Factorization of positive-semidefinite operators with absolutely summable entries

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Works:

1. R. Balan and F. Jiang. *Factorization of positive-semidefinite operators with absolutely summable entries*. arXiv:2409.20372
2. R. Balan, K. Okoudjou, A. Poria, *On a Feichtinger Problem*, Operators and Matrices vol. 12(3), 881-891 (2018) http://dx.doi.org/10.7153/oam-2018-12-53
3. R. Balan, K.A. Okoudjou, M. Rawson, Y. Wang, R. Zhang, *Optimal l1 Rank One Matrix Decomposition*, in "Harmonic Analysis and Applications", Rassias M., Ed. Springer (2021)

Feichtinger-Heil-Larson Framework

Consider an infinite matrix $A = (A_{m,n})_{m,n\geq 0}$ so that $\|A\|_1 := \sum_{m,n\geq 0} |A_{m,n}| < \infty$.
This implies that A acts on $l^2(\mathbb{N})$ as a trace-class compact operator.
Assume additionally $A = A^* \geq 0$ as a quadratic form.
Question: Is there a factorization $A = \sum_{k\geq 0} f_k f_k^*$ so that $\sum_{k\geq 0} \|f_k\|_1^2 < \infty$?

Sources:

1. H. Feichtinger, Oberwolfach Reports 2004.
2. C. Heil and D. Larson, *Operator theory and modulation spaces*. In Frames and Operator Theory in Analysis and Signal Processing, Contemporary Mathematics. American Mathematical Society, 2006.

Optimal Decompositions

Let $Sym^+(\mathbb{C}^n) = \{A \in \mathbb{C}^{n\times n}\ ,\ A^* = A \geq 0\}$. For $A \in Sym^+(\mathbb{C}^n)$, denote

$$\gamma_+(A) := \inf_{A=\sum_{k\geq 1} x_x x_k^*} \sum_k \|x_k\|_1^2$$

It is easy to show that $\|A\|_1 := \sum_{i,j} |A_{ij}| \leq \gamma_+(A)$.

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The *matrix problem*: For every $n \geq 1$ find the best constant C_n such that, for every $A \in Sym^+(\mathbb{C}^n)$,

$$\gamma_+(A) \leq C_n \|A\|_1 := C_n \sum_{k,l=1}^{n} |A_{k,l}|$$

That is, we are interested in finding:

$$C_n = \sup_{A\geq 0} \frac{\gamma_+(A)}{\|A\|_1}$$

Properties of $\gamma_+(A)$

The infimum is achieved:

$$\gamma_+(A) := \inf_{A=\sum_{k\geq 1} x_x x_k^*} \sum_k \|x_k\|_1^2 = \min_{A=\sum_{k=1}^{n^2} x_x x_k^*} \sum_k \|x_k\|_1^2.$$

Upper bounds:

$$\gamma_+(A) \leq n\, trace(A) \leq n\|A\|_1 = n \sum_{k,j} |A_{k,j}|$$

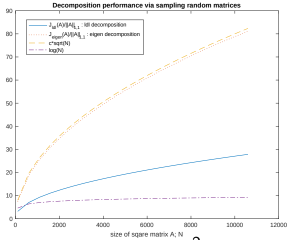
$$C_n \leq n$$

Lower bounds:

$$\|A\|_1 = \min_{A=\sum_{k\geq 1} x_x y_k^*} \sum_k \|x_k\|_1 \|y_k\|_1 \leq \gamma_+(A)$$

Convexity: for $A, B \in Sym^+(\mathbb{C}^n)$ and $t \geq 0$,

$$\gamma_+(A+B) \leq \gamma_+(A) + \gamma_+(B) \quad , \quad \gamma_+(tA) = t\gamma_+(A)$$



Maximum of $\sum_k \|x_k\|_1^2 / \|A\|_1$ over 30 random noise realizations, where $x_k's$ are obtained from the eigendecomposition, or the LDL factorization.

Properties of $\gamma_+(A)$

Lower bound is achieved, $\gamma_+(A) = \|A\|_1$ in the following cases:

1. If $A = xx^*$ is of rank one.
2. If $A \geq 0$ is a diagonally dominant matrix, $A_{ii} \geq \sum_{k \neq i} |A_{i,k}|$.
3. If $A \geq 0$ admits a Non Negative Matrix Factorization (NNMF), $A = BB^T$ with $B_{ij} \geq 0$.

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Continuity, Lipschitz and linear program reformulation:

1. $\gamma_+ : Sym^+(\mathbb{C}^n) \to \mathbb{R}$ is continuous.
2. If $A, B \geq \delta I$ and $trace(A), trace(B) \leq 1$ then
$$|\gamma_+(A) - \gamma_+(B)| \leq \left(\frac{n}{\delta^2} + n^2\right) \|A - B\|_{Op}.$$
3. Let $S_1 = \{x \in \mathbb{C}^n \ , \ \|x\|_1 = 1\}$ denote the compact unit sphere with respect to the l^1 norm, and let $\mathcal{B}(S_1)$ denote the set of Borel measures over S_1. Then:
$$\gamma_+(A) = \inf_{\mu \in \mathcal{B}(S_1) : \int_{S_1} xx^* d\mu(x) = A} \mu(S_1) \ , \ \mu^*(x) = \sum_{k=1}^{m} \lambda_k \delta(x - g_k)$$
where $\gamma_+(A) = \sum_{k=1}^m \lambda_k$ and $A = \sum_{k=1}^m \lambda_k g_k g_k^*$ is the optimal factorization.

Primal and dual problems for γ_+

The linear program is a convex optimization problem (which is great), but it is defined on an infinite dimensional space (not so great!).
Its dual problem enjoys strong duality (this may not be obvious due to infinite dimensional technical issues):

Theorem

Assume $A \geq 0$. Its associated primal (min) & dual (max) problems are:

$$\max_{T=T^*:\langle Tx,x\rangle\leq 1\ ,\ \forall \|x\|_1\leq 1} trace(TA) = \min_{\mu\in\mathcal{B}(S_1):\int_{S_1} xx^*\,d\mu(x)=A} \mu(S_1) = \gamma_+(A)$$

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Note the quantity (quadratic bound):

$$\rho_1(T) = \max_{x:\|x\|_1\leq 1} \langle Tx, x\rangle$$

The dual problem and C_n turn into:

$$\gamma_+(A) = \max_{T=T^*:\rho_1(T)\leq 1} trace(TA)$$

$$C_n = \max_{A\geq 0:\|A\|_1\leq 1} \gamma_+(A) = \max_{A\geq 0:} \max_{T=T^*:} trace(TA) = \max_{A\geq 0:} \max_{T=T^*:} \frac{trace(TA)}{\|A\|_1\,\rho_1(T)}$$

The bound ρ_1

Recall the definition of quadratic bound for a hermitian $T = T^*$:

$$\rho_1(T) = \max_{x:\|x\|_1 \le 1} \langle Tx, x\rangle.$$

How to compute it?
Easy cases:

1. If $T \le 0$ then $\rho_1(T) = 0$
2. If $T \ge 0$ then

$$\rho_1(T) = \max_k T_{k,k} = \max_{i,j} |T_{i,j}| =: \|T\|_\infty$$

This resembles the *numerical radius* of a matrix, $r(T) = \max_{\|x\|_2=1} |\langle Tx, x\rangle|$, which for hermitian matrices equals the largest singular value (operator norm). Note differences: (i) $\|\cdot\|_2 \to \|\cdot\|_1$; (ii) no absolute value $|.|$.

The bound ρ_1 - cont'd

Assume $\lambda_{max}(T) \geq 0$, i.e. T is NOT negative definite. Then:

$$\rho_1(T) = \max_{x:\|x\|_1=1} \langle Tx, x\rangle = \max_{\substack{A \geq 0 : \\ rank(A)=1 \\ \|A\|_1 = 1}} trace(TA) = \max_{\substack{A \geq 0 : \\ rank(A)=1 \\ \|A\|_1 \leq 1}} trace(TA)$$

The bound ρ_1 - cont'd

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Note the convex relaxation:

$$\pi_+(T) := \max_{\substack{A \geq 0 : \\ \|A\|_1 \leq 1}} trace(TA)$$

which defines a semi-definite program (SDP). Thus:

$$\rho_1(T) \leq \pi_+(T).$$

Primal and dual problems for ρ_1

The SDP enjoys strong duality:

Theorem

Assume $T = T^*$. The primal-dual programs have strong duality:

$$\pi_+(T) = \max_{\substack{A \geq 0 : \\ \|A\|_1 \leq 1}} trace(TA) = \min_{Y \geq 0} \|T + Y\|_\infty$$

where $\|Z\|_\infty = \max_{i,j} |Z_{i,j}|$.

The proof of this theorem is based on the Von Neumann's min-max theorem.

Closing the loop

Theorem

$$C_n := \max_{\substack{A \geq 0 \\ A \neq 0}} \frac{\gamma_+(A)}{\|A\|_1} = \max_{\substack{T = T^* \\ \rho_1(T) \neq 0}} \frac{\pi_+(T)}{\rho_1(T)}.$$

Consequently, for $T = T^*$ that are not negative definite:

$$\max_{x:\|x\|_1=1} \langle Tx, x\rangle \leq \max_{\substack{A \geq 0 : \\ \|A\|_1 \leq 1}} trace(TA) \leq C_n \max_{x:\|x\|_1=1} \langle Tx, x\rangle.$$

Remark. Compare this double inequality to Grothendieck related max-cut inequality: For $T = T^* \geq 0$, or $T = T^*$ and $diag(T) = 0$,

$$\max_{x \in \{+1,-1\}^n} \langle Tx, x\rangle \leq \max_{\substack{A \geq 0 \\ A_{ii} = 1}} trace(TA) \leq 2K_G \max_{x \in \{+1,-1\}^n} \langle Tx, x\rangle.$$

The BMS Result and its Consequence

Theorem (Afonso Bandeira, Dustin Mixon, Stefan Steinerberger - Oberwolfach 2024; ACHA 2024)

There are $\alpha > 0$, $N_0 > 1$ so that for any $n \geq N_0$,

$$C_n \geq \alpha\sqrt{n}$$

Consequence: No-go result for the Feichtinger-Heil-Larson Problem:

Theorem

There exists $A = A^* \geq 0$ and infinite hermitian matrix $A = (A_{i,j})_{i,j\in\mathbb{N}}$ so that $\sum_{i,j\in\mathbb{N}} |A_{i,j}| < \infty$ and yet, for any factorization $A = \sum_{k\geq 1} x_k x_k^*$, $\sum_{k\geq 1} \|x_k\|_1^2 = \infty$.

Thank you for listening! ... QUESTIONS?

Motivation

A Feichtinger Problem

At a 2004 Oberwolfach meeting, Hans Feichtinger asked the following question:
(Q1) Given a positive semi-definite trace-class operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $Tf(x) = \int K(x,y)f(y)dy$, with $K \in M^1(\mathbb{R}^d \times \mathbb{R}^d)$, and its spectral factorization, $T = \sum_k \langle \cdot, h_k \rangle h_k$, must it be $\sum_k \|h_k\|_{M^1}^2 < \infty$?

A modified version of the question is:
(Q2) Given T as before, i.e., $T = T^* \geq 0$, $K \in M^1(\mathbb{R}^d \times \mathbb{R}^d)$, is there a factorization $T = \sum_k \langle \cdot, g_k \rangle g_k$ such that $\sum_k \|g_k\|_{M^1}^2 < \infty$?

Problem Reformulation

Matrix Language

Consider an infinite matrix $A = (A_{m,n})_{m,n \geq 0}$ so that $\|A\|_\wedge := \|A\|_1 := \sum_{m,n \geq 0} |A_{m,n}| < \infty$.
This implies that A acts on $l^2(\mathbb{N})$ as a trace-class compact operator.
Assume additionally $A = A^* \geq 0$ as a quadratic form.
Let $(e_k)_{k \geq 0}$ denote an orthogonal set of eigenvectors normalized so that $A = \sum_{k \geq 0} e_k e_k^*$. It is easy to check that $e_k \in l^1(\mathbb{N})$, for each k.
Equivalent reformulations of the two problems (Heil, Larson '08):

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Equivalent reformulations of the two problems (Heil, Larson '08):
Q1: Does it hold $\sum_{k\geq 0} \|e_k\|_1^2 < \infty$? Answer: Negative in general! (see [1])

Q2: Is there a factorization $A = \sum_{k\geq 0} f_k f_k^*$ so that $\sum_{k\geq 0} \|f_k\|_1^2 < \infty$?

Problem Reformulation

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Consider an infinite matrix $A = (A_{m,n})_{m,n\geq 0}$ so that $\|A\|_\wedge := \|A\|_1 := \sum_{m,n\geq 0} |A_{m,n}| < \infty$.
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Q2: Is there a factorization $A = \sum_{k\geq 0} f_k f_k^*$ so that $\sum_{k\geq 0} \|f_k\|_1^2 < \infty$?

Using previous equivalence and Bandeira et all's result: The answer to Q2 is negative in general!

Linear program result

Optimal Factorization from a Measure Theory Perspective

Let $S_1 = \{x \in \mathbb{C}^n \ , \ \|x\|_1 = 1\}$ denote the compact unit sphere with respect to the l^1 norm, and let $\mathcal{B}(S_1)$ denote the set of Borel measures over S_1. For $A \in \mathrm{Sym}(\mathbb{C}^n)^+(\mathbb{C}^n)$ consider the optimization problem:

$$(p^*, \mu^*) = inf_{\mu \in \mathcal{B}(S_1): \int_{S_1} xx^* d\mu(x) = A} \ \mu(S_1) \quad (M)$$

Theorem (Optimal Measure)

For any $A \in Sym^+(\mathbb{C}^n)$ the optimization problem (M) is convex and its global optimum (minimum) is achieved by

$$p^* = \gamma_+(A) \ \ , \ \ \mu^*(x) = \sum_{k=1}^{m} \lambda_k \delta(x - g_k)$$

where $A = \sum_{k=1}^m (\sqrt{\lambda_k} g_k)(\sqrt{\lambda_k} g_k)^*$ is an optimal decomposition that achieves $\gamma_+(A) = \sum_{k=1}^m \lambda_k$.

Super-resolution and Convex Optimizations

$$\gamma_+(A) = \min_{x_1,\ldots,x_m \,:\, A=\sum_k x_k x_k^*} \sum_{k=1}^{m} \|x_k\|_1^2 \;,\;\; m = n^2 \quad (P)$$

$$p^* = \inf_{\mu \in \mathcal{B}(S_1) \,:\, A=\int_{S_1} xx^* d\mu(x)} \int_{S_1} d\mu(x) \quad (M)$$

Remarks

1. *The optimization problem (P) is non-convex, but finite-dimensional. The optimization problem (M) is convex, but infinite-dimensional.*
2. *If $g_1, \ldots, g_m \in S_1$ in the support of μ^* are known so that $\mu^* = \sum_{k=1}^{m} \lambda_k \delta(x - g_k)$, then the optimal $\lambda_1, \ldots, \lambda_m \geq 0$ are determined by a linear program. More general, (M) is an infinite-dimensional linear program.*
3. *Finding the support of μ^* is an example of a super-resolution problem. One possible approach is to choose a redundant dictionary (frame) that includes the support of μ^*, and then solve the induced linear program.*

Proof of the Optimal Measure Result

Recall: we want to show the following problems admit same solution:

$$\gamma_+(A) = \min_{x_1,\ldots,x_m \,:\, A=\sum_k x_k x_k^*} \sum_{k=1}^{m} \|x_k\|_1^2 \;, \;\; m = n^2 \quad (P)$$

$$p^* = \inf_{\mu\in\mathcal{B}(S_1) \,:\, A=\int_{S_1} xx^*\,d\mu(x)} \int_{S_1} d\mu(x) \quad (M)$$

a. Assume $A = \sum_{k=1}^m x_k x_k^*$ is a global minimum for (P). Then $\mu(x) = \sum_{k=1}^m \|x_k\|_1^2 \delta(x - \frac{x_k}{\|x_k\|_1})$ is a feasible solution for (M). This shows $p^* \leq \gamma_+(A)$.

b. For reverse: Let μ^* be an optimal measure in (M). Fix $\varepsilon > 0$. Construct a disjoint partition $(U_l)_{1\leq l\leq L}$ of S_1 so that each U_l is included in some ball $B_\varepsilon(z_l)$ of radius ε with $\|z_l\|_1 = 1$. Thus $U_l \subset B_\varepsilon(z_l) \cap S_1$.

For each l, compute $x_l = \frac{1}{\mu^*(U_l)} \int_{U_l} x\,d\mu^*(x) \in B_\varepsilon(z_l)$. Let $g_l = \sqrt{\mu^*(U_l)} x_l$.

Proof: The Optimal Measure Result (cont)

Key inequality:

$$0 \leq R_l := \int_{U_l} (x - x_l)(x - x_l)^* \, d\mu^*(x) = \int_{U_l} xx^* \, d\mu^*(x) \; - \mu^*(U_l) x_l x_l^*$$

Sum over l and with $R = \sum_{l=1}^{L} R_l$ get

$$A = \sum_{l=1}^{L} \int_{U_l} xx^* \, d\mu^*(x) \leq \sum_{l=1}^{L} g_l g_l^* + R$$

By sub-additivity and homogeneity:

$$\gamma_+(A) \leq \sum_{l=1}^{L} \|g_l\|_1^2 + \gamma_+(R) \leq \sum_{l=1}^{L} \mu^*(U_l) \|x_l\|_1^2 + n \, trace(R)$$

But $\|x_l - z_l\|_1 \leq \varepsilon$ and $\|x - x_l\|_1 \leq 2\varepsilon$ for every $x \in U_l$. Hence $\|x_l\|_1 \leq 1 + \varepsilon$ and $trace(R_l) \leq 4\mu^*(U_l)\varepsilon^2$. (In fact, $\|x_l\|_1 \leq 1$ by triangle inequality)

Proof: The Optimal Measure Result (end)

Thus:

$$\gamma_+(A) \leq \mu^*(S_1) + (2\varepsilon + \varepsilon^2 + 4n\varepsilon^2)\mu^*(S_1)$$

Since $\varepsilon > 0$ is arbitrary, it follows

$$\gamma_+(A) \leq \mu^*(S_1) = p^*$$

This ends the proof of the measure result. □

Second New Result: The Continuity Property

Theorem (The Continuity Property)

The map $\gamma_+ : (Sym^+(\mathbb{C}^n), \|\cdot\|) \rightarrow \mathbb{R}$ is continuous.

Remarks

1. This statement extends the continuity result from $Sym^{++}(\mathbb{C}^n) = \{A = A^* > 0\}$ to $Sym^+(\mathbb{C}^n) = \{A = A^* \geq 0\}$.
2. Proof is based on a (new?) comparison result between non-negative operators.
3. Global Lipschitz is still open.

The Continuity Property

The proof is based on the following two lemmas:

Lemma (L1)

Let $A \in Sym^+(\mathbb{C}^n)$ of rank $r > 0$. Let $\lambda_r > 0$ denote the r^{th} eigenvalue of A, and let $P_{A,r}$ denote the orthogonal projection onto the range of A. For any $0 < \varepsilon < 1$ and $B \in Sym^+(\mathbb{C}^n)$ such that $\|A - B\|_{Op} \leq \frac{\varepsilon\lambda_r}{1-\varepsilon}$, the following holds true:

$$A - (1-\varepsilon)P_{A,r}BP_{A,r} \geq 0 \qquad (1)$$

Lemma (L2)

Let $A \in Sym^+(\mathbb{C}^n)$ of rank $r > 0$. Let $\lambda_r > 0$ denote the r^{th} eigenvalue of A. For any $0 < \varepsilon < \frac{1}{2}$ and $B \in Sym^+(\mathbb{C}^n)$ such that $\|A - B\|_{Op} \leq \varepsilon\lambda_r$, the following holds true:

$$B - (1-\varepsilon)P_{B,r}AP_{B,r} \geq 0 \qquad (2)$$

where $P_{B,r}$ denotes the orthogonal projection onto the top r eigenspace of B.

Proof of Continuity of γ_+

Fix $A \in Sym^+(\mathbb{C}^n)$. Let $(B_j)_{j\geq 1}$, $B_j \in Sym^+(\mathbb{C}^n)$, be a convergent sequence to A. We need to show $\gamma_+(B_j) \to \gamma_+(A)$.
Let $A = \sum_{k=1}^{n^2} x_k x_k^*$ be the optimal decomposition of A such that $\gamma_+(A) = \sum_{k=1}^{n^2} \|x_k\|_1^2$.
If $A = 0$ then $\gamma_+(A) = 0$ and

$$0 \leq \gamma_+(B_j) \leq n\, trace(B_j) \leq n^2 \|B_j\|_{Op}.$$

Hence $\lim_j \gamma_+(B_j) = 0$.
Assume $rank(A) = r > 0$ and let $\lambda_r > 0$ denote the smallest strictly positive eigenvalue of A. Let $\varepsilon \in (0, \frac{1}{2})$ be arbitrary. Let $J = J(\varepsilon)$ be so that $\|A - B_j\|_{Op} < \varepsilon \lambda_r$ for all $j > J$. Let $B_j = \sum_{k=1}^{n^2} y_{j,k} y_{j,k}^*$ be the optimal decomposition of B_j such that $\gamma_+(B_j) = \sum_{k=1}^{n^2} \|y_{j,k}\|_1^2$.
Let $\Delta_j = A - (1-\varepsilon) P_{A,r} B_j P_{A,r}$. By Lemma L1, for any $j > J$,

$$\gamma_+(A) \leq (1-\varepsilon)\gamma_+(P_{A,r} B_j P_{A,r}) + \gamma_+(\Delta_j) \leq (1-\varepsilon) \sum_{k=1}^{n^2} \|P_{A,r} y_{j,k}\|_1^2 + n\, trace(\Delta_j)$$

Proof of Continuity of γ_+ (cont)

Pass to a subsequence j' of j so that $y_{j',k} \to y_k$, for every $k \in [n^2]$, and $\gamma_+(B_{j'}) \to \liminf_j \gamma_+(B_j)$. Then $\lim_{j'} P_{A,r} y_{j',k} = P_{A,r} y_k = y_k$ and

$$\lim_{j'} \sum_{k=1}^{n^2} \|P_{A,r} y_{j',k}\|_1^2 = \lim_{j'} \sum_{k=1}^{n^2} \|y_{j',k}\|_1^2 = \liminf_j \gamma_+(B_j)$$

On the other hand, $\lim_j trace(\Delta_j) = \varepsilon\, trace(A)$. Hence:

$$\gamma_+(A) \leq (1-\varepsilon) \liminf_j \gamma_+(B_j) + \varepsilon\, trace(A)$$

Since $\varepsilon > 0$ is arbitrary, it follows $\gamma_+(A) \leq \liminf_j \gamma_+(B_j)$.
The inequality $\limsup_j \gamma_+(B_j) \leq \gamma_+(A)$ follows from Lemma L2 similarly: with $\Delta_j = B_j - (1-\varepsilon) P_{B_j,r} A P_{B_j,r}$ and $A = \sum_{k=1}^{n^2} x_k x_k^*$ optimal,

$$\gamma_+(B_j) \leq (1-\varepsilon)\gamma_+(P_{B_j,r} A P_{B_j,r}) + n\, trace(\Delta_j) = (1-\varepsilon) \sum_{k=1}^{n^2} \|P_{B_j,r} x_k\|_1^2 + n\, trace(\Delta_j).$$

Next take limsup of lhs by noticing $P_{B_j,r} \to P_{A,r}$ and $\limsup_j \|\Delta_j\|_{Op} = \varepsilon \|A\|_{Op}$: $\limsup_j \gamma_+(B_j) \leq (1-\varepsilon)\gamma_+(A) + n^2 \varepsilon \|A\|_{Op}$. Take $\varepsilon -> 0$ and result follows. □

Proof of Lemmas

Proof of Lemma L1

Let $P = P_{A,r}$. and $\Delta = A - (1-\varepsilon)P_{A,r}BP_{A,r}$. For any $x \in \mathbb{C}^n$:

$$\langle \Delta x, x\rangle = \langle APx, Px\rangle - (1-\varepsilon)\langle BPx, Px\rangle = \langle (A-(1-\varepsilon)B)Px, Px\rangle =$$

$$= \varepsilon\langle APx, Px\rangle + (1-\varepsilon)\langle (A-B)Px, Px\rangle \geq \varepsilon\lambda_r \|Px\|^2 - (1-\varepsilon)\|A-B\|_{Op}\|Px\|^2 \geq 0$$

because $\|A-B\|_{Op} \leq \frac{\varepsilon\lambda_r}{1-\varepsilon}$.

Proof of Lemma L2

Let $P = P_{B,r}$ and $\Delta = B - (1-\varepsilon)P_{B,r}AP_{B,r}$. Let $C = B - P_{B,r}BP_{B,r} \geq 0$. Let μ_r be the r^{th} eigenvalue of B. Note $|\mu_r - \lambda_r| \leq \|A-B\|_{Op} \leq \varepsilon\lambda_r$. Thus $\mu_r \geq (1-\varepsilon)\lambda_r$. For any $x \in \mathbb{C}^n$:

$$\langle \Delta x, x\rangle = \langle Cx, x\rangle + \langle BPx, Px\rangle - (1-\varepsilon)\langle APx, Px\rangle = \langle Cx, x\rangle + \varepsilon\langle BPx, Px\rangle +$$

$$+(1-\varepsilon)\langle (B-A)Px, Px\rangle \geq \langle Cx, x\rangle + (\varepsilon\mu_r - (1-\varepsilon)\|A-B\|_{Op})\|Px\|^2 \geq 0$$

because $\|A-B\|_{Op} \leq \varepsilon\lambda_r \leq \frac{\varepsilon\mu_r}{1-\varepsilon}$.

Third new result: Strong duality for γ_+

Theorem

For every $A \geq 0$,

$$\max_{\substack{T = T^* \\ \langle Tx, x\rangle \leq 1 \ , \ \forall \|x\|_1 \leq 1}} trace(TA) = \min_{\substack{\mu \in \mathcal{B}(S_1) \\ \int_{S_1} xx^* d\mu(x) = A}} \mu(S_1) = \gamma_+(A)$$

Proof [Fushuai "Black" Jiang]

The second equality was established earlier as a "super-resolution" result.

For the first equality:

1. Let $A = \sum_{k=1}^m x_k x_k^*$ be its optimal decomposition such that $\gamma_+(A) = \sum_{k=1}^m \|x_k\|_1^2$, and let $T = T^*$ be a generic matrix so that $\langle Ty, y\rangle \leq 1$ for all $\|y\|_1 \leq 1$. Denote $y_k = \frac{x_k}{\|x_k\|_1}$. Then

$$trace(TA) = \sum_{k=1}^m \langle Tx_k, x_k\rangle = \sum_{k=1}^m \|x_k\|_1^2 \langle Ty_k, y_k\rangle \leq \sum_{k=1}^m \|y_k\|_1^2 = \gamma_+(A)$$

Proof of strong duality for γ_+ (2)

2. For the reverse inequality, let $H \subset Sym^+(\mathbb{C}^n) \times \mathbb{R}$ denote the set

$$H = \left\{ (\int_{S_1} zz^* \, d\mu(z), r + \int_{S_1} d\mu) \ \ , \ \ \mu \in \mathcal{B}(S_1) \ , \ \ r \geq 0 \right\}$$

Claim 1: H is closed.
Use Banach-Alaoglou theorem that the set of unit Borel measures is weak-* compact.
Claim 2: H is convex. – immediate
Let $q = max_{T=T^*} \ trace(TA)$ subject to $\langle Tx, x \rangle \leq 1$ for all $\|x\|_1 \leq 1$.
Claim 3: $(A, q) \in H$, which establishes the theorem.
Assume the contrary: $(A, q) \notin H$. Then it is separated by a hyperplane from H:

$$trace\left(R \int_{S_1} xx^* \, d\mu(z) \right) + a(r + \int_{S_1} d\mu) \geq c_0 > trace(AR) + aq \ \ , \ \ \forall \mu \in \mathcal{B}(S_1), r \geq 0$$

Deduce: $a \geq 0$, $c_0 \leq 0$. If $a = 0$ then contradiction for $\mu = \mu^*$. Rescale by dividing through a. Denote $T_0 = -R/a$.

Proof of strong duality for γ_+ (3)

We obtained:

$$\int_{S_1} (1 - \langle T_0 x, x\rangle) d\mu \geq c_0 > q - trace(AT_0)$$

for every Borel measure $\mu \in \mathcal{B}(S_1)$. This means $\langle T_0 x, x\rangle \leq 1$ for all $\|x\| = 1$. This also implies $\langle T_0 x, x\rangle \leq 1$ for all $\|x\|_1 \leq 1$. On the other hand $q < trace(AT_0) + c_0 \leq trace(AT_0)$ which contradicts the optimality of q. Q.E.D.