On Homogeneous Polynomial Approximations of Nolinear Control Systems

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November 21, 1993

Abstract

In this paper we are going to study the chain of some distributions required by nonlinear control problems associated to homogeneous polynomial nonlinear systems. The idea is due to Ruiz and Nijmeijer and the basic facts are strongly inspiered by their paper. The distribution studied here is the controlled invariant distribution used in the problems of local disturbance decoupling and input-output decoupling of nonlinear systems.

1 Construction of the non-standard splitting of modules

We shall give some details about the construction of a graduation of the Lie algebra of the homogeneous polynomial vector field induced by a dilation. For a more extensive treatement of this subject, the reader is referred to [Bacc92] (Appendix B) or [Good76].

1. Let $\{e_i\}_{1\leq i\leq n}$ denote the canonical basis of $\mathbf{R}^n$. For any n-tuple of natural numbers $(w_1, w_2, ..., w_n)$, ordered nondecreasing $(1 \leq w_1 \leq ... \leq w_n \stackrel{\text{def}}{=} r)$ we define a family of maps $\delta_\lambda : \mathbf{R}^n \mapsto \mathbf{R}$ by:

$$\delta_\lambda(x) = \sum_{i=1}^{n} \lambda^{w_i} x_i e_i$$

Such a family is called a *dilation* on $\mathbf{R}^n$. Let $K[X]$ denote the ring of formal power series in variables $x_1, x_2, ..., x_n$. A polynomial $p \in K[X]$ is said to be *homogeneous* of weight l with respect to δ_λ if

$$p(\delta_\lambda(x)) = \lambda^l p(x) \quad , \quad \lambda \geq 0$$

For a monomial $x_1^{k_1}...x_n^{k_n}$ the associated weight is $w_1k_1 + w_2k_2 + \cdots + w_nk_n$. If p is not homogeneous, the weight of p is defined as the maximal weight of its homogeneous components. The weight of p will be denoted by $wgt(p)$. We call *subdegree* the minimal weight of its homogeneous components (we denote it by $sbd(p)$). We consider the following sets:

$$H_l = \{p \in K[X] | deg(p) = sbd(p) = l\}$$

$$\mathcal{P}_w = \{p \in K[X] | deg(p) \leq w\}$$

$$\mathcal{J}_j = \{p \in K[X] | sbd(p) > j\}$$

The first two are K-vector subspace of $K[X]$ while the last is an ideal in the ring $K[X]$. By this way we obtain a graduation and a filtration on $K[X]$ induced by the dilation:

$$K[X] = \bigoplus_{0 \leq l} H_l = \bigcup_{0 \leq w} \mathcal{P}_w = \mathcal{P}_w \bigoplus \mathcal{J}_w \quad , \quad \mathcal{P}_w = \bigoplus_{o \leq l \leq w} H_l$$

2. We now construct the vector fields over $K[X]$ that are derivations of the ring $K[X]$, i.e.: $v : K[X] \to K[X]$ which is: (i) a K-linear map and (ii) $v(p_1 p_2) = v(p_1)p_2 + p_1 v(p_2)$. We denote by $\mathcal{X}$ the Lie algebra of the vector fields. We say that X is an *homogeneous polynomial vector field* (h.p.v.f.) with respect to the dilation δ_λ, with the weight m if:

$$X H_l \subset H_{l-m} \quad , \quad for\ every\ l \geq 0$$

The weight of h.p.v.f. may be positive as well as negative. Always $x_i \frac{\partial}{\partial x_i}$ is a zero weight vector field regardless the weight of x_i. We denote by $wgt(X)$ the weight of the vector field X. If X is not an homogeneous p.v.f. the weight of X is defined as the maximal weight of its homogeneous components). As before we call *subdegree* of X the minimal weight of its homogeneous components and we denote it by $sbd(X)$.

Let us consider the following sets:

$$\mathcal{X}_i = \{v \in \mathcal{X} | wgt(v) = sbd(v) = i\}$$

$$Q_i = \{v \in \mathcal{X} | wgt(v) \geq r - i\}$$

$$T_i = \{v \in \mathcal{X} | sbd(v) < r - i)\}$$

The sets $\mathcal{X}_i$ and Q_i are K-vector subspaces of $\mathcal{X}$ while T_i is a $K[X]$-submodule of $\mathcal{X}$ (we consider that 0 could be added to any set). Obviously:

$$\mathcal{X} = \bigoplus_{i \leq r} \mathcal{X}_i = \bigcup_{i \geq 0} Q_i = Q_m \bigoplus T_m$$

and:

$$\mathcal{X}_{r-i} = \frac{Q_i}{Q_{i-1}} \quad , \quad Q_l = \bigoplus_{i=0}^{l} \mathcal{X}_{r-i}$$

3. The dual of $\mathcal{X}$ is the $K[X]$-module of the 1-forms denoted by Λ:

$$\Lambda = \{\omega : \mathcal{X} \to K[X] | \omega\ is\ a\ K[X] - linear\ map\ \}$$

Let $\omega \in \Lambda$. We say that ω is an *homogeneous polynomial* $1 - forms$ (h.p.1-f.) of weight s if:

$$\omega(Q_m) \subset H_{s-m} \quad , \quad for\ every\ m \leq r$$

We denote by $wgt(\omega)$ the weight of ω. If ω is not an homogeneous p.1-f., $wgt(\omega)$ denotes the weight of the maximal weight of its homogeneous component. We call *subdegree* the minimal weight of its homogeneous components and we denote it by $sbd(\omega)$. As in the previous case we consider the following sets:

$$\Lambda_i = \{\omega \in \Lambda | wgt(\omega) = sbd(\omega) = i\}$$

$$R_i = \{\omega \in \Lambda | wgt(\omega) \leq i\}$$

$$S_i = \{\omega \in \Lambda | sbd(\omega) > i\}$$

The first two sets are K-vector subspace while the last is a $K[X]$-submodule of Λ.Obviously:

$$\Lambda = \bigcup_{i \geq 0} R_i = \bigoplus_{s \geq 0} \Lambda_s = R_m \bigoplus S_m$$

and:

$$\Lambda_i = \frac{R_i}{R_{i-1}} \quad , \quad R_l = \bigoplus_{i=0}^{l} \Lambda_i$$

For any function of $\mathcal{F}(\mathbf{R}^n)$, $\mathcal{C}^\infty$ vector field on $\mathbf{R}^n$ or $\mathcal{C}^\infty$ 1-form on $\mathbf{R}^n$ we could associate its Taylor polynomial approximation. We could arrange this polynomial in terms of homogeneous components. Thus we obtain an expansion in homogeneous Taylor polynomials of the object. We shall use such expansions.

EXAMPLE 1.1 *Let in $\mathbf{R}^2$ be the dilation given by $\delta_\lambda(x_1, x_2) = (\lambda x_1, \lambda^2 x_2)$. Then we have for the following h.p.v.fs:*

$$X_1 = \frac{\partial}{\partial x_1} \; X_2 = \frac{\partial}{\partial x_2} \; \textit{and} \; X_3 = x_2 \frac{\partial}{\partial x_1} + x_1^3 \frac{\partial}{\partial x_2}$$

the weights: $wgt(X_1) = 1$, $wgt(X_2) = 2 = r$, $wgt(X_3) = -1$, and for the h.p.1-fs:

$$\omega_1 = dx_1 \; , \; \omega_2 = dx_2 \; , \; \omega_3 = x_2^2 dx_1 + x_1^3 dx_2$$

the weights: $wgt(\omega_1) = 1$, $wgt(\omega_2) = 2$, $wgt(\omega_3) = 5$. $\diamond$

The basical properties of the homogeneous objects defined above are given by the following result:

PROPOSITION 1.2 *1) If h is an homogeneous polynomial, X an h.p.v.f. and ω an h.p.1-f then:*

$$wgt(hX) = wgt(X) - wgt(h) \quad \textit{and} \quad wgt(h\omega) = wgt(\omega) + wgt(h)$$

2) If $h \in H_l$ then $dh \in R_l$.
3) If $X_1 \in Q_{m_1}, X_2 \in Q_{m_2}$ then $[X_1, X_2] \in Q_{m_1+m_2}$ ($[X_1, X_2]$ denotes the Lie bracket of the two vector fields).
4) If $\omega \in R_s$ and $X \in Q_m$ then $L_X \omega \in R_{s-m}$. $\Box$

2 Systems-Jets

We introduce some factorization on the previous modules:
· For the polynomials: $p/k = \{q \in K[X] | p - q \in \mathcal{J}_k\}$.
· For the vector fields: $v/k = \{w \in \mathcal{X} | v - w \in T_{k+1}\}$.
· For the 1-forms: $\omega/k = \{\varphi \in \Lambda | \omega - \varphi \in S_k\}$.

Definition Any two K-vector subspace or $K[X]$-submodules S_1,S_2 are K-equivalent iff $\forall x_1 \in S_1$ $\exists x_2 \in S_2$ such that $x_1/k = x_2/k$ and $\forall x_2 \in S_2$ $\exists x_1 \in S_1$ such that $x_1/k = x_2/k$.

Let Σ^1 and Σ^2 be two affine polynomial nonlinear control systems given by:

$$\Sigma^1 \left\{ \begin{array}{rcl} \dot{x} & = & f^1(x) + \sum_{i=1}^m g_i^1(x) u_i \\ y_j & = & h_j^1(x) \end{array} \right. \quad \Sigma^2 \left\{ \begin{array}{rcl} \dot{x} & = & f^2(x) + \sum_{i=1}^m g_i^2(x) u_i \\ y_j & = & h_j^2(x) \end{array} \right.$$

on the same state-space ($\mathbf{R}^n$) with the same number of inputs (m) and outputs (p).

Definition We say that $\Sigma^1 \overset{k}{\sim} \Sigma^2$ iff $f^1 - f^2 \in T_{k+1}$ $g_i^1 - g_i^2 \in T_k$ and $h_j^1 - h_j^2 / in \mathcal{J}_{k+1}$ for any $i = \overline{1,m}$ and $j = \overline{1,p}$.

Fact: It is an equivalence relation on the set of affine polynomial control systems with the same number of inputs, outputs and the same state dimension, respectively.

We call the class of the systems k-related by the above equivalence relation the *k-systems jet*. We will denote such a k-systems jet by Σ/k.

With the above notations: $\Sigma^1 \overset{k}{\sim} \Sigma^2$ iff $f^1/k = f^2/k$, $g_i^1/k-1 = g_i^2/k-1$ and $h_j^1/k+1 = h_j^2/k+1$. Our problem is to characterize some objects constructed starting from a member of a k-systems jet (to be more precise, the problem is to find out the class of an object associated to a member of a k-systems jet.

3 The basic Lemma

The enabled operations with jets of vector fields and 1-forms are given by the following Lemma:

LEMMA 3.1 *1. Let Θ^1 and Θ^2 be two free $K[X]$-submodules of Λ having the same rank ($= s$) and such that $\Theta^1/j = \Theta^2/j$ for some $j \geq r$. Let G^1 and G^2 be two free $K[X]$-submodules of $\mathcal{X}$ having the same rank ($= m$) and such that $G^1/u = G^2/u$ for some $u > j$. We suppose that the distributions spanned by G^1 and G^2 and the codistributions spanned by Θ^1 and Θ^2 as well as the codistributions spanned by $\Phi^1 = \Theta^1 \bigcap (G^1)^\perp$ and $\Phi^2 = \Theta^2 \bigcap (G^2)^\perp$ are constant dimensional (regular). Then: $\Phi^1/j-r = \Phi^2/j-r$*

2. If Θ^1,Θ^2,G^1 and G^2 are as above, then:

$((\Theta^1)^\perp \bigcap G^1)/j-r = ((\Theta^2)^\perp \bigcap G^2)/j-r$.

3) Let Ψ^1 and Ψ^2 be two free $K[X]$-submodules of 1-forms with the same rank, say s, such that $\Psi^1/j = \Psi^2/j$, for some $j \geq r$, and X^1, X^2 be two vector fields such that $X^1/u = X^2/u$ for some $u > j$.

Then: $(\Psi^1 + L_{X_1}\Psi^1)/j-r = (\Psi^2 + L_{X_2}\Psi^2)/j-r$.

Proof*(Sketch) Let us consider the basis as follows:*

$$\{\omega^1, \ldots \omega^s\} \subset \Theta^1 \quad and \quad \{\pi^1, \ldots \pi^s\} \subset \Theta^2$$

$$\{v_1, \ldots, v_m\} \subset G^1 \quad and \quad \{w_1, \ldots, w_m\} \subset G^2$$

We shall denote with Π and Ω the matrices having as entries the components of π_i and, respectively, ω_i ordered according with the rows and with V and W the matrices having as entries the components of v_i and, respectively, w_i ordered according with the columns. The basis are chosen such that [1]: $\pi_i/j = \omega_i/j$, $w_i/u = v_i/u$ and:

$$\Pi \cdot W = (\pi_i(w_j))_{ij} = \left[\begin{array}{c|c} \frac{I}{A} & 0 \end{array} \right] \; ; \; dim(I) = s - dim\Theta^2 \bigcap (G^2)^\perp$$

Then:

$$\Omega \cdot V = (\omega_i(v_j))_{ij} = \left[\begin{array}{c|c} \frac{I+\varepsilon_1}{A} & \varepsilon_2 \end{array} \right]$$

such that: $\Omega V/j-r = \Pi W/j-r$ and so:

$\varepsilon_1/j-r = 0$; $\varepsilon_2/j-r = 0$; $B/j-r = A/j-r$.

[1] We have to point out that we can obtain the identity matrix in the upper-left corner of $\Pi \cdot W$ only if the ring over which are defined the modules is the ring of formal power series. Just here we need to consider the construction over the ring of formal power series.

Since the codistributions spanned by $\Theta^1 \bigcap (G^1)^\perp$ and $\Theta^2 \bigcap (G^2)^\perp$ are constant dimensional and coincides at the origin then $[\varepsilon_2]$ has columns linearly dependent on the columns of

$$\left[\frac{I+\varepsilon_1}{B}\right] \qquad (*)$$

Let us take $\pi \in \Theta^2 \bigcap (G^2)^\perp$. Then $\pi = \sum_{i=1}^{s} \varphi_i \pi_i$ where the functions φ_i compound the row-vector φ given by: $\varphi = g[-A\ I_q]$ for some $g \in K^q[X]$.

We wish to prove that for any $\omega \in \Theta^1 \bigcap (G^1)^\perp$ there exists an 1-form Δ such that:

$$\begin{cases} \omega + \Delta \in \Theta^2 \bigcap (G^2)^\perp \\ \Delta / j - r = 0 \end{cases}$$

Then it results that $\omega/j - r \in (\Theta^2 \bigcap (G^2)^\perp)/j - r$. For the converse we can follow the same way and we obtain the statement.

Let us consider the map:

$$\Delta : K^q[X] \to \Lambda \ , \ \Delta(g) = g \cdot [-A\ I]\Pi - \omega$$

For any $\omega \in \Theta^1 \bigcap (G^1)^\perp$ we can write: $\omega = f \cdot \Omega$ such that $f \Omega V = 0$ (f has the same meanning as φ from above). Then: $f[\frac{I+\varepsilon_1}{B}] = 0$ and we partition f according with $(*)$: $f = [f_1\ f_2]$. Then $f_1 + f_1\varepsilon_1 + f_2 B = 0 \Rightarrow f_1/j - 2 = -f_2 B/j - r = -f_2 A/j - r$ and: $\Delta(f_2)/j - r = [-f_2 A\ f_2]\Pi/j - r\ - [f_1\ f_2]\Omega/j - r\ = 0$
$\square$

4 Main Result

We consider the Controlled Invariant Distribution Algorithm that gives for a system of the form:

$$\Sigma \begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{cases}$$

($x \in M \subset \mathbf{R}^n$, M being an open neighborhood of the origin) the maximal controlled invariant distribution included in the kernel of the differential of the output mapping:

<u>Step 0</u>: $\Omega_0 = span\{dh\}$

<u>Step k</u>: $\Omega_k = \Omega_{k-1} + L_f(\Omega_{k-1} \bigcap G^\perp) + \sum_{i=1}^{m} L_{g_i}(\Omega_{k-1} \bigcap G^\perp)$

where $G = span\{g_i | i = \overline{1,m}\}$

Let us consider the following assumptions for a generic system Σ:

<u>*Assumptions*</u>

(H1) The distribution spanned by G has maximal and constant rank m.

(H2) The codistribution spanned by $\{dh\}$ has maximal and constant rank p.

(H3) At every step of the previous algorithm, the codistributions spanned by Ω_{k-1} and $\Omega_{k-1} \bigcap G^\perp$ have constant dimension on M.

Under (H1)-(H3) we know that there exists $\mu_\Sigma^* \in \mathbf{N}$ such that:

$$\Omega_k = \Omega_{k+1} = \cdots = \Omega_* \ , \quad for any k \geq \mu_\Sigma^*$$

THEOREM 4.1 *Let us suppose Σ_1 and Σ_2 belong to the same a-system jet ($\Sigma_1 \overset{a}{\sim} \Sigma_2$). Let us suppose that (H1)-(H3) are fulfilled by Σ_1 and Σ_2. Then:*

a) For any k such that $a - 2 - 2rk \geq r$ we have that:

$$\Omega_k^1/(a - 2 - 2rk) = \Omega_k^2/(a - 2 - 2rk)$$

b) If $max(\mu^*_{\Sigma_1}, \mu^*_{\Sigma_2}) = \mu^* \geq [\frac{a-2-r}{2r}]$ *then:*

$$\Omega^1_*/(a-2-2r\mu^*) = \Omega^2_*/(a-2-2r\mu^*)$$

In particular:

$$\Omega^1_*|_0 = \Omega^2_*|_0$$

c) If $\mu^* = max(\mu^*_{\Sigma_1}, \mu^*_{\Sigma_2}) > [\frac{a-2-r}{2r}]$ *but, for instance,* $\mu^*_{\Sigma_1} > [\frac{a-2-r}{2r}] \geq \mu^*_{\Sigma_2}$ *we obtain:*

$$\Omega^1_*|_0 \supseteq \Omega^2_*|_0$$

Proof *(i) We take* $\Theta^1 = \Omega^1_0$, $\Theta^2 = \Omega^2_0$, $j = a-2$, $u = a-1$. *Applying first point of Lemma we obtain that* $(\Omega^1_0 \bigcap (G^1)^\perp)/a-2-r = (\Omega^2_0 \bigcap (G^2)^\perp)/a-2-r$. *From Assumption 2 we obtain that* $\Omega^1_0 \bigcap (G^1)^\perp$ *and* $\Omega^2_0 \bigcap (G^2)^\perp$ *have the same rank (because still* $a-2-r \geq r$*). We apply the second point of Lemma for* $X_1 = f^1$,$X_2 = f^2$ *and then for* $X_1 = g^1_i$,$X_2 = g^2_i$. *We will obtain now that:* $\Omega^1_1/a-2r-2 = \Omega^2_1/a-2r-2$. *We see that the "depth" of the common zone where the equality between the two modules holds decreases with* $2r$. *Now, by induction, we obtain the statement. The number* μ *is obtained in such a manner that the last equality must hold for a minimum of "depth"* x: $\Omega^1_\mu/x = \Omega^2_\mu/x$ *where:* $x = a-2-\mu \cdot 2r \geq r \Rightarrow \mu$. *(ii) and (iii) are immediately now.*

5 Example

The following example shows the possibility of avoiding of a singularity when it is used a non-standard dilation:

$$\begin{cases} \dot{x} & = & \begin{bmatrix} -1 \\ 2x_1 \end{bmatrix} u \\ y & = & x_2 + (x_1^2 + x_2^2) + 2x_1^2 x_2 + x_1^4 \end{cases}$$

Using the standard dilation ($w_1 = w_2 = 1$) we obtain the following approximations:

$$\Sigma^1 \begin{cases} \dot{x} & = & \begin{bmatrix} -1 \\ 0 \end{bmatrix} u \\ y & = & x_2 \end{cases}$$

$dh(g) = 0$.

$$\Sigma^2 \begin{cases} \dot{x} & = & \begin{bmatrix} -1 \\ 2x_1 \end{bmatrix} u \\ y & = & x_2 + x_1^2 + x_2^2 \end{cases}$$

$dh(g) = 4x_1x_2 \not\equiv 0$. Then $G^\perp \bigcap \Omega_0$ is singular !

$$\Sigma^3 \begin{cases} \dot{x} & = & \begin{bmatrix} -1 \\ 2x_1 \end{bmatrix} u \\ y & = & x_2 + x_1^2 + x_2^2 + 2x_1^2 x_2 \end{cases}$$

$dh(g) = 4x_1^3 \not\equiv 0$. Then $G^1 \bigcap \Omega_0$ is singular !

$$\Sigma^4 \begin{cases} \dot{x} & = & \begin{bmatrix} -1 \\ 2x_1 \end{bmatrix} u \\ y & = & x_2 + x_1^2 + x_2^2 + 2x_1^2 x_2 + x_1^4 \end{cases}$$

$dh(g) = 0$.

While, using a non-standard dilation ($w_1 = 1$ and $w_2 = 2$) we obtain as follows:

$$\Sigma_p^1 \begin{cases} \dot{x} &= \begin{bmatrix} -1 \\ 2x_1 \end{bmatrix} u \\ y &= 0 \end{cases} \quad (the\ class\ k = 0)$$

$dh(g) = 0$

$$\Sigma_p^2 \begin{cases} \dot{x} &= \begin{bmatrix} -1 \\ 2x_1 \end{bmatrix} u \\ y &= x_2 + x_1^2 \end{cases} \quad (the\ class\ k = 1)$$

$dh(g) = 0$

$$\Sigma_p^3 \begin{cases} \dot{x} &= \begin{bmatrix} -1 \\ 2x_1 \end{bmatrix} u \\ y &= x_2 + x_1^2 \end{cases} \quad (the\ class\ k = 2)$$

$dh(g) = 0$

$$\Sigma_p^4 \begin{cases} \dot{x} &= \begin{bmatrix} -1 \\ 2x_1 \end{bmatrix} u \\ y &= x_2 + (x_1^2 + x_2^2) + 2x_1^2 x_2 + x_1^4 \end{cases} \quad (the\ class\ k = 3)$$

$dh(g) = 0$

We see that in the later sequence of approximations we have avoided the singularity that has occurs in the former sequence.

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