DISTRIBUTIONS WITH SINGULARITIES: Punctual and Local Study

R. BALAN
Polytechnic Institute of Bucharest
Department of Automatic Control and Computers
313, Splaiul Independentei, Bucharest , Romania

15 June 1992

Abstract

In this paper we complete the theory of punctual and local integrability of smooth and analytic distributions starting with the classical Hermann's and Nagano's results (of which we give new proofs). Then we discuss Stefan's and Sussmann's papers (where we assert that there are some errors) and we give a different version of a theorem. Finally we give a new proof of Cerveau's theorem that is a complete characterization of finitely-generated involutive $\mathcal{F}(M)$-module of smooth vector fields.

1 Introduction

Let M be a $\mathcal{C}^r$ finite-dimensional, connected and paracompact manifold ($r = \infty$ or ω , by the case); let $\mathcal{F}(M)$ denote the ring of the $\mathcal{C}^r$ real-valued functions defined on M and let $V^r(M)$ be the $\mathcal{F}(M)$-module of $\mathcal{C}^r$ vector fields on M. We put $n = \dim M$.

We call *distribution* on M , the mapping :

$$L : x \in M \longrightarrow L(x) \subset T_x M$$

where $L(x)$ is a vector subspace of the tangent space to M at x. The *dimension*(or *rank*) of the distribution is $\dim L(x)$ (it is punctually defined).

Let S be a set of $\mathcal{C}^r$ vector fields everywhere defined. The distribution generated by the set S is :

$$L(x) = span_{\mathbf{R}}\{ \, v|_x \quad , v \in S \, \} \quad \forall x \in M.$$

We call $\mathcal{C}^r$ *-distribution on* M , a distribution L generated by a set S of $\mathcal{C}^r$ vector fields.

The distribution L is called *integrable at* $x_0 \in M$ if there exists a submanifold $N_{x_0} \overset{i}{\hookrightarrow} M$ (i being the canonical inclusion) passing through x_0, such that:

$$T_x N_{x_0} = L(x) \text{ , for all } x \in N_{x_0}.$$

(more precisely, we have: $i_{*,x}(T_x N_{x_0}) = L(x)$, $\forall\ x \in N_{x_0}$, where $i_{*,x}$ is the differential of i in x). N_{x_0} is called an *integral manifold* of the distribution and we say that L is *punctually integrable* in x_0. From the definition it follows directly that dim $N_{x_0} = \dim L(x_0)$ and L is also punctually integrable in every $q \in N_{x_0}$.

The distribution is called *locally integrable* (or to have *the integral manifold property* if for each point in M there is an integral manifold of the distribution L (namely if it is punctually integrable at every point of M).

Let us consider the distribution L and a point $x_0 \in M$. If there exists a neighborhood of x_0 where the distribution has constant dimension then the point x_0 is called an *ordinary point* (or a *regular point*), otherwise it is called a *singular point*. If the distribution has singular points then we say that it is a *distribution with singularities*.

Our goal is to find criteria of punctual and local integrability of a distribution generated by a $\mathcal{F}(M)$-module of $\mathcal{C}^r$ vector fields (this distribution may be a distribution with singularities).

In §2 we discuss Stefan's and Sussmann's papers pointing out some errors. Also we give a few examples about involutivity of modules and distributions.

In §3 we construct a split of distribution that will be useful through the whole paper. This split was suggested us by the Nagano's paper ([Na66]).

Based on this construction we will prove results about punctual integrability (in §4): Theorem(4.4) which represents the punctual version of Nagano's theorem (the result appears in [Fr78] but with an algebraic proof), Theorem(4.6) which represents a reformulation of a theorem presented in [Su73, St74, St80] but always with different statements, a new proof of a theorem presented in [St74], as well as criteria involving various conditions (for example the involutivity). We point out that the Theorem(4.4) works in the analytic case while Theorem(4.6) requires only $r = \infty$.

By extending the study on an open subset of the manifold, we will obtain results about local integrability (§5):the well-known Nagano's theorem (in the analytic case), Hermann's theorem (with a new proof) and a normal form of the finitely-generated involutive module. From the last theorem Hermann's theorem follows as a simply corollary and we can give another proof of Nagano's theorem using a known algebraic result (the notherianity of the module of analytic vector fields).

Since our study is punctual or local, we point out that the integral manifolds are always regular embedding submanifolds.

2 Preliminary definitions and results

If S is a set of vector fields everywhere defined on M then we denote by $S^{\#}$ the $\mathcal{F}(M)$-module generated by S (i.e. the smallest $\mathcal{F}(M)$-module which includes S). We observe that the distribution generated by S is the same with the distribution generated by $S^{\#}$.

2.1 Discussion about Stefan's and Sussmann's papers

We assert that the implication $e \Rightarrow d$ of Theorem 4.2 from Sussmann's paper ([Su73]) and Theorem 4 from Stefan's paper ([St80]) are not correct (implicitly Theorem 5 -in [St80]- has a wrong proof). However, the equivalences
$a \Leftrightarrow b \Leftrightarrow c \Leftrightarrow d \Leftrightarrow f$ in [Su73] are correct. To prove this we will use an example given by Stefan himself in [St80] relative to a wrong theorem by Lobry ([Lo70]). We refer now to Stefan's paper and we begin by recalling the definition of local subintegrability. A set S of $\mathcal{C}^\infty$ vector fields is *locally subintegrable at* $x_0 \in M$ if there exists a neighborhood Ω of x_0 in M and a subset S^b of S which satisfies the following conditions :

(LS.1) $L^b(x_0) = L(x_0)$ and S^b is integrable on Ω

(LS.2) For every vector field X in S there exists $\varepsilon > 0$ such that

$$dX^t(x_0).L^b(x_0) = L^b(X^t(x_0)) \text{ for } |t| < \varepsilon$$

(we have denoted by L the distribution generated by S, by L^b the distribution generated by S^b and by $X^t(x_0)$ the flow generated by the vector field X).
We say that S is *locally subintegrable* (on M) if it is locally subintegrable at every $x \in M$. We remark that the choice of the subset S^b may depend on the point x_0. The theorem that we assert it is not correct, is the following

"**Theorem 4** *(from [St80]) A set S of $\mathcal{C}^\infty$ vector fields is integrable if and only if the set $S^{\#}$ is locally subintegrable on M.* "□

The example that proves this is the following:

EXAMPLE 2.1 Let $M = \mathbf{R}^2$ and let S be the set of all vector fields of the form:

$$\frac{\partial}{\partial x} + \Phi(x,y)\frac{\partial}{\partial y}$$

where Φ is an arbitrary smooth (i.e. $\mathcal{C}^\infty$) function which satisfies two requirements:

1) $\Phi(0,0)=0$

2) $\frac{\partial \Phi}{\partial x} \equiv 0$ in some neighborhood of the origin depending on Φ. ◇

The distribution generated by S will be of the form:

$$L(x_0) = \begin{cases} T_{x_0}\mathbf{R}^2 & , \ x_0 \neq (0,0) \\ span_{\mathbf{R}}\{\frac{\partial}{\partial x}\mid_{(0,0)}\} & , \ x_0 = (0,0) \end{cases}$$

So:

$$\dim L(x_0) = \begin{cases} 2 & , \ x_0 \neq (0,0) \\ 1 & , \ x_0 = (0,0) \end{cases}$$

It is clear that L is not integrable in the origin. We will prove that $S^{\#}$ is locally subintegrable on $M = \mathbf{R}^2$. To this purpose we will use Proposition(6.3) (from [St80])-first point:

Proposition 6.3 *For a set S of $\mathcal{C}^\infty$ vectorfields to be locally subintegrable at $x_0 \in M$ it is sufficient that there exist a neighborhood Ω of x_0 in M and vector fields $Y_1, Y_2, \ldots, Y_p$ in S which satisfy the following conditions:*

(a) The vectors $Y_1, Y_2, \ldots, Y_p$ span $L(x_0)$

(L denotes the distribution generated by S)

(b) There exist continuous functions $\lambda_{ijk} : \Omega \longrightarrow \mathbf{R}$ such that, for every $y \in \Omega$ and $1 \leq i, j \leq p$,

$$[Y_i, Y_j](y) = \sum_k \lambda_{ijk}(y) Y_k(y)$$

(c) Given $X \in S$ there exist $\varepsilon > 0$ and continuous functions $\lambda_{ik} : [-\varepsilon, \varepsilon] \to \mathbf{R}$ such that, for $|t| \leq \varepsilon$ and $1 \leq i \leq p$,

$$[X, Y_i](x_t) = \sum_k \lambda_{ik}(t) Y_k(x_t) \quad , \text{ where } x_t = X^t(x_0), \text{ as above.} \ \Box$$

Now, the proof that $S^{\#}$ is locally subintegrable on $M = \mathbf{R}^2$.

Let $x_0 \in \mathbf{R}^2$, $x_0 \neq (0,0)$.Let Ω be a neighborhood of x_0 such that $O(0,0) \notin \overline{\Omega}$ ($\overline{\Omega}$ denote the closure of Ω). Then there exists a function Φ so that $\Phi(q) \neq 0, \forall q \in \Omega$ and $\Phi(x,y)\frac{\partial}{\partial y} \in S^{\#}$. Let $\Psi : \mathbf{R}^2 \to \mathbf{R}$ be a smooth function such that $\Psi(q) \neq 0, \forall q \in \mathbf{R}^2$ *and* $\Psi(q) = \Phi(q)$ $\forall q \in \Omega$ (Ψ can be found , possibly by reducing the neighborhood Ω and using the partition of unit) and let $Y_2 = \Psi^{-1}\Phi\frac{\partial}{\partial y} \in S^{\#}$. On Ω we have $Y_2|_\Omega = \frac{\partial}{\partial y}|_\Omega$. Let $Y_1 = \frac{\partial}{\partial x} \in S^{\#}$. We verify (a)–(c) of the previous proposition using Y_1, Y_2 *and* Ω like above.

(a) $\{Y_1(x_0), Y_2(x_0)\} = \{\frac{\partial}{\partial x}|_{x_0}, \frac{\partial}{\partial y}|_{x_0}\}$ and it spans $T_{x_0}\mathbf{R}^2 = L(x_0)$

(b) $[Y_1, Y_2]|_\Omega = 0$

(c) For $X \in S$ we can choose $\varepsilon > 0$ such that $x_t = X^t(x_0) \in \Omega$, $\forall |t| < \varepsilon$. Then the functions $\lambda_{ik}(t)$ will be even the components of $[X, Y_i]$ in the local frame $\{\frac{\partial}{\partial x}|_\Omega, \frac{\partial}{\partial y}|_\Omega\}$.

Now, let $x_0 = (0,0)$. We put $Y_1 = \frac{\partial}{\partial x} \in S^{\#}$ and let Ω be an arbitrary neighborhood of the origin. We verify (a),(b),(c):

(a) $Y_1(x_0) = \frac{\partial}{\partial x}|_{x_0}$ and spans $L(0)$

(b) $[Y_1, Y_1] = 0$

(c) Let $X \in S^{\#}$. Then X is of the form:

$$X = f_1 \frac{\partial}{\partial x} + f_2 \Phi \frac{\partial}{\partial y}$$

where $f_1, f_2 : \mathbf{R}^2 \to \mathbf{R}$ are arbitrarly smooth functions and Φ satisfies the two requirements. Remark that $\exists \mu > 0$ such that $\Phi(x, 0) = 0$, for all $|x| < \mu$.
We find the integral curve of the vector field X passing through the origin. We have the system:

$$\begin{cases} \dot{x} = f_1(x, y) & , \ x(0) = 0 \\ \dot{y} = f_2(x, y)\Phi(x, y) & , \ y(0) = 0 \end{cases}$$

We obtain a solution x=x(t) at least continuous. We choose $\varepsilon > 0$ such that we have: $|x(t)| < \mu$, for all $|t| < \varepsilon$.
Since $y(t) = 0, \ |t| < \varepsilon$ is a particular solution of the second equation and using the theorem of existence and unicity of the Cauchy problem we obtain the system solution:

$$\begin{cases} x = x(t) & , \ |t| < \varepsilon \ \textit{such that} \ |x(t)| < \mu \\ y = 0 \end{cases}$$

So we have:

$$[X, Y_1] = [f_1 \frac{\partial}{\partial x} + f_2 \Phi \frac{\partial}{\partial y}, \frac{\partial}{\partial x}] = -\frac{\partial f_1}{\partial x} \frac{\partial}{\partial x} - (\frac{\partial f_2}{\partial x} \Phi + f_2 \frac{\partial \Phi}{\partial x}) \frac{\partial}{\partial y}$$

By choosing $\lambda_{11}(t) = -\frac{\partial f_1}{\partial x}(x(t), 0)$ and because $\Phi(x(t), 0) = 0$ and $\frac{\partial \Phi}{\partial x}(x(t), 0) = 0$ we obtain:

$$[X, Y_1](x_t) = \lambda_{11}(t) Y_1(x_t) \ , \ \textit{for all} \ |t| < \varepsilon.$$

So we have checked that $S^\#$ is locally subintegrable (we can check it also directly using the definition — respectively the conditions $(LS.1)$ and $(LS.2)$). The mistake (in [St80]) consists in the next assertion of the Lemma(6.2) : "...it is easy to produce a subsequence (s_m) of (t_m) and a $\mathcal{C}^\infty$ vectorfield V on Ω such that $V \in S^\#$, $PV = 0$ and $V(\sigma(s_m)) \neq 0$ for all m."(here PV means the projection in the bases given by $Y_1, ..., Y_d \in S^b$, $d = dim \ L(x_0)$ and $\sigma(s_m) = X^{s_m}(x)$, for some $X \in S$ and $x \in M$). In the above example this is equivalent to ask for a sequence $(s_m) \to 0$, where $(\sigma(s_m))$ is a sequence of points taken on an integral curve through the origin $(\sigma(0) = 0)$, $V = f_1 \frac{\partial}{\partial x} + f_2 \Phi \frac{\partial}{\partial y}$ and $PV = f_1 \frac{\partial}{\partial x} = 0$. So $V = f_2 \Phi \frac{\partial}{\partial y}$ and the assertion requires $\Phi(\sigma(s_m)) \neq 0$ for all m. But we have seen that for every vector field $X \in S^\#$ the integral curve through the origin has a piece which is included into the axis Ox. We take (s_m) such as: $\sigma(s_m) = (x_m, 0)$. On the other hand, giving Φ (like above) there exists $\mu > 0$ such that $\Phi(x, 0) = 0$, for all $|x| < \mu$. Then we obtain that there exists an integer N_μ such that $\Phi(\sigma(s_m)) = 0$, for all $m > N_\mu$.

The solution that we propose in section 4 is to reformulate the condition of the existence of ε (see the point (c) above) in such a way that it becomes independent of every other conditions (that means there exists an $\varepsilon > 0$ "good" for all vector fields). This happens , for example, in the case when $S^\#$ is finitely

generated, because we choose $\varepsilon = \min_i \varepsilon_{X_i}$, where $\{X_i\}_{i=\overline{1,p}}$ spans the module. With this condition, theorem(5) for [St80] is proved, but we will give a proof without the criterion of local subintegrability (modified).
Now we turn to Sussmann's paper. Even though the implication e⇒d is false, the other equivalences are true. We prove this directly on the Sussmann's proof (for this we suppose that the reader is familiar with the Sussmann's paper — [Su73]): We will prove that from (a) it results (d) (in Theorem 4.2). The implication (a)⇒(e) is true (for example it is included in Theorem (4.6) of this paper) and from both (a) and (e) we will obtain (d). We have that $W^1(t), \ldots, W^k(t) \in \Delta(X_t(m))$ are independent. Since $X_t(m)$ belongs to the integral manifold of Δ passing through m it results $\dim \Delta(X_t(m)) = \dim \Delta(m)$ and so $W^1(t), \ldots, W^k(t)$ form a basis for $\Delta(X_t(m))$. Now the proof is complete.

2.2 Discussion about involutivity

Let $\mathcal{L}$ be a $\mathcal{F}(M)$-module of C^r vector fields and L be the distribution generated by $\mathcal{L}$. We say that a vector field X *belongs to the distribution* L (we write $X \in L$) if for all $p \in M$, $X(p) \in L(p)$. We say that the module $\mathcal{L}$ is *involutive* if for every $X, Y \in \mathcal{L}$ we have that $[X,Y] \in \mathcal{L}$. A distribution L is called *involutive* if for every two vector fields $X, Y \in L$ we have $[X,Y] \in L$.

If the distribution has not any singularities then the problem of integrability is completely solved by Frobenius' theorem:

THEOREM 2.2 *If L is a C^r-distribution without singularities then the following conditions are equivalent:*
(a) L is locally integrable
(b) L is involutive
(c) $\mathcal{L}$ is involutive. □

In the general case of the distributions with singularities, the following result is obvious:

PROPOSITION 2.3 *If L is a locally integrable distribution then L is involutive.* □

On the contrary, the following examples show that if L is integrable then $\mathcal{L}$ need not to be involutive.

EXAMPLE 2.4 (smooth case) *Let $X_1 = \varphi(x,y)\frac{\partial}{\partial x}$ and $X_2 = (x^2+y^2)\frac{\partial}{\partial y}$, where*

$$\varphi(x,y) = \begin{cases} e^{-\frac{1}{x^2+y^2}} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

We have: $\mathcal{L} = span_{\mathcal{F}(M)}\{X_1, X_2\} = \{f_1\varphi\frac{\partial}{\partial x} + f_2(x^2+y^2)\frac{\partial}{\partial y}\ ,\ f_1, f_2 \in \mathcal{F}(M)\ \}$; $M = \mathbf{R}^2$ *and*

$$L(p) = \begin{cases} T_p\mathbf{R}^2 & , p \neq (0,0) \\ \{0\} & , p = (0,0) \end{cases}$$

The distribution L is punctually integrable at every point $p \in \mathbf{R}^2$ (in the origin the integral manifold is the point $O(0,0)$), but

$$[X_1, X_2] = 2x\varphi(x,y)\frac{\partial}{\partial y} - (x^2+y^2)\frac{\partial \varphi}{\partial y}\frac{\partial}{\partial x} = 2x\frac{\varphi(x,y)}{x^2+y^)}X_2 - \frac{(x^2+y^2)\frac{\partial \varphi}{\partial y}(x,y)}{\varphi(x,y)}X_1$$

It is very easy to prove that: $2x\frac{\varphi(x,y)}{x^2+y^2} \in \mathcal{F}(M)$, *but:* $\frac{(x^2+y^2)\frac{\partial\varphi}{\partial y}}{\varphi(x,y)} = \frac{2y}{x^2+y^2}$; $(x,y) \neq 0$ *does not admit a limit at x=y=0. So* $[X_1, X_2] \notin \mathcal{L}$ $\diamond$

EXAMPLE 2.5 (analytic case) *Let* $X_1 = (x^2+y^2)\frac{\partial}{\partial x}$, $X_2 = (x^4+y^4)\frac{\partial}{\partial y}$ *and* $M = \mathbf{R}^2$.
We put $\mathcal{L} = span_{\mathcal{F}(M)}\{X_1, X_2\}$ *and the distribution generated by* $\mathcal{L}$ *is:*

$$L(p) = \begin{cases} T_p\mathbf{R}^2 & , p \neq (0,0) \\ \{0\} & , p = (0,0) \end{cases}$$

The distribution L is integrable on M (is the same distribution that in the previous example) but: $[X_1, X_2] = 4x^3(x^2+y^2)\frac{\partial}{\partial y} - 2y(x^4+y^4)\frac{\partial}{\partial x} = f_1X_1 + f_2X_2$, *where* $f_1 = -\frac{2y(x^4+y^4)}{x^2+y^2}$, $f_2 = \frac{4x^3(x^2+y^2)}{x^4+y^4}$ *and* $\frac{\partial f_2}{\partial x} = \frac{4x^8-4x^6y^2+20x^4y^4+12x^2y^6}{x^8+2x^4y^4+y^8}$.
Since $\frac{\partial f_2}{\partial x}|_{(x,0)} = 4$ $\frac{\partial f_2}{\partial x}|_{(0,y)} = 0$, f_2 *is not a* $\mathcal{C}^1$ *function and* $[X_1, X_2] \notin \mathcal{L}$. $\diamond$

If we note $smt_r(L) = \{X \in V^r(M),$ *such that* $X \in L\}$ then the explanation is that we have the inequality: $smt_r(L) \neq \mathcal{L}$ though $\mathcal{L} \subset smt_r(L)$ ($r = \infty$ or $r = \omega$). From the Proposition(2.3) it follows immediately:

PROPOSITION 2.6 *If L is a locally integrable distribution, then $smt_r(L)$ is involutive ($r = \infty$ or $r = \omega$).* $\square$

In the case of punctual inegrability the following example shows that the module may be not involutive:

EXAMPLE 2.7 (see [Fr78]) *Let* $X_1 = xz\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ *and* $X_2 = \frac{\partial}{\partial z}$. *Then* $\mathcal{L}$=$span_{\mathcal{F}(M)}\{X_1, X_2\}$, $L(p) = \mathcal{L}|_p$ *for all* $p \in M = \mathbf{R}^3$.
Since $\dim L(p) = 2, \forall p \in M$, *$L$ is a distribution without singularities. Let* $p = (0,0,0)$. *The submanifold* $N_0 = \{(x,y,z) \in \mathbf{R}^3 | x = 0\}$ *(the plane Oyz) is a maximal integral manifold through p. But* $[X_1, X_2] = -x\frac{\partial}{\partial x} \notin \mathcal{L}$. *So $\mathcal{L}$ is not involutive.* $\diamond$

The following example shows that if $\mathcal{L}$ is involutive it does not necessarely follow that L is involutive:

EXAMPLE 2.8 (see [Na66]) *Let* $M = \mathbf{R}^2$, $X_1 = \frac{\partial}{\partial x}$, $X_2 = \varphi(x)\frac{\partial}{\partial y}$, *where*

$$\varphi(x) = \begin{cases} e^{-\frac{1}{x^2}} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Then: $\mathcal{L}$=$span_{\mathcal{F}(M)}\{\frac{\partial}{\partial x}, \varphi\frac{\partial}{\partial y}, \varphi'(x)\frac{\partial}{\partial y}, \cdots, \varphi^{(n)}(x)\frac{\partial}{\partial y}, \cdots\}$ *is involutive. The distribution generated by* $\mathcal{L}$ *is:*

$$L(x,y) = \begin{cases} T_{(x,y)}\mathbf{R}^2 & , \; x \neq 0 \\ span_{\mathbf{R}}\{\frac{\partial}{\partial x}|_{(0,y)}\} & , \; x = 0 \end{cases}$$

We have that $Y = x\frac{\partial}{\partial y} \in \mathcal{L}$, *but* $[X_1, Y] = \frac{\partial}{\partial y} \notin L$. $\diamond$

If the distribution is without singularities then it is easy to prove that his rank (that is the dimension of the vector subspace) is constant on M (we have supposed that M is connected). Also we can prove that the set of ordinary (or regular) points of distribution is an open dense subset of M.

3 Split of the distribution

Let $\mathcal{L}$ be a $\mathcal{F}(M)$-module of $\mathcal{C}^r$ vector fields and let L denote the associated distribution. Let $x_0 \in M$ be a fixed point. Let $k = \dim L(x_0) \leq n = \dim M$. Then there exist k vector fields $\tilde{a}_1, \ldots, \tilde{a}_k \in \mathcal{L}$ such that $\tilde{a}_1|_{x_0}, \ldots, \tilde{a}_k|_{x_0}$ are independent. We assume that a system of local coordinates at x_0 has been fixed, and we put $\tilde{A}(x) \stackrel{\text{def}}{=} [\tilde{a}_1|_x, \ldots, \tilde{a}_k|_x] \in \mathbf{R}^{n\times k}$, where $\tilde{a}_i|_x$ are the components of the vector field $\tilde{a}_i$ evaluated at x and ordered according to the column. Since rank$\tilde{A}(x_0) = k$ there exists a neighborhood $\mathcal{U}$ where $\tilde{A}$ is a full-rank matrix. We suppose, possibly renumbering the coordinates, that the first k rows of A are independent. We partition the matrix $\tilde{A}$ as follows:

$$\tilde{A}(x) = \begin{bmatrix} \tilde{A}_1(x) \\ \cdots \\ \tilde{A}_2(x) \end{bmatrix} \begin{matrix} \}k \\ \\ \\ \end{matrix}$$

where $\tilde{A}_1(x)$ is nonsingular on $\mathcal{U}$. From now on will agree implicitly that $x \in \mathcal{U}$. Let

$$A(x) \stackrel{\text{def}}{=} \tilde{A}(x)\tilde{A}_1^{-1}(x) = \begin{bmatrix} I_k \\ \cdots \\ A_2(x) \end{bmatrix} = [a_1|_x \ldots a_k|_x] \tag{1}$$

In local coordinates we have:

$$a_i = \frac{\partial}{\partial x^i} + \sum_{j=k+1}^{n} A_{2,ji}\frac{\partial}{\partial x^j}$$

We associate to A the family $\mathcal{F}_\varepsilon$ defined by:

$$\mathcal{F}_\varepsilon = \{a_\alpha \in V^r(M) | a_\alpha \stackrel{\text{def}}{=} \sum_{i=1}^{k} \alpha_i a_i \;,\; \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbf{R}^k, |\alpha| \stackrel{\text{def}}{=} \sum_{i=1}^{k} < \varepsilon\}$$

$\mathcal{F}_\varepsilon$ can be identified with a ball in a k-dimensional space. For $\varepsilon > 0$ small enough we know that $\exp : \mathcal{F}_\varepsilon \to M$ is a regular embedding. So $\exp \mathcal{F}_{\varepsilon.x_0} \subset M$ is a submanifold in M of dimension k ($\exp \mathcal{F}_{\varepsilon.x_0} \stackrel{\text{def}}{=} \{\exp a_\alpha.x_0 |\ a_\alpha \in \mathcal{F}_\varepsilon\}$ and $\exp a_\alpha.x_0$ denotes $x(1)$ where $x(t)$ is the solution of the differential system $\dot{x}(t) = a_\alpha|_{x(t)}$ with the initial condition $x(0) = x_0$).

PROPOSITION 3.1 *If L is punctually integrable at x_0, then $\mathcal{N}_{\varepsilon.x_0} \stackrel{\text{def}}{=} \exp \mathcal{F}_{\varepsilon.x_0}$ is an integral manifold of L passing through x_0.*

Proof

Let $a_\alpha \in L$ as above and let $\tilde{\mathcal{N}}_{x_0}$ be an integral manifold of L passing through x_0. Then:

$$a_\alpha|_x \in T_x\tilde{\mathcal{N}}_{x_0}\ ,\ \ for\ all\ x \in \tilde{\mathcal{N}}_{x_0}.$$

A piece of integral curve of a_α, passing through x_0, will be included in $\tilde{\mathcal{N}}_{x_0}$. So $\exp ta_\alpha.x_0 \in \tilde{\mathcal{N}}_{x_0}$ for $|t| < \mu(\alpha)$. We consider the unit ball in k-real space: $B^1 \stackrel{\text{def}}{=} \{\alpha \in \mathbf{R}^k|\ |\alpha| = 1\}$ and we obtain $\mu : B^1 \to \mathbf{R}$ a continuous function ($\alpha \mapsto \mu(\alpha)$). Since B^1 is a compact set we obtain that there exists $\alpha_0 \in B^1$ such that $\inf \mu(B^1) = \min \mu(B^1) = \mu(\alpha_0) > 0$. Let $\varepsilon = \mu(\alpha_0)$. We conclude $\mathcal{N}_{\varepsilon.x_0} \subset \tilde{\mathcal{N}}_{x_0}$ and, since $\mathcal{N}_{\varepsilon.x_0}$ is a k-dimensional manifold like $\tilde{\mathcal{N}}_{x_0}$, it is an open subset of $\tilde{\mathcal{N}}_{x_0}$. So $\mathcal{N}_{\varepsilon.x_0}$ is an integral manifold too. Q.E.D. □

The problem to be solved is to determine conditions which guarantee that $\mathcal{N}_{\varepsilon.x_0}$ is an integral manifold of L (that means, for all $x \in \mathcal{N}_{\varepsilon.x_0}$ $L(x) = T_x\mathcal{N}_{\varepsilon.x_0}$).

Relied on the vector fields, we will construct now a split of the distribution. Let

$$\mathcal{G} \stackrel{\text{def}}{=} \{b \in \mathcal{L} | b = \sum_{j=k+1}^{n} b^j(x)c_j(x), where\ b^j(x) \in \mathcal{F}(M)\ and\ c_j(x)|_{\mathcal{U}} = \frac{\partial}{\partial x^j}\}$$

$$L_{(-1)} \stackrel{\text{def}}{=} \{a_\alpha \in V^r(M) | \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbf{R}^k\} = \mathcal{F}_\infty$$

.

It is very easy to prove the following lemma:

LEMMA 3.2 *The distribution generated by $\mathcal{G} \oplus L_{(-1)}$ coincides locally with L. That means: $L(x) = \mathcal{G}|_x \oplus L_{(-1)}|_x$, for all $x \in \mathcal{U}$ ($\oplus$ denotes a direct sum).* □

By dimensional relation: $\dim L(x_0) = \dim L_{(-1)}|_{x_0}$ and it results: $\mathcal{G}|_{x_0} = \{0\}$. We have obtained two algebraic structures which generate locally the distribution: $L_{(-1)}$, which is a k-dimensional $\mathbf{R}$- vector subspace, and $\mathcal{G}$, wich is a $\mathcal{F}(M)$-module and we say that $(L_{(-1)}, \mathcal{G})$ is a *split* of the distribution generated by $\mathcal{L}$.

4 Punctual results

We have the following result:

PROPOSITION 4.1 *The distribution L is punctually integrable at x_0 if and only if we have the relations:*
R_1. $L_{(-1)}|_x = T_x\mathcal{N}_{\varepsilon.x_0}$ *, for all* $x \in \mathcal{N}_{\varepsilon.x_0}$
R_2. $\mathcal{G}|_{\mathcal{N}_{\varepsilon.x_0}} = 0$ *(that means* $\mathcal{G}|_x = 0$ *, for all* $x \in \mathcal{N}_{\varepsilon.x_0}$*).*

Proof

" $\Rightarrow$ " Let L be integrable at x_0. It follows from Proposition(3.1) that $\mathcal{N}_{\varepsilon.x_0}$ is an integral manifold. Using Lemma(3.2) we obtain:

$$\mathcal{G}|_x \oplus L_{(-1)}|_x = T_x\mathcal{N}_{\varepsilon.x_0} \ , \ for\ all\ x \in \mathcal{N}_{\varepsilon.x_0}$$

From $\dim L_{(-1)}|_x = \dim L|_{x_0} = k = \dim T_x\mathcal{N}_{\varepsilon.x_0}$ *it results* $L_{(-1)}|_x = T_x\mathcal{N}_{\varepsilon.x_0}$. *If there exists* $x \in \mathcal{N}_{\varepsilon.x_0}$ *such that* $\mathcal{G}|_x \neq \{0\}$ *then also there exists* $b \in L(x)$ *such that* $b^T = [0\ b_2^T]$ *with* $b_2 \neq 0$. *But then* $b \notin L_{(-1)}|_x$. *This implies that:*

$$k = \dim L(x) = \dim(\mathcal{G}|_x \oplus L_{(-1)}|_x) > \dim L_{(-1)}|_x = \dim T_x\mathcal{N}_{\varepsilon.x_0} = k$$

A contradiction. Therefore $\mathcal{G}|_{\mathcal{N}_{\varepsilon.x_0}} = 0$.
" $\Leftarrow$ " It results that:

$$L(x) = \mathcal{G}|_x + L_{(-1)}|_x \stackrel{R_2}{=} L_{(-1)}|_x \stackrel{R_1}{=} T_x\mathcal{N}_{\varepsilon.x_0} \ , \ for\ all\ x \in \mathcal{N}_{\varepsilon.x_0}$$

So $\mathcal{N}_{\varepsilon.x_0}$ *is an integral manifold of L passing through x_0.* *Q.E.D.* □

Following the proof of Nagano's theorem we have the next lemma (see [Na66] for proof):

LEMMA 4.2 *Let* $u, v \in V^\infty(M), x_0 \in M$ *and* $s, t \in \mathbf{R}, |\mathbf{s}|, |\mathbf{t}| < \varepsilon$, *where* $\varepsilon > 0$ *is so small that* $f(s,t) = \exp[t(su+v)].x_0$ *makes sense. If* $[u,v]|_{\exp tv.x_0} = 0$, $\forall |t| < \varepsilon$ *then* $\frac{\partial f(s,t)}{\partial s}|_{s=0} = tu|_{f(0,t)}$, $\forall |t| < \varepsilon$. □

The above result allows us to find a condition equivalent to R_1 of Proposition(4.1).

COROLLARY 4.3 *The distribution L is punctually integrable at x_0 if and only if:*
1) $[u,v]|_{\exp tv.x_0} = 0$, *for all* $u, v \in L_{(-1)}$ *and* $|t| < \varepsilon$, ε *depending on* v.
2) $\mathcal{G}|_{\mathcal{N}_{\varepsilon.x_0}} = 0$

Sketch of Proof

" $\Leftarrow$ " From Lemma(4.2) and by dimensional reasons we obtain: $L_{(-1)}|_x = T_x\mathcal{N}_{\varepsilon.x_0}$, *for all* $x \in \mathcal{N}_{\varepsilon.x_0}$.
" $\Rightarrow$ " Let $u, v \in L_{(-1)}$ *of the form:*

$$u = \sum_{i=1}^{k} \alpha_i \frac{\partial}{\partial x^i} + \sum_{j=k+1}^{n} u_j \frac{\partial}{\partial x^j}$$

$$v=\sum_{i=1}^{k}\beta_i\frac{\partial}{\partial x^i}+\sum_{j=k+1}^{n}v_j\frac{\partial}{\partial x^j}$$

where $\alpha_i,\beta_i\in\mathbf{R}$ *and* $u_j,v_j\in\mathcal{F}(M)$, *such that* $u,v\in L_{(-1)}$. *We obtain:*

$$[u,v]=\sum_{j=k+1}^{n}[u(v_j)-v(u_j)]\frac{\partial}{\partial x^j}$$

Let $x=\exp tv.x_0$. *Since* $[u,v]|_x\in L(x)$ *we have* $[u,v]|_x=0$. *Q.E.D.* □

We emphasize from the second part of proof the relation:

$$[u,v]=\sum_{j=k+1}^{n}[u(v_j)-v(u_j)]\frac{\partial}{\partial x^j}\ \textit{for all}\ u,v\in L_{(-1)} \tag{2}$$

The same relation is also valid for $u,v\in\mathcal{G}$ or $u\in L_{(-1)}$ and $v\in\mathcal{G}$.

Let $\mathcal{L}$ be a $\mathcal{F}(M)$-module of $\mathcal{C}^r$ vector fields. We denote by $L^\infty\mathcal{L}$ the Lie closure of $\mathcal{L}$ (i.e. the minimal involutive module that contains $\mathcal{L}$). We can state the following result due to Freeman ([Fr78]) (here we use a proof borrowed from Nagano):

THEOREM 4.4 *Let* $\mathcal{L}$ *be an analytic* $\mathcal{F}(M)$*-module of vector fields and let* L *denote the associated distribution. Then* L *is punctually integrable at* x_0 *if and only if:* $L^\infty\mathcal{L}|_{x_0}=L(x_0)$

Sketch of Proof

We will apply the Corollary(4.3).

" $\Rightarrow$ " It follows directly by using the Proposition (2.3) *and the fact that any vector field from* $L^\infty\mathcal{L}$ *is written as a finite combination of Lie brackets of vector fields from* $\mathcal{L}$.

" $\Leftarrow$ " Let $L^\infty_{(-1)}$ *and* $\mathcal{G}^\infty$ *be obtained by splitting the distribution generated by* $L^\infty\mathcal{L}$. *From* $L_{(-1)}\subset L^\infty_{(1)},\mathcal{G}\subset\mathcal{G}^\infty$ *and* $L^\infty\mathcal{L}|_{x_0}=L(x_0)$ *we have that:*

$$L_{(-1)}=L^\infty_{(-1)}\ \textit{and}\ \mathcal{G}=\mathcal{G}^\infty\cap\mathcal{L}$$

From involutivity of $L^\infty\mathcal{L}$ *and relations(2) we have that:*

$$[L_{(-1)},L_{(-1)}]\subset\mathcal{G}^\infty,[L_{(-1)},\mathcal{G}^\infty]\subset\mathcal{G}^\infty,[\mathcal{G}^\infty,\mathcal{G}^\infty]\subset\mathcal{G}^\infty$$

and from analyticity and Taylor series we obtain: $\mathcal{G}|_{\mathcal{N}_{\epsilon.x_0}}=0$ *and* $[u,v]|_{\mathcal{N}_{\epsilon.x_0}}=0$ *for all* $u,v\in L_{(-1)}$ *Q.E.D.* □

Now we pass to the smooth case ($r=\infty$). First we need the following lemma:

LEMMA 4.5 *Let* $a_1,\ldots,a_k\in V^\infty(\mathcal{U})$ *($\mathcal{U}$ being an open neighborhood of x_0) be smooth vector fields and let* $Q=span_{\mathcal{F}(\mathcal{U})}\{a_1,\ldots,a_k\}$. *Let* $Z\in V^\infty(\mathcal{U})$

and $\{b_1, \ldots, b_k\} \subset Q$ such that $b_i = \sum_{j=1}^k f_{ij} a_j$ and $a_i = \sum_{j=1}^k g_{ij} b_j$ where $f_{ij}, g_{ij} : \mathcal{U} \to \mathbf{R}$ are smooth functions.
If there exist $\mathcal{C}^\infty$ functions $\lambda_i^j : (-\varepsilon, \varepsilon) \to \mathbf{R}$, $i, j = \overline{1,k}$ such that:

$$[Z, a_i]|_{\exp tZ.x_0} = \sum_{j=1}^k \mu_i^j(t) a_j|_{\exp tZ.x_0}$$

then there exist $\mathcal{C}^\infty$ functions $\mu_i^j : (-\varepsilon, \varepsilon) \to \mathbf{R}$, $i, j = \overline{1,k}$ such that:

$$[Z, b_i]|_{\exp tZ.x_0} = \sum_{j=1}^k \mu_i^j(t) b_j|_{\exp tZ.x_0}$$

Proof
We obtain:

$$[Z, b_i]|_{\exp tZ.x_0} = \sum_{l=1}^k [\sum_{j=1}^k g_{jl} Z(f_{ij}) + \sum_{j,s=1}^k g_{sl} \lambda_j^s f_{ij}] b_l|_{\exp tZ.x_0} = \sum_{l=1}^k \mu_i^l b_l|_{\exp tZ.x_0}$$

Q.E.D. □

Next theorem is our version of the result from [St74][St80][Su73]:

THEOREM 4.6 *Let $\mathcal{L}$ be a $\mathcal{F}(M)$-module of $\mathcal{C}^\infty$ vector fields on M and let L denote the associated distribution. Let $x_0 \in M$ and $k = \dim L(x_0)$. Then L is punctually integrable at x_0 if and only if there exist $\varepsilon > 0$, vector fields $a_1, \ldots, a_k \in \mathcal{L}$ and a neighborhood $\mathcal{U}$ of x_0 that satisfy the following conditions:*
1) In the point x_0 $a_1|_{x_0}, \ldots, a_k|_{x_0}$ span $L(x_0)$
2) For all smooth vector field $Z \in \mathcal{L}$, there exist smooth functions $\lambda_i^j : (-\mu_Z, \mu_Z) \to \mathbf{R}$ such that for all $t \in (-\mu_Z, \mu_Z)$ and $1 \leq i \leq k$ we have:

$$[Z, a_i]|_{\exp tZ.x_0} = \sum_{j=1}^k \lambda_i^j(t) a_j|_{\exp tZ.x_0} \tag{3}$$

where: $\mu_Z \stackrel{\text{def}}{=} \sup\{\nu | \nu \leq \varepsilon$ and $\exp tZ.x_0 \in \mathcal{U}$ for all $|t| < \nu\}$
Proof
Lemma(4.5) shows that (3) is invariant under a change of the basis. Then we choose for $\{a_i\}$ the vector fields which form the basis of $L_{(-1)}$ obtained by splitting of L. Moreover, let ε as in the definition of $\mathcal{N}_{\varepsilon.x_0}$.
$'' \Rightarrow ''$ We choose $\mathcal{U}$ as in §3.
1) It is checked by construction of vector fields $\{a_i\}$
2) Let $Z \in \mathcal{L}$. Then $Z = \sum_{j=1}^k f_j a_j + b$ where $b \in \mathcal{G}$ and $f_j \in \mathcal{F}(M)$. We obtain:

$$[Z, a_i] = \sum_{j=1}^k f_j [a_j, a_i] + [b, a_i] - \sum_{j=1}^k a_i(f_j) a_j$$

Since L is integrable at x_0 and $tZ|_x \in L(x), \forall x \in \mathcal{N}_{\varepsilon.x_0}$ we have $x_t = \exp tZ.x_0 \in \mathcal{N}_{\varepsilon.x_0}$ and we obtain: $[Z, a_i]|_{x_t} = \sum_{j=1}^{k} \lambda_i^j(t) a_j|_{x_t}$ *for all* $|t| < \mu_Z$.
$'' \Leftarrow ''$ *We apply the Corollary*(4.3)
a) We show that for all $a_1, a_2 \in L_{(-1)}$, $[a_1, a_2]|_{\exp ta_1.x_0} = 0$, *with* $|t| < \varepsilon = \mu_{a_1}$.
We write the given relation for $Z = a_1$ *and* $a_i = a_2$.
On one hand we have: $[a_1, a_2] = \sum_{j=k+1}^{n} \pi_j \frac{\partial}{\partial x^j}$
on the other hand: $\sum_{j=1}^{k} \lambda_2^j(t) a_j = \lambda_2^1(t) \frac{\partial}{\partial x^1} + \cdots + \lambda_2^k(t) \frac{\partial}{\partial x^k} + \sum_{s=k+1}^{n} \Theta_2^s(t) \frac{\partial}{\partial x^s}$
From $[Z, a_i]|_{\exp tZ.x_0} = \sum_{j=1}^{k} \lambda_i^j(t) a_j|_{\exp tZ.x_0}$ *we obtain:* $[a_1, a_2]|_{\exp tZ.x_0} = 0$, $|t| < \varepsilon$.
b) We show that $\mathcal{G}|_{\mathcal{N}_{\varepsilon.x_0}} = 0$ *Let* $X \in \mathcal{G}$. *We put* $Z_i = X + a_i$ *and write:* $[Z_i, a_i] = [X, a_i]$. *Then, as above, we obtain:* $[X, a_i]|_{\exp tZ_i.x_0} = 0, |t| < \mu_{Z_i}$.
Obviously: $[X, X]|_{\exp tZ_i.x_0} = 0$. *Then:*

$$[X, (X + a_i)]|\exp t(X + a_i).x_0 = 0 \text{ or } [Z_i, X]|_{\exp tZ_i.x_0} = 0, \ |t| < \mu_{Z_i}$$

We can apply a formula from 3.2 ([*St80*]) *and we obtain:*

$$\frac{d}{dt} X(x_t) = DZ_i \circ X|_{x_t}$$

(where $x_t = \exp tZ_i.x_0$ *and* DZ_i *is the jacobian matrix of* Z_i*) with the initial condition:* $X(u(0)) = X(x_0) = 0$ *(recall that* $X \in \mathcal{G}$*). Using the theorem of existence and unicity of the Cauchy problem, we obtain:* $X|_{\exp tZ_i.x_0} = 0$. *But then* $Z_i|_{\exp tZ_i.x_0} = (a_i + X)|_{\exp tZ_i.x_0} = a_i|_{\exp tZ.x_0}$. *So:* $\exp tZ_i.x_0 = \exp ta_i.x_0$. *That means:* $X|_{\exp ta_i.x_0} = 0$, *and* $|t| < \mu_{Z_i} = \mu_{X+a_i} = \mu_{a_i} = \varepsilon$. *So:* $\mathcal{G}|_{\mathcal{N}_{\varepsilon.x_0}} = 0$

Q.E.D. □

We can also give a new proof of Theorem 6 from [St74]:

THEOREM 4.7 *Let $\mathcal{L}$ be a smooth $\mathcal{F}(M)$-module of vector fields and L the distribution generated. Let $x_0 \in M$ be a fixed point. Then L is punctually integrable at x_0 if and only if for every $X \in \mathcal{L}$ there exist $\varepsilon > 0$, a finite set $\{X_1, \ldots, X_p\} \subset \mathcal{L}$ and continuous functions $\lambda_{ij} : (-\varepsilon, \varepsilon) \to \mathbf{R}$ $(1 \le i, j \le p)$ such that:*
(a) The vectors $X_1|_{x_0}, \ldots, X_p|_{x_0}$ span $L(x_0)$.
(b) For every $t \in (-\varepsilon, \varepsilon)$ and $1 \le j \le p$, $[X, X_i](x_t) = \sum_{j=1}^{p} \lambda_{ij}(t) X_j(x_t)$ where $x_t = \exp tX : x_0$
(c) The vectors $X_i(x_t)$ span $L(x_t)$.
Remark *Here, ε depends of X but there exists the point (c) that is a very strong condition.*
Proof
First we split the distribution: $L = L_{(-1)} \oplus \mathcal{G}$. Let $X = a_\alpha \in L_{(-1)}$. We apply Lemma(4.5) *and obtain a set $\{a_1, \ldots, a_k, Y_{k+1}, \ldots, Y_p\} \subset \mathcal{L}$ where $\{a_1, \ldots, a_k\}$ are the vector fields give by* (1) *and $Y_1, \ldots, Y_p \in \mathcal{G}$.*

Using (b) *from hypotesis we obtain the system (see also relations* (2)*):*

$$[X,Y_i](x_t) = \sum_{j=k+1}^{p} \mu_{ij} Y_j|_{x_t}$$

Using again formula borrowed from 3.2 ([*St80*]) *we obtain the differential system:*

$$\frac{d}{dt} Y_i(x_t) = \sum_{j=k+1}^{p} \pi_{ij} Y_j|_{x_t} \ , \ k+1 \le i \le p$$

with initial conditions: $Y_i(x_0) = 0$, $k+1 \le i \le p$. *So:* $Y_i(x_t) = 0$, *for all* $-\varepsilon < t < \varepsilon$. *That means the dimension of* L *is constant on the integral curve of* a_α. *For every* α *we obtain an* $\varepsilon(\alpha) > 0$ *such that :*

$$\dim L(\exp t a_\alpha . x_0) = k \ , \ for\ all\ |t| < \varepsilon(\alpha)$$

By a compactness argument we obtain an $\varepsilon > 0$ *such that* $\varepsilon(\alpha) \ge \varepsilon > 0$ *for all* $|\alpha| = 1$. *Then we take* $\mathcal{F}_\varepsilon$ *and* $\mathcal{N}_{\varepsilon . x_0}$ *as in* §3 *and the proof is complete.*

Q.E.D. □

In the case when the module is involutive, the punctual integrability is solved by the following result:

PROPOSITION 4.8 *If* $\mathcal{L}$ *is an involutive* $\mathcal{F}(M)$*-module of smooth vector fields then* L *is punctually integrable at* $x_0 \in M$ *if and only if* $\mathcal{G}|_{\mathcal{N}_{\varepsilon . x_0}} = 0$.

Proof

For all $u, v \in L_{(-1)}$ *we have* $[u,v] \in \mathcal{G}$ *(from involutivity and relations* (2)*). So:* $[u,v]|_{\exp tv . x_0} = 0$, $|t| < \varepsilon$. *Applying Corollary*(4.3) *we obtain the statement.*

Q.E.D. □

5 Local results

We can give local versions of Proposition(4.1) and Corollary(4.3) under the condition that the hypotesis hold for all $x_0 \in M$. But the following three theorems are remarkable.

First Nagano's result ([Na66]) whose proof is immediate if we use Theorem (4.4)

THEOREM 5.1 *Let* $\mathcal{L}$ *be an analytic involutive* $\mathcal{F}(M)$*-module of vector fields. Then the associated distribution* L *is locally integrable.* □

An equivalent statement is the case when $\mathcal{L}$ is an analytic **R**-Lie algebra of vector fields (see[Na66]).
The following theorem is due to Hermann (Theorem 2.2 in [Hen62]) but with a proof invoking invariant distributions). Here we have two proofs (proof of Theorem(5.2) and Corollary(5.5)).

THEOREM 5.2 *Let $\mathcal{L}$ be an involutive and finitely-generated $\mathcal{F}(M)$-module of $\mathcal{C}^\infty$ vector fields. Then the associated distribution L is locally integrable.*

Proof

We apply Proposition(4.8) and it remains to prove that $\mathcal{G}|_{\mathcal{N}_{\varepsilon.x_0}} = 0$, for all $x_0 \in M$. Let $x_0 \in M$, $\{a_1, \ldots, a_k, a_{k+1}, \ldots, a_p\} \subset \mathcal{L}$ a set of generators for $\mathcal{L}$, where $\{a_1, \ldots, a_k\}$ are of the form (1) and $a_{k+1}, \ldots, a_p \in \mathcal{G}$. We have to prove that: $a_i|_{\exp ta_\alpha.x_0} = 0$, for all $k+1 \le i \le p, a_\alpha \in \mathcal{F}_\varepsilon$ and $-1 < t < 1$. Let $a_i = \sum_{j=k+1}^n a_{ij}\frac{\partial}{\partial x^j}$ and $f_{ij}(t) = a_{ij}(\exp ta_\alpha.x_0)$. We have $f_{ij}(0) = 0$. By using the formula from (3.2) ([St80]), we obtain:

$$\begin{aligned} L_{a_\alpha} a_i|_{\exp ta_\alpha.x_0} &= \textstyle\sum_{j=k+1}^n \dot{f}_{ij}(t)\frac{\partial}{\partial x^j} + Da_\alpha|_{\exp ta_\alpha.x_0} \cdot a_i|_{\exp ta_\alpha.x_0} \\ &= \textstyle\sum_{j=k+1}^n (\dot{f}_{ij}(t) + \sum_{s=k+1}^n \pi_{ijs} f_{is})\frac{\partial}{\partial x^j} \end{aligned}$$

On the other hand:

$$L_{a_\alpha} a_i|_{\exp ta_\alpha.x_0} = \sum_{j=k+1}^p g_{ij} a_j = \sum_{s=k+1}^n \sum_{j=k+1}^p g_{ij} f_{js} \frac{\partial}{\partial x^s}$$

We obtain the system of differential equations:

$$\dot{f}_{ij}(t) = \sum_{s=k+1}^p g_{is} f_{sj} - \sum_{s=k+1}^n \pi_{ijs} f_{is} \ , \ k+1 \le i \le p, k+1 \le j \le n$$

By the theorem of existence and unicity of solution of Cauchy problem, we obtain:

$$f_{ij}(t) \equiv 0 \longrightarrow a_i|_{\exp ta_\alpha.x_0} = 0 \qquad Q.E.D. \ \square$$

COROLLARY 5.3 *If $\mathcal{L}$ is a $\mathcal{F}(M)$-module of $\mathcal{C}^\infty$ vector fields and $x_0 \in M$ such that:*

1) $L^\infty \mathcal{L}|_{x_0} = \mathcal{L}|_{x_0}$;

2) $L^\infty \mathcal{L}$ is finitely-generated;

then the distribution L generated by $\mathcal{L}$ is punctually integrable at x_0

Proof

Since $L^\infty \mathcal{L}$ is finitely-generated module, it is integrable at x_0. Let N_{x_0} denote an integral manifold of $L^\infty \mathcal{L}$ passing through x_0. Applying the rank theorem we obtain that there exists a neighborhood $\mathcal{U}$ of x_0 such that:
$\dim L(x) \ge \dim L(x_0) = k$, *for all $x \in \mathcal{U}$. But $\mathcal{L} \subset L^\infty \mathcal{L}$ so:*

$$k \le \dim L(x) = \dim \mathcal{L}|_x \le \dim L^\infty \mathcal{L}|_x = \dim T_x N_{x_0} = \dim T_{x_0} N_{x_0} = \dim L(x_0) = k$$

$$\Longrightarrow \dim L(x) = k, \ \ for \ all \ \ x \in \mathcal{U} \cap N_{x_0}$$

Since $L(x) \subset L^\infty \mathcal{L}|_x = T_x N_{x_0}$ we obtain: $L(x) = T_x N_{x_0}$, for all $x \in \mathcal{U} \cap N_{x_0}$. Therefore $N_{x_0} \cap \mathcal{U}$ is also an integral manifold of L. $\qquad Q.E.D. \ \square$

Let $\mathcal{F}_{x_0}(M)$ denote the ring of germs of $\mathcal{C}^\infty$ functions in x_0. A normal form of finitely-generated involutive module is given by the Cerveau's theorem (this theorem is the Theorem 1.1 from [Ce79], but here we give a new proof):

THEOREM 5.4 *If $\mathcal{L}$ is an involutive finitely-generated $\mathcal{F}_{x_0}(M)$-module of germs of vector fields in $x_0 \in M$, then there exists a coordinate system $(y^1, \ldots, y^n)$ such that a system of generators for $\mathcal{L}$ is given by:*

$$a_i = \frac{\partial}{\partial y^i} \ , \quad 1 \leq i \leq k = \dim \mathcal{L}|_{x_0}$$

$$b_i = \sum_{j=k+1}^{n} g_{ij} \frac{\partial}{\partial y^j} \ , \quad k+1 \leq i \leq p, \ p \in \mathbf{N}$$

where: $g_{ij} = g_{ij}(y^{k+1}, \ldots, y^n)$ and $g_{ij}(y_0^{k+1}, \ldots, y_0^n) = 0, \ k+1 \leq i \leq p$; $k+1 \leq j \leq n$, $(g_0^\alpha)_{1 \leq \alpha \leq n}$ are the new coordinates of x_0.

COROLLARY 5.5 *The distribution L generated by the module $\mathcal{L}$ as above, is integrable at x_0, an integral manifold being given by: $y^{k+1} = y_0^{k+1}, \ldots, y^n = y_0^n$.*
□

Note that by means of Theorem(5.4) we have obtained a new proof of Theorem(5.2).

Proof of Theorem

The case k=0 is obvious.
The case k=n: There exists a neighborhood $\mathcal{U}$ where the generated distribution has constant dimension equal with n. Then the proof is obvious.
Let $1 \leq k \leq n$ and let $\{a_1, \ldots, a_k, a_{k+1}, \ldots, a_p\}$ be a set of generators like in the proof of Theorem(5.2).
1. We will apply the flow-box theorem.
Let $y_1 = y_1(x), z_2 = z_2(x), \ldots, z_n = z_n(x)$ be a coordinate system such that: $a_1 = \frac{\partial}{\partial y^1}$. The other vector fields will be modified too:

$$a'_i = a'_{i1} \frac{\partial}{\partial y^1} + \sum_{j=2}^{n} a'_{ij} \frac{\partial}{\partial z^j} \ ; \ 2 \leq i \leq p \ , \ a'_{ij} = a'_{ij}(y_1, z)$$

We consider the set of the generators of the form:

$$a_1, b_i = a'_i - a'_{i1} \cdot a_1 = \sum_{j=2}^{n} a'_{ij} \frac{\partial}{\partial z^j} \ ; \ 2 \leq i \leq p$$

2. We will prove that we can find a set of generators:

$$a_1, a_i = \sum_{j=2}^{n} a_{ij} \frac{\partial}{\partial z^j} \ ; \ 2 \leq i \leq p$$

such that $a_{ij} = a_{ij}(z) = a_{ij}(z_2, \ldots, z_n)$.
We have: $[a_1, b_i] = \sum_{j=2}^{n} \frac{\partial a'_{ij}}{\partial y^1} \cdot \frac{\partial}{\partial z^j} \in \mathcal{L}$. *So:*

$$[a_1, b_i] = \sum_{s=2}^{p} f_{is} b_s = \sum_{j=2}^{n} (\sum_{s=2}^{p} f_{is} a'_{sj}) \frac{\partial}{\partial z^j} \Longrightarrow \frac{\partial a'_{ij}}{\partial y^1} = \sum_{s=2}^{p} f_{ij} a'_{sj}$$

Explicitely, the system can be written in the following way:

$$\frac{\partial}{\partial y^1} \begin{bmatrix} a'_{22} \; a'_{23} \ldots a'_{2n} \\ \vdots \qquad\quad \vdots \\ a'_{p2} \; a'_{p3} \ldots a'_{pn} \end{bmatrix} = \begin{bmatrix} f_{22} & \cdots & f_{2p} \\ \vdots & & \vdots \\ f_{p2} & \cdots & f_{pp} \end{bmatrix} \cdot \begin{bmatrix} a'_{22} \; a'_{23} \ldots a'_{2n} \\ \vdots \qquad\quad \vdots \\ a'_{p2} \; a'_{p3} \ldots a'_{pn} \end{bmatrix}$$

Each row is composed by the elements of the vector fields b_i. *The solution of the differential system is expressed by:*

$$\left. \begin{bmatrix} a'_{22} & \cdots & a'_{2n} \\ \vdots & & \vdots \\ a'_{p2} & \cdots & a'_{pn} \end{bmatrix} \right|_{(y^1,z)} = \mathbf{F} \cdot \left. \begin{bmatrix} a'_{22} & \cdots & a'_{2n} \\ \vdots & & \vdots \\ a'_{p2} & \cdots & a'_{pn} \end{bmatrix} \right|_{(0,z)} ;$$

$$\mathbf{F} = \exp \int_0^{y^1} \left. \begin{bmatrix} f_{22} & \cdots & f_{2p} \\ \vdots & & \vdots \\ f_{p2} & \cdots & f_{pp} \end{bmatrix} \right|_{(t,z)} dt$$

Let $\mathbf{H} = \mathbf{F}^{-1} = (h_{ij})_{2 \le i,j \le p}$, *because* F *is invertible. We redefine :*

$$a_i = \sum_{j=2}^{p} h_{ij} b_j = \sum_{j=2}^{n} a'_{ij}(0,z) \frac{\partial}{\partial z^j} \equiv \sum_{j=2}^{n} a_{ij}(z) \frac{\partial}{\partial z^j}$$

So: $\{a_1, a_i | 2 \le i \le p\}$ *is a set of generators.*
3. We rename the coordinates z *with* x *(so:* $x_2 = z_2, \ldots, x_n = z_n$*) and we apply the construction from §3 for* $\mathcal{L}' = span_{\mathcal{F}_{x_0}(x_2,\ldots,x_n)}\{a_i | 2 \le i \le p\}$ *(* $\mathcal{F}_{x_0}(x_2, \ldots, x_n)$ *being the ring of* $\mathcal{C}^\infty$ *germs of functions in the variables* $(x_2, \ldots, x_n)$*). The dimension of* $\mathcal{L}'|_{x_0}$ *is* $k-1$ *(like vector subspace in* $T_{x_0}M$*). Now we apply again the described algorithm beginning with* $y_2 = y_2(x), z_3 = z_3(x), \ldots, z_n = z_n(x)$. *After a* k*-th application of the algorithm, we obtain the statement.* *Q.E.D.* □

6 Conclusions

When we have a differentiable distribution we can distinguish three types of analysis:

1) Global integrability
2) Local integrability
3) Punctual integrability

The global study supposes the search of the maximal integral manifolds. An important result (in $\mathcal{C}^\infty$ case) is given by the Theorem 4.1 from [Su73] due to Sussmann. In the analytic case, Nagano's theorem solves the problem (see in [Na66]). The connection between global and local integrability is given by Theorem 4.2 due also to Sussmann (about the proof see the remark from §2). In this paper we have not been interested in global study (the analysis of the foliations with singularities or of the stratifications) but in the other two points.

In the local study the results can be stated using germs of functions, vector fields, manifolds etc. . We have three interesting results: Nagano's theorem (Theorem (5.1)) in the analytic case, Hermann's theorem (Theorem (5.2)) in the smooth case ($\mathcal{C}^\infty$) and Theorem (5.4) that gives a complete characterization of finitely-generated, involutive module (also in $\mathcal{C}^\infty$ case).

The punctual study brings out many results: the criterion given by Corollary (4.3), Theorem (4.6) (both in the $\mathcal{C}^\infty$ case) and the Theorem (4.4) (in the analytic case). About the last one we have the following remark:

We consider that the initial object is the analytic module: $\mathcal{O}b$=$\mathcal{L}$. To this object we associate a module of vector fields: $D\mathcal{O}b$=$\mathcal{L}$ (again the initial object). We carry on the iterative sequence:

$$\mathcal{L}^{k+1} \stackrel{\text{def}}{=} \mathcal{L}^k + span_{\mathcal{F}(M)} L_{D\mathcal{O}b}\mathcal{L}^k, \ for \ k \geq 0, \ \mathcal{L}^0 = \mathcal{O}b$$

$(L_{D\mathcal{O}b}\mathcal{L}^k \stackrel{\text{def}}{=} \{L_X Y = [X, Y] | X \in D\mathcal{O}b, Y \in \mathcal{L}^k\})$
Then $\mathcal{L}^\infty = L^\infty\mathcal{L}$ (by notation from Theorem (4.4)) and the theorem requires that: $\mathcal{L}^\infty|_{x_0} = L(x_0)$ for integrability.

The same tehnique can be applied for the study of the codistributions or of the systems of k-forms too.

References

[Ce79] D. Cerveau, *Distributions Involutives Singulières*, Ann.Inst.Fourier, Grenoble, **29**(1979), 261–294

[ChSt76] D. Chillingworth, P. Stefan, *Inegrability of singular distributions on Banach manifolds* , Math. Proc. Camb. Phil. Soc., **79**(1976),117, 117–128

[Fr78] M. Freeman, *Integration of analytic differential systems with singularities and some applications to real submanifolds of C^n*, J.Math.Soc.Japan, **30**(1978), 571–578

[Fr84] M. Freeman, *Fully integrable Pfaffian systems*, Annals of Math., **119**(1984), 465–510

[Hen62] R. Hermann, *The Differential Geometry of Foliations,II*, J.Math.Mech., **11**(1962), 303–316

[Hes86] H. Hermes, *Involutive Subdistributions and Canonical Forms for Distributions and Control Systems*, in *Theory and Appl. of Nonlin. Control Syst. (H.J.Sussmann ed.) (1986)*, 123–135

[Ku82] A. Kumpera, *Intégration des distributions singuliéres sur les variétés banachiques*, Pubblicazioni dell'instituto di Analisi Globali e Applicazioni, Firenze(1982)

[Lo70] C. Lobry, *Controlabilité des Systemes Non Lineaires*, SIAM J.Control, **8**(1970), 573–605

[Ma68] M. Matsuda, *An Integration Theorem for Completely Integrable Systems with Singularities*, Osaka J. Math., **5**(1968), 279–283

[Na66] T. Nagano, *Linear differential systems with singularities and an application to transitive Lie algebras*, J.Math. Soc.Japan, **118**(1966), 398–404

[So90] E.D. Sontag, *Integrability of Certain Distributions Associates with Actions on Manifolds and Applications to Control Problems*, in *Nonlin. Contr. and Optim. Control (H.J.Sussmann ed.) (1990)* , 81–131

[St74] P. Stefan, *Accesible Sets, Orbits and Foliations with Singularities*, Proc.London Math.Soc, **29**(1974), 699–713

[St80] P. Stefan, *Integrability of Systems of Vectorfields* , J.London Math. Soc.(2), **21**(1980), 544–556

[Su73] H.J. Sussmann, *Orbits of Families of Vectorfields and Integrability of Distributions*, Trans.Amer.Math.Soc., **180**(1973) , 171–188