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Working Group: **The Potential Value of Information and Data**
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Value of Information through an Applied Mathematician's Looking-Glass

Consider these quotes:
"The whole is greater than the sum of the parts." attributed to Aristotle (Metaphysics, Book VIII)[1]
"The whole is greater than the part." Euclid (Elements, Book I)
How to formalize mathematically?

[1] https://se-scholar.com/se-blog/2017/6/23/who-said-the-whole-is-greater-than-the-sum-of-the-parts

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How to formalize mathematically?
Recall the concept of measure:
A map $m : \mathcal{P}(\{1, 2, \ldots, N\}) \rightarrow \mathbb{R}$ such that:

1. $m(\emptyset) = 0$.
2. $m(A) \geq 0$, for every $A \subset \{1, 2, \ldots, N\}$.
3. $m(A \cup B) = m(A) + m(B) - m(A \cap B)$, for every $A, B \subset \{1, 2, \ldots, N\}$.

Immediate consequences:
If $A \cap B = \emptyset$ then $m(A \cup B) = m(A) + m(B)$ (additivity)
If $B \subset A$ then $m(B) \leq m(A)$ (monotonicity)

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Value of Information as a Super-Additive Measure

Inspired by generalized concepts of measure such as *super-additive* and *sub-additive* measures, **Amos Golan** and R.B. define:

Definition

A value of information function is a map $v : \mathcal{P}(\{1,2,\dots,N\}) \to \mathbb{R}$ such that:

1. (null value) $v(\emptyset) = 0$.
2. (normalization) $v(\{1,2,\dots,N\}) = 1$.
3. (positivity) $v(A) \geq 0$, for all $A \subset \{1,2,\dots,N\}$.
4. (super-additivity) $v(A \cup B) \geq v(A) + v(B) - v(A \cap B)$, for all $A, B \subset \{1,2,\dots,N\}$.

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3. (positivity) $v(A) \geq 0$, for all $A \subset \{1,2,\dots,N\}$.
4. (super-additivity) $v(A \cup B) \geq v(A) + v(B) - v(A \cap B)$, for all $A, B \subset \{1,2,\dots,N\}$.

In particular:

- If $A \cap B = \emptyset$ then $v(A \cup B) \geq v(A) + v(B)$ ("the whole is greater than the sum of the parts")
- If $B \subset A$ then $v(B) \leq v(A)$ ("the whole is greater than the part")

Open questions

The fundamental question we have: Is there an objective way of defining a value of information function?
In this context, we looked at geometric and analytic properties of the set of value of information functions, a.k.a. normalized super-additive measures:

$$S = \{v : \{1,2,\ldots,N\} \to [0,1], v(\emptyset) = 0, v(A \cup B) + v(A \cap B) \geq v(A) + v(B), \forall A, B\}$$

Remarks:

- This is a compact convex set in $\mathbb{R}^{2^N}$ of dimension $2^N - 2$. What are its extreme points? Is there a *distinguished extreme point* that can serve as "objective value of information"?
- Are there natural properties that can be added to the definition to narrows down the list of value of information functions?
- Connections to Choquet potential theory, and the imprecise probability theory.