, C. Heil, D. F. Walnut, "Differentiation and the Balian-Low Theorem", *J.Fourier Anal.Appl.* vol. 1, no.4, pp. 355–402, 1995.

[ChDeHe99] O. Christensen, B. Deng, C. Heil , *Density for Gabor Frames*, Applied and Computational Harmonic Analysis

[Chui93] C.K.Chui, X.Shi, *Inequalities of Littlewood-Paley Type for Frames and Wavelets*, SIAM J.Math.Anal. **24**, no.1 (1993), 263–277

[Daub90] I. Daubechies, " The Wavelet Transform, Time-Frequency Localization and Signal Analysis", *IEEE Trans.Inform.Theory*, vol. 36, no.5, pp.961–1005, 1990.

[DaLaLa96] I. Daubechies, H. J. Landau, Z. Landau, "Gabor Time-Frequency Lattices and the Wexler-Raz Identity", *J.Fourier Anal.Applic.*, vol. 1, no.4, pp.437–478, 1995.

[Feich89] H. G. Feichtinger, *Atomic Characterization of Modulation Spaces Through Gabor-Type Representations*, Rocky Mount.J.Math. **19**, no.1 (1989), 113–125

[FeiZimm98] H. G. Feichtinger, G. Zimmerman, "A Banach space of test functions for Gabor analysis", Chapter 3 in "Gabor Analysis and Algorithms. Theory and Applications", H. .G. Feichtinger, T. Strohmer Eds., Birkhäuser 1998.

[FoSt85] J. J. F. Fournier, J. Stewart, " Amalgams of L^p and l^q", *Bull.AMS*, vol. 13 , no.1, pp.1–21, 1985.

[Gröch00] K. Gröchenig, "Foundations of Time-Frequency Analysis", Birkhäuser, 2000.

[GrHe00] K. Gröchenig, C. Heil, *Gabor Meets Littlewood-Paley: Gabor Expansion in* $L^p(\mathrm{R}^d)$, preprint August 2000.

[GrHeOk01] K. Gröchenig, C. Heil, K. Okoudjou, *Gabor Analysis in Amalgam Spaces*, preprint 2001.

[GrLe01] K. Gröchenig, M. Leinert, *Wiener's Lemma for Twisted Convolution and Gabor Frames*, preprint 2001.

[HeWa89] C. Heil, D. Walnut, " Continuous and Discrete Wavelet Transforms", *SIAM Review*, vol. 31, no.4, pp.628–666, 1989.

[HlMaKiKo00] F. Hlawatsch, G. Matz, H. Kirchauer, W. Kozek, "Time-Frequency Formulation, Design, and Implementation of Time-Varying Optimal Filters for Signal Estimation", to appear in IEEE Trans. on Sig. Proc. 2000.

[Jans82] A. J. E. M. Janssen, " Bargmann Transform, Zak Transform, and Coherent States", *J.Math.Phys.* , vol. 23, pp.720–731, 1982.

[Jans88] A. J. E. M. Janssen, " The Zak Transform: a Signal Transform for Sampled Time-Continuous Signals", *Philips J.Res.*, vol. 43, pp.23–69, 1988.

[Jans95] A. J. E. M. Janssen, " Duality and Biorthogonality for Weyl-Heisenberg Frames", *J.Fourier Anal.Appl.*, vol. 1, no.4, pp.403–436, 1995.

[Kato80] T. Kato, *Perturbation Theory for Linear Operators* , Springer-Verlag 2nd Ed. 1980.

[Koz98] W. Kozek, "Nonorthogonal Pulseshapes for Multicarrier Communications in Doubly Dispersive Channels", IEEE J. on Selected Areas in Comm., vol. 16, no.8, pp.1579–1589, 1998.

[Li95] S. Li, "On General Frame Decompositions", Num.Funct.Anal.and Optim. 16(9&10), pp. 1181-1191, 1995

[Munch92] N. J. Munch, " Noise Reduction in Tight Weyl-Heisenberg Frames", *IEEE Trans.Info.Th.*, vol. 38, no.2, pp.608–616, 1992.

[Strohmer98] Th. Strohmer, "Numerical algorithms for discrete Gabor expansions", Chapter 8 in "Gabor Analysis and Algorithms. Theory and Applications", H. .G. Feichtinger, T. Strohmer Eds., Birkhäuser 1998.

[Waln92] D. F. Walnut, " Continuity Properties of the Gabor Frame Operator", *J.Math.Anal.Appl.*, vol. 165, no.2, pp.608–616, 1992.

[ZiZe97] M. Zibulski, Y. Y. Zeevi, "Analysis of Multi-Window Gabor-Type Schemes by Frame Methods", Appl.Comp.Harm.Anal., vol. 4, pp.188–221, 1997