Density and Redundancy of the Noncoherent Weyl-Heisenberg Superframes

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Abstract

In this paper I shall present the construction of Weyl-Heisenberg superframes and density results related to the noncoherent case. A superframe is a collection of r-frames $\mathcal{F}^1 = \{f_i^1, i \in \mathbf{I}\} \subset H_1, \ldots, \mathcal{F}^r = \{f_i^r, i \in \mathbf{I}\} \subset H_r$ all having the same countable index set $\mathbf{I}$ such that $\mathcal{F} = \{f_i^1 \oplus \cdots \oplus f_i^r, i \in \mathbf{I}\}$ is a frame for the Hilbert space $H = H_1 \oplus \cdots \oplus H_r$. For the Weyl-Heisenberg superframes we set $H_1 = \cdots = H_r = L^2(\mathbf{R})$, $f_i^l = g^l_{z,a,b}(x) := z_l e^{2\pi i a_l x} g^l(x - b_l)$ and $(z, a, b) \in \mathbf{I} := \Lambda \subset T^r \times \mathbf{R}^{2r}$. We study the density of superframes in the case Λ is a subset of the $r+2$ subgroup $T^r \times E_{\alpha,\beta}$. Our approach is inspired by a recent work of O.Christensen, B.Deng and C.Heil. In the special case of coherent WH superframes, we prove that its redundancy is given by $1/\alpha \cdot \beta$ (where the lattice is $\Lambda = \{(m\alpha, n\beta); m, n \in \mathbf{Z}\}$).

1 Superframes

We start by recalling the standard frame theory. Let H be a (separable, complex) Hilbert space and $\mathbf{I}$ a countable index set.

DEFINITION 1 *A set of vectors $\mathcal{F} = \{f_i, i \in \mathbf{I}\} \subset H$ is called a* frame for H *if there are two positive constants $0 < A \leq B < \infty$ such that:*

$$A\|x\|^2 \leq \sum_{i \in \mathbf{I}} | < x, f_i > |^2 \leq B\|x\|^2$$

for every $x \in H$. The constants A, B are called frame bounds *and if we can choose $A = B$, the frame is called* tight.

To a frame $\mathcal{F}$ we associate the following objects:

the *analysis operator*, $T : H \to l^2(\mathbf{I})$, $T(x) = \{< x, f_i >\}_{i \in \mathbf{I}}$

the *synthesis operator*, $T^* : l^2(\mathbf{I}) \to H$, $T^*(c) = \sum_{i\in\mathbf{I}} c_i f_i$

the *coefficient range*, $E = Ran\, T$ (it is a closed subspace of $l^2(\mathbf{I})$);

the *frame operator*, $S : H \to H$, $S = T^*T$, $S(x) = \sum_{i\in\mathbf{I}} < x, f_i > f_i$ (it is selfadjoint and $A \cdot \mathbf{1} \leq S \leq B \cdot \mathbf{1}$);

the *standard dual frame*, $\tilde{\mathcal{F}} = \{\tilde{f}_i; i \in \mathbf{I}\}$, $\tilde{f}_i = S^{-1} f_i$; it is a frame with bounds $\frac{1}{B}, \frac{1}{A}$ having the same coefficient range as $\mathcal{F}$such that the following *reconstruction formula* holds true:

$$x = \sum_{i\in\mathbf{I}} < x, f_i > \tilde{f}_i = \sum_{i\in\mathbf{I}} < x, \tilde{f}_i > f_i$$

DEFINITION 2 *A frame $\mathcal{F}^d = \{f_i^d, i \in \mathbf{I}\}$ in H is called an alternate* dual *of $\mathcal{F}$ if the reconstruction formula holds true for $(\mathcal{F}, \mathcal{F}^d)$.*

the *associated tight frame*, $\mathcal{F}^{\#} = \{f_i^{\#}, i \in \mathbf{I}\}$, $f_i^{\#} = S^{-1/2} f_i$; it is a tight frame with bound 1, having the same coefficient range as $\mathcal{F}$.

Suppose now we have a collection of Hilbert frames $(\mathcal{F}^1, \ldots, \mathcal{F}^r)$, in Hilbert spaces H^l, $\mathcal{F}^l \subset H^l$, and all having the same index set $\mathbf{I}$.

To this collection $(\mathcal{F}^1, \ldots, \mathcal{F}^r)$ we associate the following set:

$$\mathcal{F} = \{f_i^1 \oplus \cdots \oplus f_i^r, i \in \mathbf{I}\} =: \mathcal{F}^1 \oplus \cdots \oplus \mathcal{F}^r$$

'sitting' in $H = H^1 \oplus \cdots \oplus H^r$. We also consider the collection of closed subspaces $(E^1, \ldots, E^r)$ in $l^2(\mathbf{R})$ of the coefficient ranges.

DEFINITION 3 *The collection $(\mathcal{F}^1, \ldots, \mathcal{F}^r)$ is a* superframe *if $\mathcal{F}$ is a frame for H.*

An equivalent condition is given in the following theorem. I thank Deguang Han for pointing out to me an error in a previous statement of this result (in fact a similar object has been considered independently in [HaLa97] as well):

THEOREM 4 *The collection $(\mathcal{F}^1, \ldots, \mathcal{F}^r)$ is a superframe iff $E^i \cap (\oplus_{j\neq i} E^j) = \{0\}$, $\forall i$, and $E^1 \oplus \cdots \oplus E^r$ is closed in $l^2(\mathbf{I})$.*

DEFINITION 5 *Two frames $\mathcal{F}^1$ and $\mathcal{F}^2$ are called* orthogonal *if their coefficient ranges are orthogonal subspaces, i.e. $E^1 \perp E^2$ in $l^2(\mathbf{I})$ (we already assumed $\mathcal{F}^1$ and $\mathcal{F}^2$ have the same index set $\mathbf{I}$).*

Suppose the superframe $(\mathcal{F}^1, \ldots, \mathcal{F}^r)$ is given. The $\mathcal{F} = \mathcal{F}^1 \oplus \cdots \oplus \mathcal{F}^r$ is a frame in H. Consider its standard dual $\tilde{\mathcal{F}}$ in H. Then $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}^1 \oplus \cdots \oplus \tilde{\mathcal{F}}^r$ for some frames $\tilde{\mathcal{F}}^1, \ldots, \tilde{\mathcal{F}}^r$ and $(\tilde{\mathcal{F}}^1, \ldots, \tilde{\mathcal{F}}^r)$ is a superframe as well.

DEFINITION 6 *The superframe $(\tilde{\mathcal{F}}^1, \ldots, \tilde{\mathcal{F}}^r)$ is called the* standard dual superframe *of $(\mathcal{F}^1, \ldots, \mathcal{F}^r)$.*

THEOREM 7 *$\tilde{\mathcal{F}}^l$ is an alternate dual of $\mathcal{F}^l$ (not necessarily the standard dual) and $\tilde{\mathcal{F}}^l$ is orthogonal to $\mathcal{F}^j$, for $l \neq j$.*

Frames are overcomplete sets. The dual notion (with respect to the overcompleteness) is the Riesz basis for its span:

DEFINITION 8 *A set of vectors $\mathcal{F} = \{f_i, i \in \mathbf{I}\} \subset H$ is called a Riesz basis for its span (or a* s-Riesz basis*) if there are two positive constants $0 < A \leq B < \infty$ such that:*

$$A \sum_{i \in \mathbf{I}} |c_i|^2 \leq \| \sum_{i \in \mathbf{I}} c_i f_i \|^2 \leq B \sum_{i \in \mathbf{I}} |c_i|^2$$

for every finite sequence $(c_i)_i \in l^2(\mathbf{I})$. The constants A, B are called s-Riesz basis bounds

Notice if we can choose $A = B$, the s-Riesz basis is an equinorm, orthogonal set.

Suppose $\mathcal{F} = \{f_i, i \in \mathbf{I}\} \subset H$ is a s-Riesz basis in H. We call $\mathcal{F}' = \{f'_i, i \in \mathbf{I}\} \subset H$ a *biorthogonal* s-Riesz basis to $\mathcal{F}$ if $< f_i, f'_j >= \delta_{ij}$. If in addition the span of $\mathcal{F}'$ coincides with the span of $\mathcal{F}$, then $\mathcal{F}'$ is called the *standard biortogonal s-Riesz basis* of $\mathcal{F}$.

The biorthogonality (as well the frame duality) is a symmetric relation. Suppose $(\mathcal{F}, \mathcal{F}')$ are biorthogonal to one another. Then the following reconstruction formula of the coefficients holds true:

$$< \sum_{i \in \mathbf{I}} c_i f_i, f'_j >=< \sum_{i \in \mathbf{I}} c_i f'_i, f_j >= c_j$$

Similarly to superframes, a superset $(\mathcal{F}^1, \ldots, \mathcal{F}^r)$ is called a *super s-Riesz basis* if $\mathcal{F} = \mathcal{F}^1 \oplus \cdots \oplus \mathcal{F}^r$ is a s-Riesz basis for $H = H_1 \oplus \cdots \oplus H_r$.

2 Weyl-Heisenberg Superframes

Consider $G^r = T^r \times \mathbf{R}^{2r}$ the direct product of r 1-dimensional Weyl-Heisenberg groups $G^1 = T^1 \times \mathbf{R}^2$ with T^1 the 1-dimensional torus identified with the unit complex circle. Let us denote by $L^{2,r} = L^2(\mathbf{R}) \oplus \cdots \oplus L^2(\mathbf{R})$ the direct sum of r copies of $L^2(\mathbf{R})$ endowed with the scalar product $< f_1 \oplus \cdots \oplus f_r, g_1 \oplus \cdots \oplus f_r >= \sum_{i=1}^r < f_i, g_i >$. Then consider the r-direct sum of r Schrödinger representations of 1-dimensional WH groups G^1:

$$\mathcal{U}(z, p, q)\mathbf{f}(x) = \oplus_{l=1}^r u(z_l, p_l, q_l) f_l(x) \quad , \mathbf{f} \in L^{2,r}$$

where $u(z_l, p_l, q_l) f_l(x) = z_l e^{-i\pi p_l q_l} e^{2\pi i p_l x} f_l(x - q_l)$.

A *Weyl-Heisenberg set* $\mathcal{WH}_{g,\Lambda}$ is obtained by discretizing the (continuous) orbit of some generator $\mathbf{g}$ with respect to a discrete set of parameters $\Lambda \subset G^r$:

$$\mathcal{WH}_{g,\Lambda} = \{\mathcal{U}(z, p, q)\mathbf{g} \ ; \ (z, p, q) \in \Lambda \subset G^r\}$$

We index Λ by a countable index set $\mathbf{I}$, $\Lambda = \{(z^i, p^i, q^i), i \in \mathbf{I}\}$. For $r = 1$ we obtain the (standard) non-coherent Weyl-Heisenberg sets:

$$\mathcal{WH}_{g;\Lambda} = \{u(z,p,q)g \ ; \ (z,p,q) \in \Lambda \subset G^1\}$$

The coherent set is obtained by choosing $\Lambda = \{(1, m\alpha, n\beta); m, n \in \mathbf{Z}\}$ for some particular $\alpha, \beta > 0$.

A collection of WH sets all indexed by the same index set $\mathbf{I}$ (called a *Weyl-Heisenberg superset*) $(\mathcal{WH}_{g^1,\Lambda_1}, \dots, \mathcal{WH}_{g^r,\Lambda_r})$ is equivalent to the WH set $\mathcal{WH}_{g,\Lambda}$ in $L^{2,r}$ given by $\mathbf{g} = g^1 \oplus \cdots \oplus g^r \in L^{2,r}$ and

$$\Lambda = \{(z_1^i, \dots, z_r^i, p_1^i, \dots, p_r^i, q_1^i, \dots, q_r^i) \ , \ i \in \mathbf{I} \ , \ (z_l^i, p_l^i, q_l^i) \in \Lambda_l\} \subset G^r$$

Thus $(\mathcal{WH}_{g^1;\Lambda_1}, \dots, \mathcal{WH}_{g^r;\Lambda_r})$ is a WH superframe (respectively a WH super s-Riesz basis) iff $\mathcal{WH}_{g,\Lambda}$ is a frame for $L^{2,r}$ (respectively a WH s-Riesz basis in $L^{2,r}$). From now on we shall concentrate on WH sets of the form $\mathcal{WH}_{g,\Lambda}$ for some $\mathbf{g} \in L^{2,r}$ and $\Lambda \subset G^r$.

For $\alpha, \beta \in (\mathbf{R}_+^*)^r$ we denote by $E_{\alpha,\beta} = \{(t\alpha, s\beta) \ , \ t, s \in \mathbf{R}\} \subset \mathbf{R}^{2r}$ a 2-dimensional linear subspace of $\mathbf{R}^{2r}$. Let us denote by $K_{\alpha,\beta}^r = T^r \times E_{\alpha,\beta}$ the $r+2$-dimensional subgroup of G^r containing $E_{\alpha,\beta}$. Recall that a unitary representation $\mathcal{U} : G \to U(H)$ of a locally compact group G on a Hilbert space H is called *square integrable* if i) there is a cyclic vector (i.e. the linear span of its orbit is dense in H) and ii) there is a $f \in H$ such that $\int_G d\mu(\lambda)| < f, \mathcal{U}(\lambda)f >|^2 < \infty$, for the left invariant measure $d\mu$ on G. Note that although $\mathcal{U} : G^r \to U(L^{2,r})$ is not square integrable, $\mathcal{U} : K_{\alpha,\beta}^r \to U(L^{2,r})$ is square integrable. This suggests to restrict our attention on $\Lambda \subset K_{\alpha,\beta}^r$, which is what we do.

Notation. For a $\lambda \in \Lambda$ we write $\lambda \in E_{\alpha,\beta}$ if $\lambda \in K_{\alpha,\beta}^r$. We call $E_{\alpha,\beta}$ a *leaf*. We say a set Λ or a WH set $\mathcal{WH}_{g,\Lambda}$ is *supported on a leaf* $E_{\alpha,\beta}$ if $\Lambda \subset K_{\alpha,\beta}^r$.

Our analysis will be done only on leaves of the phase space.

3 Densities and Main Results

Suppose $\alpha, \beta \in (\mathbf{R}_+^*)^r$ and $\Lambda \subset K_{\alpha,\beta}^r$ are given. For $h > 0$ and $(p,q) \in \mathbf{R}^{2r}$ we denote

$$Q_h(p,q) \in \{(z,a,b) \in G^r \mid |a_i - p_i| < \frac{h}{2}, |b_i - q_i| < \frac{h}{2}, i = 1, \dots, r\}$$

the cube of size length h. For a discrete set M we denote by $\#M$ the number of points it contains. Let:

$$\nu^+(h) = \sup_{(p,q)\in E_{\alpha,\beta}} \#(Q_h(p,q) \cap \Lambda) \quad , \quad \nu^-(h) = \inf_{(p,q)\in E_{\alpha,\beta}} \#(Q_h(p,q) \cap \Lambda)$$

Following [ChDeHe97], the upper and lower densities of Λ are defined by:

$$D^+(\Lambda) = \limsup_{h\to\infty} \frac{\nu^+(h)}{\mu(Q_h(0,0) \cap E_{\alpha,\beta})}$$

$$D^-(\Lambda) = \liminf_{h\to\infty} \frac{\nu^-(h)}{\mu(Q_h(0,0)\cap E_{\alpha,\beta})}$$

where $\mu(Set) = Aria(Set) = \frac{\mu_{Haar}(T^r\times Set)}{(2\pi)^r}$, for $Set \subset E_{\alpha,\beta}$, is the 2 dimensional Lebesgue measure of Set, or the normalized Haar measure of $T^r \times Set$.

If $D^+(\Lambda) = D^-(\Lambda)$ then Λ is said to have *uniform density* $D(\Lambda) = D^+(\Lambda) = D^-(\Lambda)$.

If Λ is the regular lattice $\{(m\alpha, n\beta); m, n \in \mathbf{Z}\}$ then $D(\Lambda) = \frac{1}{|\alpha|\cdot|\beta|}$, with $|\alpha| = \sqrt{\sum_{l=1}^r \alpha_l^2}, |\beta| = \sqrt{\sum_{l=1}^r \beta_l^2}$.

Λ is said to be δ-*uniformly separated* if for any $(z,p,q)\in\Lambda$, $\#(Q_{2\delta}(p,q)\cap\Lambda) \le 1$.

Λ is said to be *relatively uniformly separated* if $\Lambda = \cup_{k=1}^{s_0}\Lambda_k$ for some $s_0 > 0$ and each Λ_k is δ_k-uniformly separated for some δ_k.

The following results extend similar results obtained in [ChDeHe97].

LEMMA 9 *Λ is relatively uniformly separated iff $D^+(\Lambda) < \infty$, iff $\nu^+(h) < \infty$, for some $h > 0$.*

The proof is presented in the next section.

For the next result, recall that $\mathcal{WH}_{g,\Lambda}$ is called a *WH Bessel set* if there is a $B > 0$ such that $\sum_{\lambda\in\Lambda} | < \mathbf{f}, U(\lambda)\mathbf{g} > |^2 \le B\|\mathbf{f}\|^2$ for every $\mathbf{f}\in L^{2,r}$.

THEOREM 10 *If $\mathcal{WH}_{g,\Lambda}$ is a WH Bessel set then $D^+(\Lambda) < \infty$, and therefore Λ is relatively uniformly separated.*

The proof is defered until the next section.

THEOREM 11 (Comparison Theorem) *Suppose $\mathcal{WH}_{g,\Lambda}$ is a frame for $L^{2,r}$ and $\mathcal{WH}_{\varphi,\Delta}$ is a Riesz basis for its span in $L^{2,r}$ with $\Lambda,\Delta\subset K^r_{\alpha,\beta}$. Then $D^+(\Lambda)\ge D^+(\Delta)$ and $D^-(\Lambda)\ge D^-(\Delta)$.*

The proof in given in the next section.

COROLLARY 12 *Suppose $\mathcal{WH}_{g,\Lambda}$ is a Riesz basis for $L^{2,r}$ supported in the leaf $E_{\alpha,\beta}$. Then Λ has uniform density $D^+(\Lambda) = D^-(\Lambda) = D(\Lambda) = \frac{\alpha\beta}{|\alpha|\cdot|\beta|} =: D_0(\alpha,\beta)$, where $|\alpha| = \sqrt{\sum_{i=1}^r \alpha_i^2}$, $|\beta| = \sqrt{\sum_{i=1}^r \beta_i^2}$.*

Proof of Corollary

The proof is based on the Comparison Theorem. Clearly any WH Riesz basis for $L^{2,r}$ supported in the leaf $E_{\alpha,\beta}$ would have the same uniform density $D_0(E_{\alpha,\beta})$. Therefore we have only to construct an example of such WH Riesz basis and to compute its density. This is done in the following:

EXAMPLE 13 *Consider $\varphi = \varphi^1\oplus\cdots\oplus\varphi^r$ with*

$$\varphi^l = \sqrt{\frac{\alpha_l}{\alpha\cdot\beta}}\mathbf{1}_{[a_l,b_l]} \quad \text{where} \quad a_l = \frac{1}{\alpha_l}\sum_{k=1}^{l-1}\alpha_k\beta_k \quad \beta_l = \frac{1}{\alpha_l}\sum_{k=1}^{l}\alpha_k\beta_k$$

and

$$\Delta = \{(e^{-i\frac{mn}{\alpha\cdot\beta}\alpha\otimes\beta}, m\frac{\alpha}{\alpha\cdot\beta}, n\beta)\ ,\ (m,n)\in\mathbf{Z}^2\}$$

The claim is that the WH set $\mathcal{WH}_{\varphi,\Delta}$ *is an orthonormal basis for* $L^{2,r}$.

Notice $a_1 = 0, a_2 = \frac{\alpha_1\beta_1}{\alpha_2}, a_3 = \frac{\alpha_1\beta_1+\alpha_2\beta_2}{\alpha_3}, \ldots, \alpha_r = \frac{\alpha\cdot\beta-\alpha_r\beta_r}{\alpha_r}$, $b_1 = \beta_1, b_2 = \frac{\alpha_1\beta_1+\alpha_2\beta_2}{\alpha_2}, \ldots, a_r = \frac{\alpha\cdot\beta}{\alpha_r}$. *Thus:*

$$\varphi_{m,n}(x) = \sqrt{\frac{\alpha_1}{\alpha\cdot\beta}}e^{2\pi im\frac{\alpha_1}{\alpha\cdot\beta}(x-n\beta_1)}\mathbf{1}_{[a_1,b_1]}(x-n\beta_1)\oplus\cdots\oplus\sqrt{\frac{\alpha_r}{\alpha\cdot\beta}}e^{2\pi im\frac{\alpha_r}{\alpha\cdot\beta}(x-n\beta_r)}\mathbf{1}_{[a_r,b_r]}(x-n\beta_r)$$

$$<\varphi_{m,n},\varphi_{m',n'}> = \delta_{n,n'}\frac{1}{\alpha\cdot\beta}\sum_{l=1}^r\alpha_l\int_{a_l}^{b_l}e^{2\pi i\frac{\alpha_l}{\alpha\cdot\beta}(m-m')x}dx$$

$$= \delta_{n,n'}\frac{1}{\alpha\cdot\beta}\int_0^{\alpha\cdot\beta}e^{2\pi i\frac{m-m'}{\alpha\cdot\beta}x}dx = \delta_{m,m'}\delta_{n,n'}$$

This shows the system is orthonormal. It remains only to prove that $\mathcal{WH}_{\varphi,\Delta}$ *is complete in* $L^{2,r}$.

Consider $\mathbf{f}\in L^{2,r}$ *such that* $<\mathbf{f},\varphi_{m,n}> = 0$ *for every* $m,n\in\mathbf{Z}$, *i.e.* $\sum_{l=1}^r < f^l,\varphi^l_{mn}> = 0$. *More specific this means:*

$$0 = \sum_{l=1}^r < f^l,\varphi^l_{mn}> = \sum_{l=1}^r\sqrt{\frac{\alpha_l}{\alpha\cdot\beta}}\int_{a_l}^{b_l}e^{2\pi im\frac{\alpha_l}{\alpha\cdot\beta}x}f^l(x+n\beta_l)dx\quad,\quad\forall m,n$$

Set $n=0$. *We shall prove that* $f^l|_{[a_l,b_l]}\equiv 0$, *for all* l. *Similarly one can obtain* $f^l|_{[a_l+n\beta_l,b_l+n\beta_l]}\equiv 0$ *for every* n *and since* $b_l - a_l = \beta_l$ *we would obtain* $f^l\equiv 0$ *which means* $\mathbf{f}\equiv 0$, *or the WH set is complete and thus an orthonormal basis.*

For $n=0$ *we change the variable* $y = \frac{\alpha_l}{\alpha\cdot\beta}x$ *and let* $H^l(y) = \sqrt{\frac{\alpha\cdot\beta}{\alpha_l}}f^l(\frac{\alpha\cdot\beta}{\alpha_l}y)$ *and* $c_l = \frac{1}{\alpha\cdot\beta}\sum_{k=1}^{l-1}\alpha_k\beta_k$. *Then:*

$$0 = \sum_{l=1}^r\int_{c_l}^{c_{l+1}}e^{2\pi imx}h^l(x)dx = \int_0^1 e^{2\pi imx}(\sum_{l=1}^r\mathbf{1}_{[c_l,c_{l+1}]}(x)h^l(x))dx\quad,\quad\forall m$$

Thus $\sum_{l=1}^r\mathbf{1}_{[c_l,c_{l+1}]}h^l\equiv 0$. *Note now* $0 = c_1 < c_2 < \cdots < c_r < c_{r+1} = 1$ *which makes the intervals* $[c_l,c_{l+1}]$ *nonoverlaping. Therefore* $h^l|_{[c_l,c_{l+1}]}\equiv 0$, *or* $f^l|_{[a_l,b_l]}\equiv 0$. $\diamond$

For this example, the density is $D(\Lambda) = \frac{1}{Cell\_aria}$ where

$$Cell\_aria = |\frac{\alpha}{\alpha\cdot\beta}|\cdot|\beta| = \frac{|\alpha|\cdot|\beta|}{\alpha\cdot\beta}.$$

Hence

$$D(\Lambda) = \frac{\alpha\cdot\beta}{|\alpha|\cdot|\beta|} = D_0(\alpha,\beta)$$

and this concludes the proof of the Corollary 12. $\square$

DEFINITION 14 *Suppose $\mathcal{WH}_{g,\Lambda}$ is a frame for $L^{2,r}$ and Λ is supported in the leaf $E_{\alpha,\beta}$ and has uniform density $D(\Lambda)$. Then by* redundancy *we mean the following number:*

$$r(\Lambda) = \frac{D(\Lambda)}{D_0(\alpha,\beta)}$$

The Comparison Theorem proves that if $(\mathcal{WH}_{g^1;\Lambda_1},\dots,\mathcal{WH}_{g^r;\Lambda_r})$ is a superframe then $r(\Lambda)\geq 1$, whereas if $(\mathcal{WH}_{g^1;\Lambda_1},\dots,\mathcal{WH}_{g^r;\Lambda_r})$ is a super s-Riesz basis then $r(\Lambda)\leq 1$. Note that $r(\Lambda)=1$ does not imply $\mathcal{WH}_{g,\Lambda}$ is a Riesz basis or a frame for $L^{2,r}$ (just add or leave out a finite number of vectors in any Riesz basis). Note also that any of the strict inequalities would not imply the set to be a frame or s-Riesz basis in $L^{2,r}$.

For a *coherent WH frame* (i.e. for one for which $\Lambda=\{(m\alpha,n\beta);m,n\in\mathbf{Z}\}$ for some $\alpha,\beta\in\mathbf{R}_+^r$),

$$r(\Lambda) = 1/\alpha\cdot\beta =: r_0(\alpha,\beta)$$

Note that always $r(\Lambda)\geq D(\Lambda)$.

Suppose $\mathcal{WH}_{g,\Lambda}$ and $\mathcal{WH}_{\mathbf{h},\Delta}$ are frames in $L^{2,r}$ supported on the same leaf $E_{\alpha,\beta}$ and having uniform densities $D(\Lambda)$, $D(\Delta)$. Then by *redundancy of Λ relative to Δ* we mean the ratio:

$$r(\Lambda,\Delta) = \frac{r(\Lambda)}{r(\Delta)} = \frac{D(\Lambda)/D_0(\alpha,\beta)}{D(\Delta)/D_0(\mu,\nu)}$$

If Λ and Δ are regular,

$$r(\Lambda,\Delta) = \frac{\mu\cdot\nu}{\alpha\cdot\beta} = \frac{r_0(\alpha,\beta)}{r_0(\mu,\nu)}$$

The definition of redundancy is justified also by the following result:

THEOREM 15 *Suppose $\mathbf{a}\in(\mathbf{R}_+^*)^r$ and denote by $R_{\mathbf{a}}:\mathbf{R}^{2r}\to\mathbf{R}^{2r}$, $R_{\mathbf{a}}(p,q)=(\mathbf{a}\otimes p,\mathbf{a}^{-1}\otimes q)$ where $\mathbf{a}^{-1}=(\frac{1}{a_1},\cdots,\frac{1}{a_r})$ and $\mathbf{a}\otimes p=(a_1p_1,\dots,a_rp_r)$. Suppose $\mathcal{WH}_{g,\Lambda}$ is a frame for $L^{2,r}$ supported on the leaf $E_{\alpha,\beta}$ and having uniform density $D(\Lambda)$. Then $R_{\mathbf{a}}(\Lambda)\subset K^r_{R_{\mathbf{a}}(\alpha,\beta)}$, $D(R_{\mathbf{a}}(\Lambda))\neq D(\Lambda)$ but $r(R_{\mathbf{a}}(\Lambda))=r(\Lambda)$.*

REMARK 16 *Note that $\mathcal{WH}_{g,\Lambda}$ is unitary equivalent to $\mathcal{WH}_{\mathbf{g}',R_{\mathbf{a}}(\Lambda)}$ for $\mathbf{g}'=V(\mathbf{a})\mathbf{g}$ where $V(\mathbf{a})$ is the unitary dilation with scales $\mathbf{a}$: $V(\mathbf{a})=\oplus_{l=1}^r v(a_l)$, $v(a_l)f^l(x)=\sqrt{a_l}f^l(a_lx)$.*

4 Proofs of the Results

4.1 Proof of Lemma 9

The proof in essentialy the same as the proof of Lemma 2.3 from [ChDeHe97]. We (re)derive here the result just for completness.

$\Rightarrow$ Suppose $\Lambda = \cup_{k=1}^{s_0} \Lambda_k$, and each Λ_k is δ_k-uniformly separated. Let $\delta = \min_{k=1,\dots,s_0}(\delta_k)$. Then any cube $Q_{\frac{\delta}{2}}(p,q)$ contains at most s_0 points of Λ. Thus $\nu^+(h) \leq s_0(\frac{h}{\delta/2})^2 = \frac{4s_0}{\delta^2}h^2$ and $\mu(Q_h(0,0) \cap E_{\alpha,\beta}) = const \cdot h^2$. Thus $D^+(\Lambda) \leq \frac{4s_0}{const\cdot\delta^2} < \infty$.

$\Leftarrow$ Suppose now $D^+(\Lambda) < \infty$, for some h. Let $N_h = \nu^+(h)$ for a fxed h. Thus each cube $Q_h(p,q)$ contains at most N_h points of Λ. Let $e_1, e_2, \dots, e_{2^d}$ be the verticies of the unit cube $[0,1]^{2d} \subset \mathbf{R}^{2d}$ and define $Z_k = (2\mathbf{Z})^{2d} + e_k$ and $B_k = \cup_{n\in Z_k} Q_h(nh)$, $k = 1, \dots, 2^{2d}$. Then $\mathbf{R}^{2d}$ is the disjoint union of the 2^{2d} sets B_k. Moreover for every $m, n \in Z_k$, $m \neq n$, $dist(Q_h(mh, Q_h(nh) \geq h$ and each cube $Q_h(nh)$ contains at most N_h elements of Λ. Thus $\Lambda \cap B_k$ can be split into N_h-uniformly separated squences and therefore Λ can be split into $2^{2d}N_h$ uniformly separated sequences.

4.2 Proof of Theorem 10

The proof is based on Theorem 3.1 from [ChDeHe97]. Consider $f \in L^2(\mathbf{R})$, $\|f\| = 1$, and $\mathbf{f} = 0 \oplus \cdots \oplus f \oplus \cdots \oplus 0$, with f on the l^{th} position. Then $B = B\|\mathbf{f}\|^2 \geq \sum_{\lambda\in\Lambda} | < \mathbf{f}, U(\lambda)\mathbf{g} > |^2$ which means each $\pi_i(\Lambda) = \Lambda_i = \{(z_i, p_i, q_i), i \in \mathbf{I}\}$ gives a Bessel set $\mathcal{WH}_{g^i;\Lambda_i}$, $g^i = \pi_i\mathbf{g}$.

Now we apply Theorem 3.1 from [ChDeHe97] and obtain that each Λ_i is relatively uniformly separated. Since $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_r$ we obtain that Λ is relatively uniformly separated and hence $D^+(\Lambda) < \infty$.

4.3 Proof of Theorem 11

The proof is based on a Homogeneous Approximation Property (HAP) for supersets that will be stated below. But first a lemma whose proof can be found in [ChDeHe97] as Lemma 3.3:

LEMMA 17 *Set $\varphi = e^{-\frac{\pi}{2}x^2}$, and let $h > 0$ be fixed. Then there is a $K = K(h) > 0$ such that for each $f \in L^2(\mathbf{R})$ and each $(z,p,q) \in T^1 \times \mathbf{R}^2 = G^1$,*

$$| < \varphi, u(z,p,q)f > |^2 \leq K(h) \int\int_{Q_h(p,q)\subset G^1} | < \varphi, u(w,x,y)f > |^2 \, d\mu_{G^1}(w,x,y)$$

Proof of Lemma 17

See Lemma 3.3 in [ChDeHe97]. $\square$

LEMMA 18 (Local HAP) *Let $\mathbf{g} \in L^{2,r}$ and $\Lambda \subset K^r_{\alpha,\beta}$ be such that $\mathcal{WH}_{g,\Lambda}$ is a frame for its span $\mathcal{E} \subset L^{2,r}$. Then for each $\mathbf{f} \in L^{2,r}$,*

$$\forall\varepsilon > 0\, \exists R > 0\, \forall(z,p,q) \in K^r_{\alpha,\beta}\ ,\ dist(\pi U(z,p,q)\mathbf{f}, W(R,p,q)) < \varepsilon \qquad (1)$$

where $W(R,p,q) = span\{\tilde{g}_\lambda, \lambda \in Q_R(p,q) \cap \Lambda\}$, $\{\tilde{g}_\lambda, \lambda \in \Lambda\}$ is the standard dual of $\mathcal{WH}_{g,\Lambda}$ and $\pi : L^{2,r} \to \mathcal{E}$ is the orthogonal projection onto $\mathcal{E}$.

Proof of Lemma 18

By Theorem 10 the assumption $\mathcal{WH}_{g,\Lambda}$ is a frame implies Λ is relatively uniformly separated. Thus we can separate Λ into subsets that are uniformly separated, all supported on the same leaf: $\Lambda = \cup_{k=1}^{r_0} \Lambda_k$, where Λ_k is δ_k-uniformly separated. Define $\delta = min\{\delta_1/2, \ldots, \delta_{r_0}/2\}$.

Let H be the set of those elements $\mathbf{f} \in L^{2,r}$ for which (1) holds. One can easily check that H is closed under finite linear combinations. It is also closed under $L^{2,r}$-norm: if $(\mathbf{f}_k)_{k\geq 1}$ is a sequence in H converging to $\mathbf{f}$ in $L^{2,r}$-sense then for any $\varepsilon > 0$ choose first a $k_\varepsilon > 1$ such that $\|\mathbf{f} - \mathbf{f}_{k_\varepsilon}\|_{L^{2,r}} < \frac{\varepsilon}{2}$ and then $R > 0$ such that $dist(\pi U(z,p,q)\mathbf{f}_{k_\varepsilon}, W(R,p,q)) < \frac{\varepsilon}{2}$ for every $(z,p,q) \in K^r_{\alpha,\beta}$. Then a triangle inequality argument shows that $\mathbf{f}$ has the HAP as well. Thus H is a closed subset of $\mathcal{E}$.

It then sufficies to show that if $\varphi(x) = e^{-\frac{\pi}{2}x^2}$ then the gaussian generator $\varphi = \varphi\oplus\cdots\oplus\varphi$ and all its time-frequency translates belong to H, i.e. $(U(z,r,s)\varphi \in H$, for every $(z,r,s) \in G^r$, for then $H = L^{2,r}$ and the result follows.

Fix $(z,r,s) \in G^r = T^r \times \mathbf{R}^{2r}$ and consider any $(z',p,q) \in G^r$. The expansion of $\pi(U(z',p,q)U(z,r,s)\varphi)$ with respect to the frame $\mathcal{WH}_{g,\Lambda}$ has the form:

$$\pi(U(z',p,q)U(z,r,s)\varphi) = \sum_{\lambda\in\Lambda} < U(z',p,q)U(z,r,s)\varphi, U(\lambda)\mathbf{g} > \tilde{\mathbf{g}}_\lambda$$

If A, B are the frame bounds of $\mathcal{WH}_{g,\Lambda}$ then $\{\tilde{\mathbf{g}}_\lambda\}$ has bounds $\frac{1}{B}$ and $\frac{1}{A}$. Hence:

$$\begin{aligned} dist(\pi(U(z',p,q)U(z,r,s)\varphi), W(R,p,q))^2 &\leq \|\pi(U(z',p,q)U(z,r,s)\varphi) - \\ &\quad \sum_{\lambda\in\Lambda\cap Q_R(p,q)} < U(z',p,q)U(z,r,s)\varphi, U(\lambda)\mathbf{g} > \tilde{\mathbf{g}}_\lambda\|_2^2 \\ &= \| \sum_{\lambda\in\Lambda\setminus Q_R(p,q)} < U(z',p,q)U(z,r,s)\varphi, U(\lambda)\mathbf{g} > \tilde{\mathbf{g}}_\lambda\|_2^2 \\ &\leq \frac{1}{A} \sum_{\lambda\in\Lambda\setminus Q_R(p,q)} | < U(z',p,q)U(z,r,s)\varphi, U(\lambda)\mathbf{g} > |^2 \\ &= \frac{1}{A}\sum_{k=1}^{r_0} \sum_{\lambda\in\Lambda_k\setminus Q_R(p,q)} | < U(z',p,q)U(z,r,s)\varphi, U(\lambda)\mathbf{g} > |^2 \end{aligned}$$

Note that for $\lambda = (z'', a, b)$,

$$| < U(z',p,q)U(z,r,s)\varphi, U(\lambda)\mathbf{g} >_{L^{2,r}} |^2 \leq \sum_{l=1}^{r} | < \varphi, u(1, a_l - p_l - r_l, b_l - q_l - s_l)g^l > |^2$$

Using Lemma 17,

$$| < \varphi(1, a_l - p_l - r_l, b_l - q_l - s_l)g^l > |^2 \leq K(\delta) \int\int_{Q_\delta(a_l-p_l-r_l, b_l-q_l-s_l)} | < \varphi, u(1,x,y)g^l > |^2 dx\, dy$$

Now we sum over $\lambda \in \Lambda_k \cap Q_R(p,q)$ and take into account that Λ_k is δ-separated we obtain:

$$\sum_{\lambda\in\Lambda_k\setminus Q_R(p,q)} \int\int_{Q_\delta(a_l-p_l-r_l,b_l-q_l-s_l)} |<\varphi, u(1,x,y)g^l>|^2 dx\, dy$$
$$\leq \int\int_{\mathbf{R}^2\setminus Q_{R-\delta}(-r_l,s_l)} |<\varphi, u(1,x,y)g^l>|^2 dx\, dy$$

Since the map $(x,y) \mapsto <\varphi, u(1,x,y)g^l>$ is in $L^2(\mathbf{R}^2)$ (i.e. φ is an admissible vector) we can choose R large enough so that for given (r,s), the integral of the right hans side above becomes smaller than $\frac{\varepsilon^2 A}{k(\delta)r_0r^2}$ for every $l = 1,\ldots,r$. Then summating over l and k we obtain:

$$dist(\pi(U(z',p,q)U(z,r,s)\varphi), W(R,p,q)) \leq \varepsilon$$

for every $(z',p,q) \in G^r$ which implies $U(z,r,s)\varphi \in H$ and thus $H = L^{2,r}$. End of proof. □

LEMMA 19 (HAP and Uniqueness) *Let* $\mathbf{g} \in L^{2,r}$ *and* $\Lambda \subset K^r_{\alpha,\beta}$ *be such that* $\mathcal{WH}_{g,\Lambda}$ *is a frame for* $L^{2,r}$. *Then for each* $\mathbf{f} \in L^{2,r}$,

$$\forall\varepsilon > 0\, \exists R > 0\, \forall(z,p,q) \in K^r_{\alpha,\beta}\ ,\ dist(U(z,p,q)\mathbf{f}, W(R,p,q)) < \varepsilon \qquad (2)$$

where $W(R,p,q) = span\{\tilde{g}_\lambda, \lambda \in Q_R(p,q)\cap\Lambda\}$ *and* $\{\tilde{g}_\lambda, \lambda \in \Lambda\}$ *is the standard dual of* $\mathcal{WH}_{g,\Lambda}$.

Proof of Lemma 19

It comes directly from Lemma 18. □

LEMMA 20 (Strong HAP) *Suppose* $\mathcal{WH}_{g,\Lambda}$ *is a frame for* $L^{2,r}$ *where* $\mathbf{g} \in L^{2,r}$, $\Lambda \subset K^r_{\alpha,\beta}$. *Then*

$$\forall \mathbf{f} \in L^{2,r}\, \forall\varepsilon > 0\, \exists R > 0\, \forall(z,p,q) \in K^r_{\alpha,\beta}\, \forall h > 0\, \forall(w,x,y) \in Q_h(p,q)\cap K^r_{\alpha,\beta}$$
$$dist(U(w,x,y)\mathbf{f}, W(h+R,p,q)) < \varepsilon \quad (3)$$

Proof of Lemma 20

Note that for $(w,x,y) \in Q_h(p,q)\cap K^r_{\alpha,\beta}$, $W(R,x,y) \subset W(h+R,p,q)$. Then, by applying Lemma 19 to $\mathbf{f}$ and $(w,x,y) \in K^r_{\alpha,\beta}$ we get $dist(U(w,x,y)\mathbf{f}, W(R,x,y)) < \varepsilon$. But $dist(U(w,x,y)\mathbf{f}, W(h+R,p,q)) \leq dist(U(w,x,y)\mathbf{f}, W(R,x,y)) < \varepsilon$ which end the proof of lemma 20. □.

Proof of Theorem 11

Let $\{\tilde{\varphi}_\delta, \delta \in \Delta\}$ be the (standard) biorthogonal s-Riesz basis of $\mathcal{WH}_{\varphi,\Delta}$, and $\{\tilde{\mathbf{g}}_\lambda, \lambda \in \Lambda\}$ be the standard dual of $\mathcal{WH}_{g,\Lambda}$. Let $W(h,p,q) = span\,\{\tilde{\mathbf{g}}_\lambda, \lambda \in Q_h(p,q)\cap\Lambda\}$, $V(h,p,q) = span\,\{\varphi_\delta, \delta \in Q_h(p,q)\cap\Delta\}$. Since Δ, Λ are relatively uniformly separated, each space $V(h,p,q)$, $W(h,p,q)$ is finite dimensional. Let

$C = \sup_{\delta\in\Delta} \|\tilde{\varphi}_\delta\| < \infty$ Fix $\varepsilon > 0$. Using the strong HAP Lemma 20 for $\mathbf{f}\varphi$, $\exists R > 0$ such that

$$\forall (p,q) \in E_{\alpha,\beta}, \forall h > 0, \forall (z,x,y) \in K^r_{\alpha,\beta} \cap Q_h(p,q) \quad , \quad dist(U(z,x,y)\varphi, W(h+R,p,q)) < \frac{\varepsilon}{C}$$

Let $P_V = P_{V(h,p,q)}$ and $P_W = P_{W(h+R,p,q)}$ denote the two orthogonal projectors onto $V(h,p,q)$, respectively $W(h+R,p,q)$. Define $T : V(h,p,q) \to V(h,p,q)$, $T = P_V P_W$. We shall evaluate the trace of T:

$$\begin{aligned}
trace\{T\} &= \sum_{\delta\in\Delta\cap Q_h(p,q)} < T\varphi_\delta, \tilde{\varphi}_\delta > \ = \sum_{\delta\in\Delta\cap Q_h(p,q)} < P_W\varphi_\delta, P_V\tilde{\varphi}_\delta > \\
&= \sum_{\delta\in\Delta\cap Q_h(p,q)} < I\varphi_\delta, \tilde{\varphi}_\delta > + < (P_W - I)\varphi_\delta, \tilde{\varphi}_\delta >= \ \#(\Delta \cap Q_h(p,q)) \\
&\quad + \sum_{\delta\in\Delta\cap Q_h(p,q)} < (P_W - I)\varphi_\delta, \tilde{\varphi}_\delta > \ \geq \ \#(\Delta \cap Q_h(p,q)) \\
&\quad - \sum_{\delta\in\Delta\cap Q_h(p,q)} \|(I - P_W)\varphi_\delta\| \cdot \|\tilde{\varphi}_\delta\| \ \geq \ \#(\Delta \cap Q_h(p,q)) - \sum_{\delta\in\Delta\cap Q_h(p,q)} \frac{\varepsilon}{C} C \\
&= (1-\varepsilon) \cdot \#(\Delta \cap Q_h(p,q))
\end{aligned}$$

On the other hand, since any eigenvalue of T, λ_T is subunital, $|\lambda_T| \leq 1$ we obtain:

$$trace\{T\} = \sum_{\text{spectrum of } T} \lambda_T \leq rank(T) \leq dim(W(h+R,p,q)) = \#(\Lambda \cap Q_{h+R}(p,q))$$

Hence

$$\#(\Lambda \cap Q_{h+R}(p,q)) \geq (1+\varepsilon) \cdot \#(\Delta \cap Q_h(p,q))$$

A simple computation shows that

$$Aria(Q_h(p,q) \cap K^r_{\alpha,\beta}) = \frac{|\alpha| \cdot |\beta|}{\|\alpha\|_\infty \cdot \|\beta\|_\infty} h^2$$

for every $(p,q) \in E_{\alpha,\beta}$ (see the proof of Theorem 15). Therefore

$$\begin{aligned}
\frac{\#(\Lambda \cap Q_{h+R}(p,q))}{Aria(Q_{h+R} \cap K^r_{\alpha,\beta})} (\frac{h+R}{h})^2 &= \frac{\#(\Lambda \cap Q_{h+R}(p,q))}{Aria(Q_{h+R} \cap K^r_{\alpha,\beta})} \cdot \frac{Aria(Q_{h+R}(p,q) \cap K^r_{\alpha,\beta})}{Aria(Q_h(p,q) \cap K^r_{\alpha,\beta})} \\
&\geq \ (1-\varepsilon) \frac{\#(\Delta \cap Q_{h+R}(p,q))}{Aria(Q_{h+R} \cap K^r_{\alpha,\beta})}
\end{aligned}$$

Now, taking the supremum over $(p,q) \in E_{\alpha,\beta}$ and the limit $h \to \infty$ we obtain $D^+(\Lambda) \geq (1-\varepsilon) D^+(\Delta)$; but $\varepsilon > 0$ was arbitrary, consequently $D^+(\Lambda) \geq D^+(\Delta)$. Similary, taking the infimum over $(p,q) \in E_{\alpha,\beta}$ and next the limit $h \to \infty$ we obtain $D^-(\Lambda) \geq (1-\varepsilon) D^-(\Delta)$ for every $\varepsilon > 0$ and therefore $D^-(\Lambda) \geq D^-(\Delta)$ which ends the proof of theorem. □

4.4 Proof of Theorem 15

We want to find the relation between $D(R_{\mathbf{a}}(\Lambda))$ and $D(\Lambda)$. Fix $\varepsilon > 0$. Let $h > 0$ be sufficiently large such that $D^+(\Lambda) \cdot \mu(Q_h(0,0) \cap E_{\alpha,\beta}) \leq \nu_\Lambda^+(h) + \varepsilon$. Let $p, q \in E_{\alpha,\beta}$ be such that $\#(Q_h(p,q) \cap \Lambda) \geq \nu_\Lambda^+(h) - \varepsilon$. Then $\#(R_{\mathbf{a}}(Q_h(p,q)) \cap R_{\mathbf{a}}(\Lambda)) = \#(Q_h(p,q) \cap \Lambda) > \nu_\Lambda^+(h) - \varepsilon$.

Next we need to find the aria $Aria(R_{\mathbf{a}}(Q_h(0,0)) \cap E_{R_{\mathbf{a}}(\alpha,\beta)})$. Note that

$$R_{\mathbf{a}}(Q_h(0,0)) = \{(z,d,e) \in G^r \mid |d_i| < \frac{a_i h}{2}, |e_i| < \frac{h}{2a_i}$$

On the other hand the two leaves are parametrized as: $E_{\alpha,\beta} = \{(t\alpha, s\beta) \mid t, s \in \mathbf{R}\}$ and respectively $E_{R_{\mathbf{a}}(\alpha,\beta)} = \{(t\mu, s\nu) \mid t, s \in \mathbf{R}\}$ for $(\mu,\nu) = R_{\mathbf{a}}(\alpha,\beta)$. We obtain:

$$Set_1 = Q_h(0,0) \cap E_{\alpha,\beta} = \{(z, t\alpha, s\beta) \; ; \; |t| < \frac{h}{2\alpha_i}, |s| < \frac{h}{2\beta_i}\}$$

$$Set_2 = R_{\mathbf{a}}(Q_h(0,0)) \cap E_{\mu,\nu} = \{(z, t\mu, s\nu) \; ; \; |t| < \frac{h}{2\alpha_i}, |s| < \frac{h}{2\beta_i}\}$$

Therefore the measures are:

$$Aria(Set_1) = h^2 \frac{|\alpha| \cdot |\beta|}{\|\alpha \cdot \|_\infty \|\beta\|_\infty}$$

$$Aria(Set_2) = h^2 \frac{|mun| \cdot |\nu|}{\|\alpha\|_\infty \cdot \|\beta\|_\infty}$$

On the other hand:

$$\begin{aligned} D^+_{R_{\mathbf{a}}(\Lambda)} \cdot Aria(Set_2) &\geq \#(R_{\mathbf{a}}(Q_h(p,q)) \cap R_{\mathbf{a}}(\Lambda)) \\ &= \#(Q_h(p,q) \cap \Lambda) \; > \; \nu_\Lambda^+(h) - \varepsilon \; \geq \; D^+(\Lambda) \cdot Aria(Set_1) - 2\varepsilon \end{aligned}$$

which implies:

$$D^+_{R_{\mathbf{a}}(\Lambda)} \geq D^+_\Lambda \frac{|\alpha| \cdot |\beta|}{|\mu| \cdot |\nu|} - \frac{2\varepsilon}{h^2} \frac{\|\alpha\|_\infty \cdot \|\beta\|_\infty}{|\alpha| \cdot |\beta|} \; ; \; \forall h \;\; \Rightarrow \;\; D^+_{R_{\mathbf{a}}(\Lambda)} \geq D^+_\Lambda \frac{|\alpha| \cdot |\beta|}{|\mu| \cdot |\nu|}$$

Similarly $D^+_\Lambda \geq D^+_{R_{\mathbf{a}}(\Lambda)} \frac{|\mu| \cdot |\nu|}{|\alpha| \cdot |\beta|}$ (considering $R_{\mathbf{a}}(\mathbf{a}^{-1})$ for instance). Also a similar argument holds for D^-_Λ and $D^-_{R_{\mathbf{a}}(\Lambda)}$ as well. Hence we obtain the following equalities:

$$D^+_{R_{\mathbf{a}}(\Lambda)} = D^+_\Lambda \frac{|\alpha| \cdot |\beta|}{|\mu| \cdot |\nu|} \quad , \quad D^-_{R_{\mathbf{a}}(\Lambda)} = D^-_\Lambda \frac{\alpha| \cdot |\beta|}{|\mu| \cdot |\nu|} \tag{4}$$

If Λ has uniform density, i.e. $D^+_\Lambda = D^-_\Lambda$, then we obtain:

$$D(R_{\mathbf{a}}(\Lambda)) = D(\Lambda)\frac{|\alpha|\cdot|\beta|}{|\mu|\cdot|\nu|} \tag{5}$$

and in terms of redundancies:

$$r(R_{\mathbf{a}}(\Lambda)) = \frac{D(R_{\mathbf{a}}(\Lambda))}{D_0(R_{\mathbf{a}}(\alpha,\beta))} = \frac{D(\Lambda)\frac{|\alpha|\cdot|\beta|}{|\mu|\cdot|\nu|}}{\frac{\mu\cdot\nu}{|\mu|\cdot|\nu|}} = \frac{D(\Lambda)}{\frac{\alpha\cdot\beta}{|\alpha|\cdot|\beta|}} = r(\Lambda)$$

where we have used $\mu\cdot\nu = \alpha\cdot\beta$. This ends the proof. $\square$

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