The Influence of Windowing on Time Delay Estimates

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1 Introduction

It is often assumed that a pure delay translates into a phase shift in frequency domain, and thus one can estimate the time delay by evaluating the phase shift for one or a set of frequencies. Such a relation holds true when the Fourier transform is used to go into the frequency domain. However, in practice, one has to use the windowed Fourier transform to obtain the frequency domain representation. In this short note we show the influence of windowing on the delay estimation problem is negligable for a reasonable large class of windows and signals. Such a result is useful when fractionary delays are to be estimated.

2 Best Delay Estimate: The Problem

Let us consider a signal s and a window g smooth enough for the following derivations to hold true. We consider a family of dilated versions of g, $g_B(X) = g(Bx)$, $B > 0$ and study the asymptotical behaviour for $B \to 0$. The *Windowed Fourier Transform* (WFT) of s with respect to g is defined by:

$$\mathcal{F}_{g_B} s(\omega, t) = \int_{-\infty}^{\infty} g_B(x-t) s(x) dx \tag{1}$$

If we denote by $T_\tau s(x) = s(x-\tau)$ the translation by τ of s, then its WFT is:

$$E_B^\tau := \mathcal{F}_{g_B} T_\tau s(\omega, t) = \int_{-\infty}^{\infty} e^{i\omega x} g_B(x-t) s(x-\tau) dx \tag{2}$$

The basic question reduces to how to find the "best" phase Φ^* ("best" in a sense to be explained later) that fits into:

$$\mathcal{F}_{g_B} T_\tau s \approx e^{i\Phi^*} \mathcal{F}_{g_B} s \tag{3}$$

In particular, we are interested in phases that belong to the polynomial class:

$$\Phi_{M,N} = \{\phi|\phi(\omega,B) = \sum_{k=0}^{N}\sum_{l=0}^{M}\gamma_{kl}B^k\omega^l\} \tag{4}$$

The "best" phase in (3) is defined as follows:

$$\Phi^* = argmin_{phase\in\Phi_{M,N}}\int_{-\infty}^{\infty}|\Im\log\frac{E_B^\tau(\omega,t)}{E_B^0(\omega,t)} - phase(\omega,B)|^2|E_B^0(\omega,t)|^2d\omega \tag{5}$$

where the criterion is a weighted distance between the actual phase difference $(\Im\log\frac{E_B^\tau}{E_B^0})$ and the predicted phase $(phase(\omega,B)$. Next we present the answer to the "best phase" question in several cases.

3 Particular Cases of Best Delays

1. The best linear phase. If the best phase has the form $\Phi^*(\omega,B) = \hat{\tau}(B)\omega$ then we obtain the following estimate for $\hat{\tau}$:

$$\hat{\tau} = \int_{-\infty}^{\infty}\omega|E_B^0(\omega,t)|^2\Im\log\frac{E_B^\tau(\omega,t)}{E_B^0(\omega,t)}d\omega/\int_{-\infty}^{\infty}\omega^2|E_B^0(\omega,t)|^2d\omega \tag{6}$$

Our analysis shows the estimate (6) for τ is biased and, for smooth windows, the first asymptotic correction is of second order in B and given by:

$$\frac{\hat{\tau}-\tau}{\tau} = \frac{1}{2}\frac{g(0)g''(0)-(g'(0))^2}{(g(0))^2}\frac{B^2}{\Omega_s^2} \tag{7}$$

where $\Omega_s^2 = \int\omega^2|\hat{s}(\omega)|^2d\omega/\int|\hat{s}(\omega)|^2d\omega$ is the mean-squared bandwidth, and $\hat{s}(\omega) = \frac{1}{\sqrt{2\pi}}\int e^{-i\omega x}s(x)dx$ is the Fourier transform of s. Thus:

$$\hat{\tau}(B) = \tau(1+\frac{1}{2}\frac{g_B(0)g_B''(0)-(g_B'(0))^2}{(g_B(0))^2}\frac{1}{\Omega_S^2}) + O(B^3) \tag{8}$$

2. The best phase as a second order polynomial in B. For $N=2$, the best phase has the form:

$$\Phi^*(\omega,B) = \tau\omega + B^2\sum_{m=0}^{P}\gamma_m\omega^{2m+1} \tag{9}$$

and γ's satisfy the following linear system:

$$\mathbf{S}\gamma = \tau\frac{g(0)g''(0)-(g'(0))^2}{(g(0))^2}\mathbf{a} \tag{10}$$

where $\gamma = (\gamma_m)_{0\le m\le P}$, $\mathbf{a} = (a_m)_{0\le m\le P}$, $\mathbf{S} = (S_{m_1,m_2})_{0\le m_1,m_2\le P}$ with $S_{m_1,m_2} = S_{m_1+m_2+1}$, $a_m = (m+\frac{1}{2})S_m$ and $S_m = \int\omega^{2m}|\hat{s}(\omega)|^2d\omega$.

3. The best phase using the first nontrivial term in B under the follwoing assumptions on g: $g'(0) = g''(0) = \cdots = g^{(N-1)}(0) = 0$ and $g^{(N)}(0) \neq 0$. The lowest order in B of the bias is B^N:

$$\Phi^* = \tau\omega + B^N\sum_{m=0}^{P}\gamma_{N,m}\omega^{2m+1} \tag{11}$$

and the unknowns $\gamma_{N,m}$ satisfies a linear system involving various moments of $|\hat{s}^{(l)}(\omega)|^2$.

4 Conclusions

Figure 1 plots the frequency dependence of the phase of $\frac{E_B^\tau}{E_B^0}$ for a cosinus-like window, delay of 10 samples and several window length. The data $s(x)$ has been taken from TIMIT databasis. Note for wide enough windows, the relative error becomes negligible, as predicted. We point out we have not considered the numerical aspect due to the discreteness of the practical data. Indeed if one replaces E_B^τ by its numerical evaluation (Discrete Windowed Fourier Transform), the difference would depend on the sampling frequency. For sufficiently dense sampling, the numerical integration error gets arbitrarily small. However this is not the focus of our paper. For the data used here, the sampling frequency was high enough to make the numerical errors negligable. For a 3KHz bandlimited signal and a Hanning window of length 20ms, the relative bias is about 0.3%.

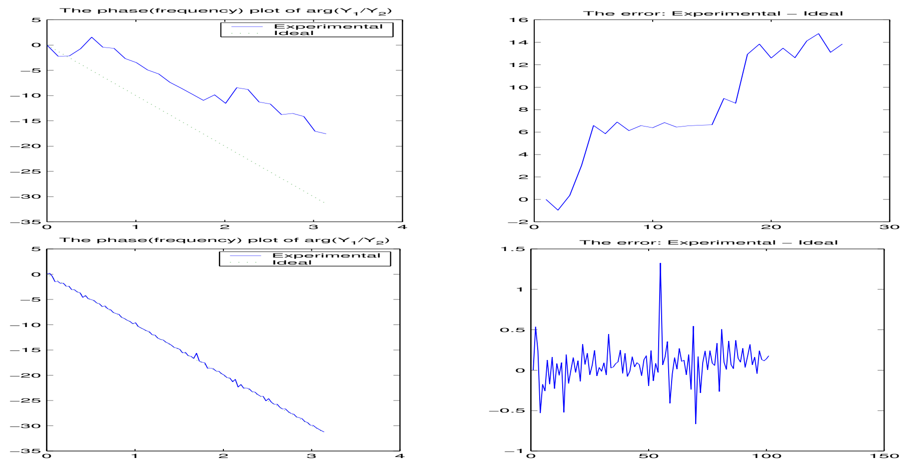

Figure 1: The experimental plots for the window $g(t) = cos(wt)$ with $-\frac{\pi}{w} \leq t \leq \frac{\pi}{w}$ and $w = 1000Hz$ (upper plot), and $w = 250Hz$ (lower plot). The mean-square signal bandwidth is about $3KHz$ and the sampling frequency is $16KHz$.

In conclusion we have obtained a closed formula for the delay estimation bias when the measured data is windowed. Numerical experiments show the bias is very small for resonably wide windows, proving that windowing in itself does not represent a major source of error. To improve the estimation, the estimated delay can be corrected by the bias given in (7).