Robustness Properties of Nonlinear Systems:

Absolute Stability of Linear Systems

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Abstract

In this paper we present a new approach to the Popov's Positivity Theorem and new statements and proofs of Absolute Stability Theorems.

In the first chapter we establish connexions between Riccati equations, Kalman-Yakubovitch-Popov systems and Lurié systems. We prove this results for stabilizable and antistabilizable solutions avoiding Youla's factorization.

In the second chapter we state the Absolute Stability Theorems that we are dealing with: Circle Criteria, General Popov Criteria and the usual Popov Criteria. Then we present the mechanism of the proofs and at the end of the chapter we give the extended proofs.

Chapter 1

The Popov Theorem of Positivity

1.1 The Objects and Statement of the Results

The main object of this paper will be the triplet of the form $\Sigma = (A, B; P) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m} \times \mathbf{R}^{(n+m)\times(n+m)}$ called a *Popov triplet* with $P = P^T$ partitioned as:

$$P = \begin{bmatrix} Q & L \\ L^T & R \end{bmatrix}$$

The appropriate interpretation of the triplet Σ is given refering to a control system:

$$\dot{x} = Ax + Bu \tag{1.1}$$

and a criterion with weighting matrix P, namely

$$J(t_1) = \int_0^{t_1} [x^T u^T] P \begin{bmatrix} x \\ u \end{bmatrix} dt \tag{1.2}$$

where $[x^T u^T] \in L^{2,n}[0, t_1] \times L^{2,m}[0, t_1]$.

The rational function:

$$\begin{aligned} \Pi_\Sigma(s) &\stackrel{\text{def}}{=} [B^T(-sI - A^T)^{-1} \;\; I] P \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix} \\ &= R + B^T(-sI - A^T)^{-1}L + L^T(sI - A)^{-1}B + \\ &\quad + B^T(-sI - A^T)^{-1}Q(sI - A)^{-1}B \end{aligned} \tag{1.3}$$

is called the *Popov function* associated to Σ. It is easy to prove that the Popov function has the following realization:

$$H_\Sigma = \left[\begin{array}{cc|c} A & 0 & B \\ -Q & -A^T & -L \\ \hline L^T & B^T & \mathrm{R} \end{array}\right] \tag{1.4}$$

We call a *Kalman-Yakubovich-Popov system* the following set of equations:

$$\begin{aligned} R &= V^T V \\ L + XB &= W^T V \\ Q + A^T X + XA &= W^T W \end{aligned} \tag{1.5}$$

and a solution of KYP, a pair $(V, W, X = X^T)$ that fulfills (1.5). For any solution of KYP system we can bring the criterion into the following form:

$$J(t_1) = -[x^T X x]|_{x(0)}^{x(t_1)} + \int_0^{t_1} \| Wx(t) + Vu(t)\|^2 dt \tag{1.6}$$

We call a *Lurié system* the following set of equations:

$$\begin{array}{r} A^T X + XA + Q + (XB + L)F = 0 \\ B^T X + L^T + RF = 0 \end{array} \tag{1.7}$$

and a solution of Lurié system, a pair $(F, X = X^T)$ that fulfills (1.7). We call F a *stabilizable solution* if $\Lambda(A + BF) \subset \mathcal{C}^-$ whereas an *antistabilizable solution* if $\Lambda(A + BF) \subset \mathcal{C}^+$ (warning! $\mathcal{C}^+ = \{z \in \mathcal{C} | Re\, z \geq 0\}$)

We see that for any solution of Lurié system we obtain a solution of KYP system (for the same X) in the following form: V is a Cholesky factor of R and $W = -VF$.

In the case $R > 0$ we consider the following equation:

$$A^T X + XA - (XB + L)R^{-1}(L^T + B^T X) + Q = 0 \tag{1.8}$$

called the *continuous-time algebraic Riccati equation (CTARE)* associated to Σ. For any solution $X = X^T$ of CTARE we have a solution of Lurié system of the form:

$$F = -R^{-1}(L^T + B^T X) \tag{1.9}$$

The solution X of (1.8) is said stabilizable or antistabilizable depending on how F, defined above, is.

To CTARE we associate the Hamiltonian matrix that is:

$$H = \begin{bmatrix} A - BR^{-1}L^T & -BR^{-1}B^T \\ -Q & -A^T + LR^{-1}B^T \end{bmatrix} \tag{1.10}$$

and has the meanning of the A-matrix of the inverse system of (1.4).

We are looking here only for the stabilizable and antistabilizable solutions of CTARE. This is why our statements shall give information about these two solutions. For the general case (i.e. for any Youla partition of the Popov function) see the original Popov's positivity theorem ([Popov73]).

Our first statement gives the result only for the case of strict positivity:

THEOREM 1 *Consider the Popov triplet $\Sigma = (A, B; P)$ and the Popov function $\Pi_\Sigma(s)$ associated to it. We supposde (A, B) controllable. Then the following are equivalent:*

A *1.* $\Pi(j\omega) > 0$, $\forall \omega \in \bar{\mathbf{R}}$ *excepting for those imaginary eigenvalues of A.*

2. The realisation (1.4) has no uncontrollable modes on the imaginary axis.

B *1.* $R > 0$

2. There are (V, W_s, X_s) and (V, W_a, X_a) the two (unique) stabilizable and, respectively, antistabilizable solutions of KYP system (1.5).

C *1.* $R > 0$

2. There exist (F_s, X_s) and (F_a, X_a) the two (unique) stabilizable and , respectively, antistabilizable solutions of the Lurié system (1.7).

D *1.* $R > 0$

2. There exists X_s and X_a the two (unique) stabilizable and ,respectively, solutions of CTARE (1.8). □

(for the proof see the next section)

If we allow not to have strict inequalities, then we obtain a weaker statement whose proof involves some perturbation techniques:

LEMMA 2 *Consider the Popov triplet $\Sigma = (A, B; P)$ and its associated Popov function Π_Σ. Suppose (A, B) controllable and $(\Pi(j\omega) \geq 0$, $\forall \omega \in \mathbf{R}$. Then, if we perturb R with εI ($R \rightarrow R' = R + \varepsilon I$ with $0 < \varepsilon \leq 1$) such that $\Pi^{'}_\Sigma(j\omega) > 0$, $\forall \omega \in \bar{\mathbf{R}}$ we obtain two solutions $(V(\varepsilon), W_s(\varepsilon), X_s(\varepsilon), F_s(\varepsilon))$ and $(V(\varepsilon), W_a(\varepsilon), X_a(\varepsilon), F_a(\varepsilon))$ as in the previous theorem and for $\varepsilon \rightarrow 0$ we obtain the following behaviour:*

$$V(\varepsilon) \rightarrow V \quad , \quad W_{s,a}(\varepsilon) \rightarrow W_{s,a} \quad , \quad X_{s,a}(\varepsilon) \rightarrow X_{s,a}(0)$$

finite, and

$$F_{s,a}(\varepsilon) \sim \frac{1}{\sqrt{\varepsilon}} M(\varepsilon)$$

possible infinite. □

Using this Lemma we obtain for the limit case the following theorem:

THEOREM 3 *Consider the Popov triplet $\Sigma = (A, B; P)$ and its associated Popov function $\Pi_\Sigma(s)$. If the pair (A, B) is controllable and $\Pi_\Sigma(j\omega) \geq 0$, $\forall \omega \in \mathbf{R}$ then there are two solutions (possibly identic) of KYP system (V, W_s, X_s), (V, W_a, X_a) such that the transmission zeros of (A, B, W, V) (i.e. the roots of $p(s) = det(V + W(sI - A)^{-1}B) \cdot det(sI - A)$) are for the first solution with $Re\, s_k \leq 0$ and for the second solution with $Re\, s_k \geq 0$.* □

1.2 Proof of Theorem 1 (The Case $R > 0$)

First it is obvious that conditions **B**,**C** and **D** are equivalent. Moreover, the relations between these solutions are:

$$F_s = -V^{-1} W_s \quad , \quad F_a = -V^{-1} W_a$$

" $\Leftarrow$ " If we suppose **B**,**C**,**D** fulfilled then by algebraic manipulation one can prove the following spectral factorization for Popov function:

$$\Pi_\Sigma(s) = S^T(-s) R S(s)$$

where:

$$S(s) = I - F(sI - A)^{-1} B = \left[\begin{array}{c|c} \text{A} & \text{B} \\ \hline \text{-F} & \text{I} \end{array}\right]$$

The transmission zeros of $S(s)$ are given by the eigenvalues of $A + BF$ because of:

$$S^{-1}(s) = \left[\begin{array}{c|c} \text{A+BF} & \text{B} \\ \hline \text{F} & \text{I} \end{array}\right]$$

Then we obtain the following facts:

i) $\Pi_\Sigma(j\omega) = S^H(j\omega) R S(j\omega) > 0$, $\forall \omega \in \mathbf{R} - j\Lambda(A)$

ii) $\Pi_\Sigma(\infty) = R > 0$

(H being the hermitian conjugacy) and moreover

iii) The realisation (1.4) has no uncontrollable (and unobservable modes on the imaginary axis).

" $\Rightarrow$ " Before starting the proof of Theorem, we need the following Lemma:

LEMMA 4 *If the realisation (1.4) has no uncontrollable modes on the imaginary axis, then it also has no unobservable modes on the imaginary axis.*

Proof: *From the definition of uncontrollable modes we have:*

$$rank \begin{bmatrix} j\omega I - A & 0 & B \\ Q & j\omega I + A^T & -L \end{bmatrix} = n \quad , \forall \omega \in \mathbf{R}$$

We complex conjugate the matrix and then we multiply the first column with $-I$:

$$rank \begin{bmatrix} j\omega I + A & 0 & B \\ -Q & -j\omega I + A^T & -L \end{bmatrix} = n$$

Now we multiply the second block row with $-I$ and then we inter-change the first two block columns:

$$rank \begin{bmatrix} 0 & j\omega I + A & B \\ j\omega I - A^T & Q & L \end{bmatrix} = n$$

We transpose the matrix:

$$rank \begin{bmatrix} 0 & j\omega I - A \\ j\omega I + A^T & Q \\ B^T & L^T \end{bmatrix} = n$$

and we interchange the block columns:

$$rank \begin{bmatrix} j\omega I - A & 0 \\ Q & j\omega I + A^T \\ L^T & B^T \end{bmatrix} = n \quad , \forall \omega \in \mathbf{R}$$

The last relation says exactly that our realisation has no unobservable modes on the imaginary axis. This ends the proof. □

Now we begin the main part of the proof for the implication $A \Rightarrow B, C, D$. We shall apply a Kucera-type scheme for proving the existance of the stabilizable and antistabilizable solutions. For, it is enough to prove the dichotomy of the Hamiltonian and to assume the pair (A, B) is controllable (and this is already assumed). The dichotomy of the Hamiltonian comes from the following facts: the eigenvalues of the Hamiltonian (1.10) are exactly the transmission zeros of the Popov function and it has no zero on the imaginary axis because of:

1) $\Pi(j\omega) > 0$, $\forall \omega \in \bar{\mathbf{R}}$

2) The realisation (1.4) has no uncontrollable or unobservable modes on the imaginary axis (from the above lemma).

So we obtain the dichotomy of the Hamiltonian and with the controllability of the pair (A, B) the proof is complete now. □

1.3 Proof of Lemma 2 (The Case $R \geq 0$)

We shall use here a variational technique which supposes to perturb R with an infinitesimal quantity (εI) and to study how the solutions of our problems depend on ε. If $\Pi_\Sigma(j\omega) \geq 0$, $\forall \omega \in \mathbf{R}$, then for $R \to \tilde{R} = R + \varepsilon I$, $\varepsilon > 0$ we obtain $\Pi'_\Sigma(j\omega) > 0$, $\forall \omega \in \bar{\mathbf{R}}$. We apply Theorem 1 and obtain $(V(\varepsilon), W_s(\varepsilon), X_s(\varepsilon), F_s(\varepsilon))$ and $(V(\varepsilon), W_a(\varepsilon), X_a(\varepsilon), F_a(\varepsilon))$ the stabilizable and, respectively, antistabilizable solutions of the Riccati problem. Let U be an orthogonal matrix such that:

$$R' = URU^T = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix} \quad , \ R_1 > 0 \tag{1.11}$$

and $R_1 \in \mathbf{R}^{m_1 \times m_1}$. We multiply the equations (1.5) and (1.7) with U and U^T and renote the variable as follows:

$$R' = URU^T \ , \ V' = UVU^T \ , \ L' = LU^T \ , \ B' = BU^T \ , \ W' = UW \ , \ F' = UF$$

Then (1.5) and (1.7) keep the form with these new matrices. To perturb the initial R with εI is equivalent to perturb this new R' because $U\varepsilon IU^T = \varepsilon I$. In that it follows we shall considere the system in this new form and we shall give up to the prime symbol:

$$R_\varepsilon = R' + \varepsilon I \quad L \leftarrow L' \quad B \leftarrow B'$$

We look now to the Riccati equation (1.8) written for R_ε:

$$A^T X_\varepsilon + X_\varepsilon A - (X_\varepsilon B + L)(R + \varepsilon I)^{-1}(X_\varepsilon B + L)^T + Q = 0 \tag{1.12}$$

We know that the stabilizable solution $X_s(\varepsilon)$ has the meanning:

$$\min_{\substack{u \in L^{2,m}[0,\infty) \ \ x \in L^{2,n}[0,\infty) \\ \dot{x} = Ax + Bu \ \ x(0) = x_0}} \int_0^\infty [x^T \ u^T] \begin{bmatrix} Q & L \\ L^T & R_\varepsilon \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt = x_0^T X_s(\varepsilon) x_0$$

Whereas the antistabilizable solution $X_a(\varepsilon)$ has the meanning:

$$\min_{\substack{u \in L^{2,m}(-\infty,0] \ \ x \in L^{2,n}(-\infty,0] \\ \dot{x} = Ax + Bu \ \ x(0) = x_0}} \int_{-\infty}^0 [x^T \ u^T] \begin{bmatrix} Q & L \\ L^T & R_\varepsilon \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt = -x_0^T X_a(\varepsilon) x_0$$

Since for $\varepsilon_1 > \varepsilon_2$ we have $R_{\varepsilon_1} > R_{\varepsilon_2}$ we also obtain: $X_s(\varepsilon_1) > X_s(\varepsilon_2)$ and $X_a(\varepsilon_1) < X_a(\varepsilon_2)$. Then the stabilizable and antistabilizable solutions of (1.12) are monotone sequences. It is enough to prove the lower boundedness of the stabilizable solution and, respectively, the upper boundedness of the antistabilizable solution in order to obtain that there exist $\lim_{\varepsilon \to 0} X_s(\varepsilon)$ and $\lim_{\varepsilon \to 0} X_a(\varepsilon)$.

We rewrite (1.12) by using from Lurié system the equation $X_\varepsilon B + L = F_\varepsilon^T R_\varepsilon$:

$$A^T X_\varepsilon + X_\varepsilon A - F_\varepsilon^T (R + \varepsilon I) F_\varepsilon + Q = 0 \tag{1.13}$$

One could verify that the effect of a state-feedback matrix $\tilde{F}$ is given by:

$$\begin{aligned} &A \to \tilde{A} = A + B\tilde{F} \\ &F_\varepsilon \to \tilde{F}_\varepsilon = F_\varepsilon - \tilde{F} \\ &Q \to \tilde{Q}_\varepsilon = Q + L\tilde{F} + \tilde{F}^T L^T + \tilde{F}^T (R + \varepsilon I)\tilde{F} \end{aligned}$$

and the Riccati equation (1.13) takes the form:

$$\tilde{A}^T X_\varepsilon + X_\varepsilon \tilde{A} - (F_\varepsilon - \tilde{F})^T (R + \varepsilon I)(F_\varepsilon - \tilde{F}) + \tilde{Q} = 0 \tag{1.14}$$

We choose $\tilde{F}$ such that $A + B\tilde{F} = \tilde{A}$ is an antistable matrix (i.e. has eigenvalues with strict positive real part). Then the following Liapunov equation has an unique solution:

$$\tilde{A}^T X_1 + X_1 \tilde{A} + \tilde{Q}_1 = 0 \tag{1.15}$$

where:

$$\tilde{Q}_1 = Q + L\tilde{F} + \tilde{F}^T L^T + \tilde{F}^T (R + I)\tilde{F}$$

We substract (1.15) from (1.14) and then we get:

$$\tilde{A}^T(X_\varepsilon - X_1) + (X_\varepsilon - X_1)\tilde{A} - (F_\varepsilon - \tilde{F})^T(R+\varepsilon I)(F_\varepsilon - \tilde{F}) - (1-\varepsilon)\tilde{F}^T\tilde{F} = 0$$

Or:

$$(-\tilde{A})^T(X_\varepsilon - X_1) + (X_\varepsilon - X_1)(-\tilde{A}) + [(F_\varepsilon - \tilde{F})^T(R+\varepsilon I)(F_\varepsilon - \tilde{F}) + (1-\varepsilon)\tilde{F}^T\tilde{F}] = 0$$

Since $-\tilde{A}$ is stable and $(F_\varepsilon - \tilde{F})^T(R+\varepsilon I)(F_\varepsilon - \tilde{F}) + (1-\varepsilon)\tilde{F}^T\tilde{F} \geq 0$ we obtain that the above Liapunov equation has a unique positive semidefinite solution and so:

$$X_\varepsilon - X_1 \geq 0 \quad \Leftrightarrow \quad X_\varepsilon \geq X_1$$

We have obtained the lower boundedness of the stabilizable solution.

For the antistabilizable solution we use the same construction but we choose $\tilde{F}$ such that $\tilde{A}$ is stable (Hurwitz). We denote by X_2 the unique solution of (1.15) (with $\tilde{A}$ in this case stable). Then the solution of the above Liapunov equation is negative semidefinite i.e. $X_\varepsilon - X_2 \leq 0$ and then $X_a(\varepsilon) \leq X_2$. This proves the upper boundedness of the antistabilizable solution.

With the previous discussion we obtain that there exist both limits $\varepsilon \to 0$ for the stabilizable and antistabilizable solutions of the Riccati equation even if we are not able to write the limit Riccati equation (because of singularity of R). In this situation it is interesting to know what happened with Kalman-Yakubovich-Popov and Lurié systems.

We start with KYP system and partition $V(\varepsilon)$ according to (1.11):

$$V = [V_1 \;\; V_2] \quad , \; V_1 \in \mathbf{R}^{m \times m_1}$$

Then:

$$\begin{bmatrix} R_1 + \varepsilon I_1 & 0 \\ 0 & \varepsilon I_2 \end{bmatrix} = \begin{bmatrix} V_1^T V_1 & V_1^T V_2 \\ V_2^T V_1 & V_2^T V_2 \end{bmatrix}$$

and we get:

$$V_1 = \begin{bmatrix} V_{11} \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} 0 \\ \sqrt{\varepsilon} I_2 \end{bmatrix}$$

with $V_{11} \in \mathbf{R}^{m_1 \times m_1}$ such that $V_{11}^T V_{11} = R_1 + \varepsilon I_1 > 0$. So:

$$V(\varepsilon) = \begin{bmatrix} V_{11}(\varepsilon) & 0 \\ 0 & \sqrt{\varepsilon} I_2 \end{bmatrix} \tag{1.16}$$

We partition $W(\varepsilon)$, L and B according to the above partitions as:

$$W(\varepsilon) = \begin{bmatrix} W_1(\varepsilon) \\ W_2(\varepsilon) \end{bmatrix} \quad L = [L_1 \;\; L_2] \quad B = [B_1 \;\; B_2]$$

From the last KYP equation we obtain:

$$L + X_\varepsilon B = [W_1^T(\varepsilon) V_{11}(\varepsilon) \;\; \sqrt{\varepsilon} W_2^T(\varepsilon)] \tag{1.17}$$

Because of boundedness of W_2, for $X_s(\varepsilon)$ and $X_a(\varepsilon)$ at limit $\varepsilon \to 0$ we have:

$$L + X(0)B = [L_1 + X(0)B_1 \;\; 0]$$

and then:

$$W_1^T(0) = (L_1 + X(0)B_1) V_{11}^{-1}(0) \tag{1.18}$$

where the above expression gives $W_{1,s}(0)$ and $W_{1,a}(0)$. For W_2 we return to (1.17) and obtain:

$$W_2^T(0)W_2(0) = Q + A^T X(0) + X(0)A - (L_1 + X(0)B_1)R_1^{-1}(L_1 + X(0)B_1)^T \geq 0 \qquad (1.19)$$

So we obtain $W_{2,s}(0)$ and $W_{2,a}(0)$ as Cholesky factor of the above matrix. The equations (1.16),(1.18) and (1.19) give us the singular solutions of KYP system. Moreover, for any Riccati solution X_ε if we have the limit $\varepsilon \to 0$ then we have also a singular solution of KYP system associated to X.

As concerned the Lurié system, we start with the equation of F:

$$F = R^{-1}(L + XB)^T = \begin{bmatrix} R_1^{-1}(\varepsilon)(L_1^T + B_1^T X_\varepsilon) \\ \frac{1}{\varepsilon}(L_2^T + B_2^T X_\varepsilon) \end{bmatrix}$$

The problem is with the last rows because of $\frac{1}{\varepsilon}$. On the other hand, developing the equation 1.12 we get:

$$A^T X_\varepsilon + X_\varepsilon A - (L_1 + X_\varepsilon B_1)R_1^{-1}(\varepsilon)(L_1 + X_\varepsilon B_1)^{-1} - \frac{1}{\varepsilon}(L_2 + X_\varepsilon B_2)(L_2 + X_\varepsilon B_2)^T + Q = 0$$

Because of boundedness of the terms, we obtain the following behaviour:

$$(L_2 + X_\varepsilon B_2)^T \sim \frac{1}{\sqrt{\varepsilon}} M(\varepsilon)$$

Then:

$$F \sim \begin{bmatrix} F_1(\varepsilon) \\ \frac{1}{\sqrt{\varepsilon}} M(\varepsilon) \end{bmatrix}$$

for which $\lim_{\varepsilon \to 0} F_1(\varepsilon) = R_1^{-1}(L_1^T + B_1^T X(0))$ is finite, but for $F_2(\varepsilon)$ we cannot say anything about the limit. In terms of Extended Hamiltonian Pencil the condition of existence of the limit for $F(\varepsilon)$ is given in [IoOa94].

Chapter 2

Absolute Stability Criteria

2.1 Statement of the Problem and the Results

Let us consider a SISO linear system described by its transfer function $H(s)$ and a scalar nonlinear feedback $v = \varphi(y)$ as in the figure. Connected with the nonlinear function φ we define the following six sectorial classes:

$$N_{\alpha,\beta} = \{\varphi : \mathbf{R} \to \mathbf{R} | \varphi \text{ piecewise continuous },\ \varphi(0) = 0 \text{ and } \alpha \le \frac{\varphi(y)}{y} \le \beta\}$$

$$SN_{\alpha,\beta} = \{\varphi : \mathbf{R} \to \mathbf{R} | \varphi \text{ continuous },\ \varphi(0) = 0 \text{ and } \alpha < \frac{\varphi(y)}{y} < \beta\}$$

$$CN_{\alpha,\beta} = \{\varphi : \mathbf{R} \to \mathbf{R} | \varphi \text{ continuous },\ \varphi(0) = 0 \text{ and } \alpha \le \frac{\varphi(y)}{y} \le \beta\}$$

$$N_{\alpha,\beta}(t) = \{\varphi : \mathbf{R} \times \mathbf{R} \to \mathbf{R} | \varphi(y,t) \text{ continuous in } t \text{ ,and } \varphi(.,t) \in N_{\alpha,\beta}\}$$

$$SN_{\alpha,\beta}(t) = \{\varphi : \mathbf{R} \times \mathbf{R} \to \mathbf{R} | \varphi(y,t) \text{ continuous in } t \text{ , and } \varphi(.,t) \in SN_{\alpha,\beta}\}$$

$$CN_{\alpha,\beta}(t) = \{\varphi : \mathbf{R} \times \mathbf{R} \to \mathbf{R} | \varphi(y,t) \text{ continuous in } t \text{ , and } \varphi(.,t) \in CN_{\alpha,\beta}\}$$

for $-\infty \le \alpha \le \beta \le \infty$. Our problem is the following:

Given α, β and a classe of the above ones, find the conditions on $H(s)$ which ensures the stability or asymptotic stability of the origin for the closed loop system for any nonlinearity φ belonging to that classe.

To be more precise let us consider a minimal realization of $H(s)$: (A, b, c^T)

$$L \begin{cases} \dot{x} &= Ax + bu \\ y &= c^T x \end{cases}$$

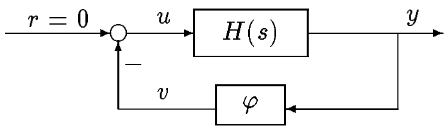

Figure 2.1: The Linear System and Nonlinear Feedback

Then, for α, β and a classe of nonlinearities given above we look for sufficiency conditions such that the equilibrium point $\bar{x} = 0$ of the following dynamics:

$$\dot{x} = Ax - b\varphi(c^T x)$$

to be stable or asymptotically stable.

For a linear system (L) and for some classe S of nonlinearities we say that it is *absolute stable (with respect to the classe S)* if for any $\varphi \in \mathcal{S}$ the origin of the closed loop system is a stable equilibrium. We say that it is *absolute asymptotical stable (with respect to the classe S)* if the origin is an asymptotical stable equilibrium for the closed loop system with any nonlinearity in that classe as feedback and moreover, the attraction domain of the equilibrium is given by the whole space.

We are going now to give results on absolute (asymptotic) stability using two types of Liapunov functions: the first results (Circle Criteria) use quadratic Liapunov functions and allow time-varying nonlinearities whereas the second type results (Popov Criteria) use Lurié type Liapunov functions (i.e. quadratic form plus integral of the nonlinearity) and allow only time-independent nonlinearities.

The Circle Criteria

THEOREM 5 (Circle Theorem of Absolute Stability) *Let us consider a linear system $H(s)$ and the time-varying classe of nonlinearities $N_{\alpha,\beta}(t)$. If the following conditions are fulfilled then $H(s)$ is absolute stable with respect to $N_{\alpha,\beta}(t)$:*

1) There exists $k \in [\alpha, \beta]$ such that the closed loop system with $\varphi(y) = ky$ is asymptotic stable (in sense of linear systems).

2) For any $\omega \in \mathbf{R}$ excepting the pur imaginary poles of $H(s)$ of the form $j\omega$:

$$Re[(1 + \alpha H(-j\omega))(1 + \beta H(j\omega))] \geq 0 \tag{2.1}$$

□

THEOREM 6 (First Circle Criterion of Absolute Stability) *Let us consider a linear system $H(s)$ and the time-varying classe of nonlinearities $SN_{\alpha,\beta}(t)$. If the above conditions are fulfilled then $H(s)$ is absolute asymptotic stable with respect to $SN_{\alpha,\beta}(t)$.*

□

THEOREM 7 (Second Circle Criterion of Absolute Asymptotic Stability) *Let us consider a linear system $H(s)$ and the time-varying classe of nonlinearities $CN_{\alpha,\beta}(t)$. If the condition 1 of Theorem 5 is fulfilled and moreover:*

3) The frequency inequality (2.1) is strict i.e.:

$$Re[(1 + \alpha H(-j\omega))(1 + \beta H(j\omega))] > 0$$

for any $\omega \in \bar{\mathbf{R}}$ excepting those poles of $H(s)$ on the imaginary axis;

4) The pair (A_e, B_e) has no uncontrollable modes on the imaginary axis, where:

$$A_e = \begin{bmatrix} A & 0 \\ \alpha\beta c c^T & -A^T \end{bmatrix} \quad ; \quad B_e = \begin{bmatrix} b \\ -\frac{\alpha+\beta}{2} c \end{bmatrix} \tag{2.2}$$

(for instance, this is true if $H(s)$ has no poles on the imaginary axis);

Then $H(s)$ is absolute asymptotic stable with respect to $CN_{\alpha,\beta}(t)$. □

Popov Criteria: The General Case

THEOREM 8 (General Popov Criterion of Absolute Stability) *Let us consider a linear system $H(s)$ and a time-independent classe of nonlinearities $N_{\alpha,\beta}$. Then, if:*

1) There exists $k \in (\alpha, \beta)$ such that the closed loop system with $u = -ky$ is asymptotic stable (as linear system);

2) There is $\alpha_0 \in \mathbf{R}$ such that:

$$1 + (\alpha + \beta) Re\, H(j\omega) + \alpha\beta |H(j\omega)|^2 + \alpha_0\, Re[j\omega\, H(j\omega)] \geq 0 \tag{2.3}$$

for any $\omega \in \mathbf{R}$ excepting those poles of $H(s)$ on the imaginary axis;

then $H(s)$ is absolute stable with respect to the classe $N_{\alpha,\beta}$. $\square$

THEOREM 9 (First General Popov Criterion of Absolute Asymptotic Stability) *Suppose $H(s)$ be a linear system. Then if the above conditions are fulfilled then $H(s)$ is absolute asymptotic stable with respect to the classe $SN_{\alpha,\beta}$.* $\square$

THEOREM 10 (Second General Popov Criterion of Absolute Asymptotic Stability) *For the linear system $H(s)$ if the condition 1 of the Theorem 8 is fulfilled and moreover:*

3) The inequality (2.3) is strict: i.e. there is $\alpha_0 \in \mathbf{R}$

$$1 + (\alpha + \beta) Re\, H(j\omega) + \alpha\beta |H(j\omega)|^2 + \alpha_0\, Re[j\omega H(j\omega)] > 0 \quad , \quad \forall \omega \in \bar{\mathbf{R}}$$

4) The following pair (A_e, B_e) has no uncontrollable modes on the imaginary axis:

$$A_e = \begin{bmatrix} A & 0 \\ -\alpha\beta c c^T & -A^T \end{bmatrix} \quad ; \quad B_e = \begin{bmatrix} b \\ -\frac{\alpha+\beta}{2} c - \frac{\alpha_0}{2} A^T c \end{bmatrix} \tag{2.4}$$

(for instance this is true if $H(s)$ has no poles on the imaginary axis) where (A, b, c^T) is a minimal realisation of $H(s)$, then $H(s)$ is absolute asymptotic stable with respect to the classe $CN_{\alpha,\beta}$. $\square$

Popov Criteria: The sector $[0, k_0]$ $k_0 > 0$

THEOREM 11 (Popov Criterion of Absolute Stability) *Consider a linear system $H(s)$ and N_{0,k_0} the classe of time-invariant nonlinearities. Then $H(s)$ is absolute stable with respect to the classe N_{0,k_0} if:*

1) There exists $0 < k < k_0$ such that the closed loop system with the feedback $u = -ky$ is asymptotic stable;

2) There exists $\tau \in \mathbf{R}$ such that:

$$\frac{1}{k_0} + Re[(1 + j\tau\omega) H(j\omega)] \geq 0 \quad \forall \omega \in \mathbf{R} \tag{2.5}$$

excepting those poles of $H(s)$ on the imaginary axis. $\square$

THEOREM 12 (First Popov Criterion of Absolute Asymptotic Stability) *Consider a linear system $H(s)$ and N_{0,k_0} a classe of time-invariant nonlinearities. Then if the above conditions are fulfilled, the linear system $H(s)$ is absolute asymptotic stable with respect to the classe SN_{0,k_0}.* $\square$

THEOREM 13 (Second Popov Criterion of Absolute Asymptotic Stability) *Consider a linear system $H(s)$ with no poles on the imaginary axis and N_{0,k_0} a classe of time-invariant nonlinearities. Then if the condition 1 from Theorem 11 is fulfilled and, moreover, we have:*

3) There exists $\tau \in \mathbf{R}$ such that (2.5) is strict:

$$\frac{1}{k_0} + Re[(1 + j\tau\omega)H(j\omega)] > 0 \quad , \quad \forall\omega \in \bar{\mathbf{R}}$$

then $H(s)$ is absolute asymptotic stable with respect to the classe CN_{0,k_0}. □

2.2 Preliminary Results

The idea behind the proofs of the previous statements is to find a Liapunov function that guarantees the stability of the system for any nonlinear feedback in some classe. This Liapunov function will be built up around the antistabilizable solution of a certain KYP system.

In the second step, in order to prove the attractivity of the origin we shall use two ways:

The first one is to consider the same Liapunov function and to apply the Barbashin-Krasovski-LaSalle Theorem on attractivity.

The second way is to look for the stabilizable solution of the same KYP system and to apply a Popov-Datko argument.

I shall develop below these ideas. We need the following definition:

Definition The Popov triplet $\Sigma = (A, B; P)$ has the property of *minimal stability* if for any $x_0 \in \mathbf{R}^n$ there exists $u \in L^{2,m}[0, \infty)$ such that:

i) $\lim_{t\to\infty} x(t) = 0$ where $x(t)$ is the solution of (1.1) with the initial condition $x(0) = x_0$;

ii) $J(\infty) = \int_0^\infty [x^T\ u^T]P[x^T\ u^T]^T dt \le 0$

which is justified by the following result:

LEMMA 14 *Consider a Popov triplet $\Sigma = (A, B; P)$, $R > 0$ and suppose the following two conditions hold:*

i) The associated KYP system has an antistabilizable solution X_a;

ii) Σ has the property of minimal stability;

Then $X_a < 0$

Proof *Because of (i), relation (1.6) becomes:*

$$J(t_1) = -[x^T X_a x]|_{x_0}^{x(t_1)} + \int_0^{t_1} \| W_a x + Vu\|^2 dt$$

Let us choose $u(.)$ to be that control such that $\lim_{t\to\infty} x(t) = 0$. We set $t_1 = \infty$ and the above relation becomes:

$$J(\infty) = x_0^T X_a x_0 + \int_0^\infty \| W_a x + Vu\|^2 dt \le 0$$

(the last inequality comes from (ii), the second property of minimal stability). This proves that $x_0^T X_a x_0 \le 0$ for any $x_0 \in \mathbf{R}^n$ and then $X_a \le 0$. We shall prove the strict inequality by contradiction. For, let $x_0 \ne 0$ be such that $X_a x_0 = 0$. Then, because of $J(\infty) \le 0$ we obtain:

$$W_a x(t) + Vu(t) \equiv 0$$

We have supposed $R > 0$ then V is invertible and $u(t) = -V^{-1}W_a x(t) = F_a x(t)$. On a hand $\Lambda(A+BF_a) \subset \mathcal{C}^+$, on the other hand $\lim_{t\to\infty} x(t) = 0$. This is a contradiction that proves that x_0 must vanish and then $X_a < 0$. □

We recall now the Barbashin-Krasovski-LaSalle Theorem:

THEOREM 15 *Let us suppose a dynamical system $\dot{x} = f(x)$, $f(0) = 0$ and a function $V : \mathbf{R}^n \to \mathbf{R}$ of classe $\mathcal{C}^1$ such that:*

i) $V(x) > 0$, $x \neq 0$, $V(0) = 0$;

ii) $\frac{dV}{dt} = L_f V \leq 0$;

iii) The set $N = \{x \in \mathbf{R}^n \mid L_f V(x) = 0\}$ does not include any positive trajectory;

iv) The function V is radially unbounded:

$$\lim_{R \to \infty} \min_{||x||=R} V(x) = \infty$$

Then the equilibrium point $\bar{x} = 0$ is a global asymptotical stable equilibrium. □

(for proof see, for instance, [Sastr94] or [Balan94]).

The second way to obtain the attractivity of the origin is the Popov-Datko argument. This is given by the following result:

THEOREM 16 *Let us suppose the linear dynamics:*

$$\dot{x} = Ax + Bu \quad , \quad x(0) = x_0$$

If:

i) A is Hurwitz (i.e. $\Lambda(A) \subset \mathcal{C}^-$)

ii) $u \in L^{2,m}[0, \infty)$

Then the trajectory is bounded (by the 2-norm of u), belongs to $L^{2,n}[0, \infty)$ and $\lim_{t \to \infty} x(t) = 0$

Proof *We have the following representation formula of the solution:*

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

Consider $M > 0$ and $a > 0$ such that $\| e^{At} \| \leq M e^{-at}$ (this inequality holds because A is Hurwitz). Then:

$$\| x(t) \| \leq M e^{-at} \| x_0 \| + M \int_0^t e^{-a(t-\tau)} \| B \| \cdot \| u(\tau) \| d\tau$$

Using Cauchy-Buniakowski-Schwartz inequality:

$$\| x(t) \| \leq M e^{-at} \| x_0 \| + M \| B \| \left(\int_0^t e^{-2a(t-\tau)} d\tau \cdot \int_0^t \| u(\tau) \|^2 d\tau \right)^{\frac{1}{2}} \leq$$

$$\leq M e^{-at} \| x_0 \| + M \| B \| \left(\int_0^\infty e^{-2a\tau} d\tau \int_0^\infty \| u(\tau) \|^2 d\tau \right)^{\frac{1}{2}} =$$

$$= M e^{-at} \| x_0 \| + \frac{M}{\sqrt{2a}} \| B \| \cdot \| u \|_2 \leq M \| x_0 \| + \frac{M \| B \|}{\sqrt{2a}} \| u \|_2$$

This proves the boundedness of the state. Now we are going to prove that $x \in L^{2,n}[0, \infty)$. For, it is enough to prove that:

$$f(t) = \int_0^t e^{-a(t-\tau)} \| u(\tau) \| d\tau \in L^2[0, \infty)$$

We have that:

$$f^2(t) = (\int_0^t e^{-\frac{a}{2}(t-\tau)} e^{-\frac{a}{2}(t-\tau)} \parallel u(\tau) \parallel d\tau)^2 \leq$$

$$\leq \int_0^t e^{-a(t-\tau)} d\tau \cdot \int_0^t e^{-a(t-\tau)} \parallel u(\tau) \|^2 d\tau \leq \frac{1}{a} \int_0^t e^{-a(t-\tau)} \parallel u(\tau) \|^2 d\tau$$

And then:

$$\int_0^\infty f^2(t)dt \leq \frac{1}{a} \int_0^\infty dt \int_0^t d\tau e^{-a(t-\tau)} \parallel u(\tau)\|^2 = \frac{1}{a} \int_0^\infty d\tau \parallel u(\tau)\|^2 \int_\tau^\infty dt e^{-a(t-\tau)} =$$

$$= \frac{1}{a} \int_0^\infty \parallel u(\tau)\|^2 \cdot \int_0^\infty e^{-as} ds = \frac{1}{a^2} \parallel u\|_2^2$$

Then $x \in L^{2,n}[0, \infty)$. Now both x and $\dot{x}$ belong to L^2. Then $x \in W^{1,2}$, where $W^{k,p}$ is the Sobolev space of functions which belong (with their first k derivatives) to L^p. It is known that the functions from Sobolev spaces are absolute continuous (see [Barbu74], Appendix 2) and because $\parallel x \parallel$ is bounded, it is also uniform continuous. Now applying the Barbălat's Lemma we obtain that $\lim_{t\to\infty} x(t) = 0$. □

2.3 Proofs of Circle Criteria

We shall prove Theorems 5,6 and 7 using the results stated in the previous section.

The Absolute Stability Result (Theorem 5)

Let us consider the Popov triplet $\Sigma = (A, b, P)$ with:

$$P = \begin{bmatrix} \alpha\beta c c^T & \frac{\alpha+\beta}{2} c \\ \frac{\alpha+\beta}{2} c^T & 1 \end{bmatrix}$$

The Popov function associated to Σ is given by:

$$\Pi_\Sigma(s) = 1 + \frac{\alpha+\beta}{2}(H(-s) + H(s)) + \alpha\beta H(-s)H(s)$$

And, evaluated on the imaginary axis, takes the form:

$$\Pi_\Sigma(s) = Re[(1 + \alpha H(-j\omega))(1 + \beta H(j\omega))]$$

Now, using Popov's Positivity Theorem (Theorem 3) and relation 2.1 we obtain the exestince of the antistabilizable solution of the KYP system associated to Σ. Our system (Σ) has the property of minimal stability if we choose $u = -kc^T x$ (see the form of $J(t)$ given below). Applying Lemma 14 we obtain that $X_a < 0$.

Let us consider the following Liapunov candidate:

$$V(x) = -x^T X_a x$$

Then the equation (1.6) takes the form:

$$J(t) = V(x(t)) - V(x_0) + \int_0^t \parallel W_a x + \mathcal{V} u\|^2 d\tau$$

and:

$$J(t) = \int_0^t [x^T \; u] P \begin{bmatrix} x \\ u \end{bmatrix} d\tau = \int_0^t (u + \alpha y)(u + \beta y) d\tau$$

We derive with respect to t the last two relations and we get:

$$(u+\alpha y)(u+\beta y)=\frac{dV}{dt}+\|\, W_a x+\mathcal{V}u\|^2$$

Or:

$$\frac{dV}{dt}=-\,\|\, W_a x+\mathcal{V}u\|^2+(u+\alpha y)(u+\beta y) \tag{2.6}$$

Now, for any time-varying nonlinearity in the classe $N_{\alpha,\beta}(t)$ we have:

$$(u+\alpha y)(u+\beta y)\leq 0$$

Then, from (2.6) we have that, for any nonlinearity in the considered classe:

$$\frac{dV}{dt}\leq 0$$

This proves that the origin is a stable equilibrium for the closed-loop with the feedback in the classe $N_{\alpha,\beta}(t)$ and ends the proof of Theorem.

The First Absolute Asymptotic Stability Result (Theorem 6)

Our system is absolute stable with respect to $SN_{\alpha,\beta}(t)$ as a consequence of the previous theorem. Moreover, because the classe $SN_{\alpha,\beta}(t)$ is defined by strict inequalities, we obtain:

$$\frac{dV}{dt}<0 \quad , \quad \text{for u, y} \neq 0$$

Since the pair (c^T, A) is observable, the only trajectory included in the set $N(t)=\{\, x\in \mathbf{R}^n \mid \frac{dV}{dt}(x(t),t)=0\,\}$ (which is time invariant) is the trivial solution $\bar{x}=0$. Now $x(t)$ is bounded which implies the boundedness of the control u (recall the sector condition for the feedback) and then the time derivative $\dot{x}$ is also bounded. That means the state $x(t)$ is uniform Lipschitz as function of t. Then $W(t)=\frac{dV}{dt}(x(t),t)$ is uniform continuous with respect to the time t. Using Barbălat'a Lemma we get:

$$\lim_{t\to\infty}\frac{dV}{dt}(x(t),t)=0$$

Then $\lim_{t\to\infty} x(t)=0$.

The Second Absolute Asymptotic Stability Result (Theorem 7)

We shall prove the attractivity of the origin using the Popov-Datko argument. The assumptions 3 and 4 of theorem ensure us that the KYP system associated to Σ (defined above) has got a stabilizable solution $(\mathcal{V}, W_s, X_s=X_s^T)$. Since $R=1>0$ the Lurié system associated to Σ has also got a stabilizable solution $(X_s, F_s=-\mathcal{V}^{-1}W_s)$. We rewrite the dynamics in the form:

$$\dot{x}=(A+bF_s)x+b(u-F_s x)$$

or:

$$\dot{x}=(A+bF_s)x+b\mathcal{V}^{-1}(\mathcal{V}u+W_s x) \tag{2.7}$$

We write the quadratic criterion associated to Σ using the stabilizable solution:

$$J(t)=-[x^T X_s x]|_{x_0}^{x(t)}+\int_0^t \|\, W_s x+\mathcal{V}u\|^2 d\tau\leq 0$$

and then:

$$\int_0^t \| W_s x + \mathcal{V}u\|^2 d\tau \le x^T(t) X_s x(t) - x_0^T X_s x_0$$

Since the closed loop system is stable (from Theorem 5), the trajectory is bounded and then in the above inequality we could make $t \to \infty$ and obatin:

$$\int_0^\infty \| W_s x + \mathcal{V}u\|^2 d\tau < \infty$$

This proves that $W_s x + \mathcal{V}u \in L^2[0,\infty)$. Now we return to (2.7) and we apply Theorem 16: $A + bF_s$ is Hurwitz and $\mathcal{V}u + W_s x \in L^2[0,\infty)$. Then $\lim_{t\to\infty} x(t) = 0$ and this ends the proof.

2.4 Proofs of Popov Criteria

First we see that Theorems 11-13 are just particular cases of Theorems 8-10 when the sector $[\alpha,\beta]$ reduces to $[0,k_0]$ ($\tau = \frac{\alpha_0}{k_0}$). Then we shall prove only Theorems 8-10. Furthermore, when $\alpha_0 = 0$ Popov's Theorems reduce to Circle Theorems that we have already proved. Then, from now on, we assume that $\alpha_0 \neq 0$.

The Absolute Stability Result (Theorem 8)

The idea if this proof is very close to that of Theorem 5, but here we use a Liapunov function of Lurié type (i.e. quadratic form plus integral of the nonlinearity).

Firstly we consider the following Popov triplets $\Sigma_1 = (A, b, P_1)$ and $\Sigma_2 = (A, b, P_2)$ where:

$$P_1 = \begin{bmatrix} \alpha\beta cc^T & \frac{\alpha+\beta}{2}c + \frac{\alpha_0}{2}A^T c \\ \frac{\alpha+\beta}{2}c^T + \frac{\alpha_0}{2}c^T A & 1 + \alpha_0 c^T b \end{bmatrix}$$

$$P_2 = \begin{bmatrix} \alpha\beta cc^T + \frac{\alpha_2\alpha-\alpha_3\beta}{2}(cc^T A + A^T cc^T) & \frac{\alpha+\beta}{2}c + \frac{\alpha_0}{2}A^T c + \frac{\alpha_2\alpha-\alpha_3\beta}{2}cc^T b \\ \frac{\alpha+\beta}{2}c^T + \frac{\alpha_0}{2}c^T A + \frac{\alpha_2\alpha-\alpha_3\beta}{2}b^T cc^T & 1 + \alpha_0 c^T b \end{bmatrix}$$

and $\alpha_2 - \alpha_3 = \alpha_0$, $\alpha_2\alpha_3 = 0$, $\alpha_2, \alpha_3 \ge 0$.

We see that P_1 and P_2 have the following form:

$$P_1 = \begin{bmatrix} Q & L \\ L^T & R \end{bmatrix} \qquad P_2 = \begin{bmatrix} Q + A^T X + XA & L + Xb \\ L^T + b^T X & R \end{bmatrix}$$

where:

$$X = -\frac{\alpha_2\alpha - \alpha_3\beta}{2}cc^T \tag{2.8}$$

This proves that Σ_1, Σ_2 have the same Popov function (see Proposition 3, §4, pp.53 from [Popov73]). For Σ_1 the Popov function takes the form:

$$\Pi_{\Sigma_1}(s) = 1 + \frac{\alpha+\beta}{2}(H(s) + H(-s)) + \alpha\beta H(-s)H(s) + \frac{\alpha_0}{2}s(H(s) - H(-s))$$

and, when we evaluate it on the imaginary axis:

$$\Pi_{\Sigma_1}(j\omega) = 1 + (\alpha+\beta) Re\, H(j\omega) + \alpha\beta|H(j\omega)|^2 + \alpha_0 Re[j\omega\, H(j\omega)]$$

And then the frequency condition (2.3) says that $\Pi_{\Sigma_2} \ge 0$.

On the other hand, after a little algebra manipulation one could prove that the quadratic criterion associated to Σ_2 takes the form:

$$J_2(t) = \int_0^t (u+\alpha y)(u+\beta y)d\tau + \alpha_2 \int_0^t (u+\alpha y)\frac{dy}{d\tau}d\tau - \alpha_3 \int_0^t (u+\beta y)\frac{dy}{d\tau}d\tau \tag{2.9}$$

(recall $\frac{dy}{d\tau} = c^T(Ax+bu)$).

Now let us choose $\varphi \in N_{\alpha,\beta}$ and consider $u = -\varphi(y)$. Then the following inequalities hold:

$$\int_0^t (u+\alpha y)(u+\beta y)d\tau \leq 0$$

$$\Psi_1(y_1) = \int_0^{y_1} (u+\alpha y)dy \leq 0$$

$$\Psi_2(y_1) = \int_0^{y_1} (u+\beta y)dy \geq 0$$

Then the quadratic criterion could be rewritten as:

$$J_2(t) = \int_0^t (u+\alpha y)(u+\beta y)d\tau + \alpha_2\Psi_1(y(t)) - \alpha_2\Psi_1(y_0) - \alpha_3\Psi_2(y(t)) + \alpha_3\Psi_2(y_0)$$

And, with $\alpha_2, \alpha_3 \geq 0$ we have the following boundednes:

$$J_2(t) \leq -\alpha_2\Psi_1(y_0) + \alpha_3\Psi_2(y_0) \tag{2.10}$$

We are going now to prove that Σ_2 has the property of minimal stability (for the original proof see 4. §25 from [Popov73]). Let us consider the case $\alpha_0 > 0$. Then $\alpha_2 = \alpha_0$ and $\alpha_3 = 0$. Consider $u = -kc^Tx + v$. Then it is enough to find $v \in L^2[0,\infty)$ such that $J_2(\infty) \leq 0$ because the dynamics is given by:

$$\dot{x} = (A - kbc^T)x + bv$$

$A - kbc^T$ is Hurwitz and using Popov-Datko argument $\lim_{t\to\infty} x(t) = 0$.

In the new variable v, the criterion becomes:

$$J_2(t) = \int_0^t ((\alpha-k)y+v)((\beta-k)y+v+\alpha_0\frac{dy}{d\tau})d\tau$$

We introduce a new variable:

$$\tilde{y} = y + \frac{1}{\alpha-k}v$$

and the criterion takes the form:

$$J_2(t) = (\alpha-k)(\beta-k)\int_0^t \tilde{y}^2 d\tau + \frac{\alpha_0(\alpha-k)}{2}\tilde{y}^2(\tau)|_0^t - \int_0^t \tilde{y}(\alpha_0\frac{dv}{dt} + (\beta-\alpha)v)d\tau$$

If we choose v to be the solution of the differential equation:

$$\alpha_0\frac{dv}{dt} + (\beta-\alpha)v = 0$$

with initial condition $v(0)$ such that $\tilde{y}(0) = 0$, then:

$$v(t) = -(\alpha-k)c^Tx(0)exp(-\frac{\beta-\alpha}{\alpha_0}t) \in L^2[0,\infty)$$

and $J_2(\infty) \leq 0$.

Now the proof goes very similar with that of Theorem 5. From Lemma 14 we obtain that the antistabilizable solution of Σ_2 $X_a^{(2)} < 0$. We factorize the criterion as:

$$\int_0^t (u+\alpha y)(u+\beta y)d\tau + \alpha_2\Psi_1(y(t)) - \alpha_2\Psi_1(y_0) - \alpha_3\Psi_2(y(t)) + \alpha_3\Psi_2(y_0) =$$

$$= -[x^T X_a^{(2)} x]|_{x_0}^{x(t)} + \int_0^t \| W_a x + \mathcal{V}u\|^2 d\tau$$

We define the Liapunov candidate as:

$$V(x) = -x^T X_a^{(2)} x - \alpha_2\Psi_1(c^T x) + \alpha_3\Psi_2(c^T x)$$

Since $\Psi_1 \leq 0$, $\Psi_2 \geq 0$ and $X_a^{(2)} < 0$ we have $V(x) > 0$, $x \neq 0$, $V(0) = 0$. Moreover, the above factorization of the criterion enables us to write:

$$V(x) = V(x_0) + \int_0^t (u+\alpha y)(u+\beta y)d\tau - \int_0^t \| W_a x + \mathcal{V}u\|^2 d\tau$$

and for the derivative:

$$\frac{dV}{dt} = (u+\alpha y)(u+\beta y) - \| W_a x + \mathcal{V}u\|^2 \tag{2.11}$$

which is identical with (2.6). With the same argument we obtain the absolute stability of $H(s)$ with respect to the classe $N_{\alpha,\beta}$.

N.B. The factorization (2.11) holds only for $u = -\varphi(y)$.

The First Absolute Asymptotic Result (Theorem 9)

We use V defined above as a Liapunov function of LaSalle type. Note that in the set $N = \{x \mid \frac{dV}{dt} = 0\}$ is included only the trivial solution $\bar{x} = 0$. Now, applying Theorem 15 (BKLS Theorem) we obtain the absolute asymptotic stability with respect to the classe $SN_{\alpha,\beta}$.

The Second Absolute Asymptotic Result (Theorem 10)

As in the proof of Theorem 5 we apply the Popov-Datko argument. For it is enough to see that the hypothesys of Theorem 10 ensures the existence of the stabilizable solution of the associated KYP system to Σ_1. The connection between J_1 and J_2, the two criteria associated to Σ_1 and Σ_2, is given by:

$$J_2(t) - J_1(t) = x^T X x|_{x(0)}^{x(t)}$$

where X is given by (2.8). Then factorizing J_1 with the help of stabilizable solution of Σ_1, $X_s^{(1)}$:

$$J_1(t) = -[x^T X_s^{(1)} x]|_{x(0)}^{x(t)} + \int_0^t \| W_s x + \mathcal{V}u\|^2 d\tau$$

we obtain for J_2 the expansion:

$$J_2(t) = x^T(X - X_s^{(1)})x|_{x(0)}^{x(t)} + \int_0^t \| W_s x + \mathcal{V}u\|^2 d\tau$$

Using the boundedness given by (2.10) we obtain for $\varphi \in CN_{\alpha,\beta}$ and $u = -\varphi(y)$ that:

$$\int_0^\infty \| W_s x + \mathcal{V}u\|^2 d\tau < \infty$$

Now the proof goes straightforward.

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