Exact Equations and Integrating Factors

(Section 2.6).

Recall how we integrate separable equations. For:

$$M(x) + N(y)\frac{dy}{dx} = 0 \qquad (*)$$

we find the primitives of M and N:

Let H_1 denote the primitive of M: $\quad H_1'(x) = M(x)$

Let H_2 denote the primitive of N: $\quad H_2'(y) = N(y)$

Set:

$$\Psi(x,y) = H_1(x) + H_2(y)$$

Remark: when $y = \phi(x)$ is a solution of $(*)$:

$$\frac{d\Psi}{dx}(x, \phi(x)) = \frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial y}\cdot\frac{dy}{dx} = H_1'(x) + H_2'(y)\cdot\frac{dy}{dx} =$$

$$= M(x) + N(y)\frac{dy}{dx} = 0$$

Hence, along solutions $y = y(x)$ of $(*)$:

$$\Psi(x,y) = \text{Constant}$$

This equation defines solutions of $(*)$ implicitly:

$$H_1(x) + H_2(y) = C$$

We want to generalize this method to more general equations.

Example Consider:

$$2x + y^2 + 2xy \cdot y' = 0$$

Note: it is not a separable equation.

Set: $\Psi(x,y) = x^2 + x \cdot y^2$

Remark:

$$\frac{\partial \Psi}{\partial x} = 2x + y^2$$

$$\frac{\partial \Psi}{\partial y} = 2xy$$

Thus the equation can be re-written:

$$\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} \cdot \frac{dy}{dx} = 0$$

Therefore, along solutions $y = y(x)$:

$$\frac{d}{dx}\left[\Psi(x, y(x))\right] = \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} \cdot \frac{dy}{dx} = 0$$

Hence:

$$\Psi(x,y) = \text{Constant}$$

or:

$$\boxed{x^2 + xy^2 = c.}$$

defines solutions implicitly.

Let's analyze in more detail when this "trick" works.

Consider:

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \qquad (\square)$$

We seek a function $\psi(x,y)$ such that:

$$\begin{cases} \dfrac{\partial \psi}{\partial x} = M(x,y) & (1) \\[2mm] \dfrac{\partial \psi}{\partial y} = N(x,y). & (2) \end{cases}$$

If such a function exists then, along solutions $y = y(x)$ of $(\square)$:

$$\frac{\partial \psi}{\partial x}(x,y) + \frac{\partial \psi}{\partial y}(x,y) \cdot \frac{dy}{dx} = 0$$

or:

$$\frac{d}{dx}\left[\psi(x, y(x))\right] = 0.$$

Hence:

$$\boxed{\psi(x,y) = C}$$

defines solutions of $(\square)$ implicitly.

An equation $(\square)$ is called exact if there exists a function ψ that satisfies (1) and (2)

The questions we need to answer are:

(1) What differential equations are exact? When the equation is exact?

(2) How to find the function ψ when the equation is exact?

(3) When the equation is not exact, can we modify it to become exact?

The answer to the first question is given by the following result:

Theorem [2.6.1] Assume functions $M(x,y)$, $N(x,y)$, $\frac{\partial M}{\partial y}(x,y)$, $\frac{\partial N}{\partial x}(x,y)$ are continuous in the rectangular region $R: \alpha < x < \beta, \ \gamma < y < \delta$. Then: the differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is exact (i.e. there exists a function $\psi(x,y)$ such that $\frac{\partial \psi}{\partial x}(x,y) = M(x,y)$ and $\frac{\partial \psi}{\partial y}(x,y) = N(x,y)$) if and only if

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$$

each point (x,y) in R. $\blacksquare$

Example Let's revisit:

$$\underbrace{2x+y^2}_{M(x,y)} + \underbrace{2xy}_{N(x,y)} \cdot y' = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y}(x,y) &= 2y \\ \frac{\partial N}{\partial x}(x,y) &= 2y \end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ at every } (x,y)$$

Hence this equation is exact. ∇

Why is this result true?

The answer comes in two parts:

1) Assume the equation is exact, that is there exists the function ψ so that $\frac{\partial \psi(x,y)}{\partial x} = M(x,y)$ and $\frac{\partial \psi}{\partial y}(x,y) = N(x,y)$

Then:

$$\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial x \partial y}$$

(check your Calculus books !)

which means:

$$\frac{\partial}{\partial y}\left[\frac{\partial \psi}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{\partial \psi}{\partial y}\right]$$

or:

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y).$$

Hence this condition is necessary.

2) For the converse, we will compute a solution ψ that shows the equation is exact:

Assume that: $\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$

We want to find a function ψ such that:

$$\begin{cases} \frac{\partial \psi}{\partial x} = M(x,y) & (1) \\ \frac{\partial \psi}{\partial y} = N(x,y). & (2). \end{cases}$$

Fix a point x_0 in (α, β).

Consider:

$$Q(x,y) = \int_{x_0}^{x} M(s,y)\, ds$$

Note:

$$\frac{\partial Q}{\partial x} = M(x,y)$$

Hence a solution of (1) is:

$$\psi(x,y) = Q(x,y) + h(y).$$

where $h(y)$ is an arbitrary function in y.

We find h by requiring to this ψ to satisfy equation (2):

$$N(x,y) = \frac{\partial \psi}{\partial y} = \frac{\partial Q}{\partial y}(x,y) + h'(y).$$

equation (2)

Thus:

$$h'(y) = N(x,y) - \frac{\partial Q}{\partial y}(x,y). \qquad (**)$$

To be able to solve this equation we need to have the right hand side a function on y only.

This requirement is satisfied because of hypothesis $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

ndeed, here is the reason: let $\gamma = N(x,y) - \frac{\partial Q}{\partial y}(x,y)$.

then: $\frac{\partial \gamma}{\partial x} = \frac{\partial N}{\partial x} - \frac{\partial^2 Q}{\partial x \partial y} = \frac{\partial N}{\partial x} - \frac{\partial^2 Q}{\partial y \partial x} = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial x}\right) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$

hence: γ is independent of x.

Returning to $(**)$: we solve this equation by integrating with respect to y:

$$h(y) = \int \left[N(x,y) - \frac{\partial Q(x,y)}{\partial y} \right] dy$$

To obtain a more compact form, fix y_0 in (γ, δ).

$$h(y) = \int_{y_0}^{y} \left[N(x,t) - \frac{\partial Q(x,t)}{\partial t} \right] dt = \int_{y_0}^{y} N(x,t)\,dt - \left[Q(x,y) - Q(x,y_0) \right]$$

$$h(y) = \int_{y_0}^{y} N(x,t)\,dt - Q(x,y) + Q(x,y_0).$$

Thus:

$$\boxed{\Psi(x,y) = Q(x,y) + h(y) = \int_{y_0}^{y} N(x,t)\,dt + Q(x,y_0) = \int_{y_0}^{y} N(x,t)\,dt + \int_{x_0}^{x} M(s,y_0)\,ds}$$

satisfies (1) and (2). (see also Problem 17).

Example

Solve: $$(y\cos x + 2xe^{y}) + (\sin x + x^2e^{y} - 1)\cdot y' = 0 \qquad (3)$$

First we need to check if this equation is exact.

$$M(x,y) = y\cos(x) + 2xe^{y}$$
$$N(x,y) = \sin(x) + x^2e^{y} - 1$$

$$\frac{\partial M}{\partial y} = \cos(x) + 2xe^{y}$$
$$\frac{\partial N}{\partial x} = \cos(x) + 2xe^{y}$$

Thus: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence the equation is exact.

Now we need to find ψ so that:

$$\begin{cases} \frac{\partial \psi}{\partial x} = y\cos(x) + 2xe^{y} \\ \frac{\partial \psi}{\partial y} = \sin(x) + x^2e^{y} - 1. \end{cases}$$

2.1: Integrate the first equation with respect to x:

$$\psi(x,y) = y\sin(x) + x^2e^{y} + h(y).$$

2.2. Plug this expression into the second equation:

$$\underbrace{\sin(x) + x^2e^{y} + h'(y)}_{\frac{\partial \psi}{\partial y}} = \sin(x) + x^2e^{y} - 1.$$

Thus: $h'(y) = -1$ and $h(y) = -y$

2.3. We obtain: $$\psi(x,y) = y\sin(x) + x^2e^{y} - y$$

The solution of (3) is given implicitly by: $\boxed{y\sin(x) + x^2e^{y} - y = C}$

Example

$$3xy + y^2 + (x^2 + xy)\, y' = 0$$

1) Check if this equation is exact:

$$M(x,y) = 3xy + y^2$$
$$N(x,y) = x^2 + xy.$$

$$\frac{\partial M}{\partial y} = 3x + 2y \neq \frac{\partial N}{\partial x} = 2x + y$$

Hence this equation is not exact.

2) However, let's try to apply the same recipe:

$$\begin{cases} \text{(a)} & \dfrac{\partial \psi}{\partial x} = 3xy + y^2 \xrightarrow{2.1} \psi(x,y) = \dfrac{3}{2}x^2 y + xy^2 + h(y). \\ \text{(b)} & \dfrac{\partial \psi}{\partial y} = x^2 + xy. \xrightarrow{2.2} \dfrac{3}{2}x^2 + 2xy + h'(y) = x^2 + xy. \end{cases}$$

Thus: $$h'(y) = \underbrace{-\frac{1}{2}x^2 - xy}$$

The problem is that the right-hand side depends on x. Hence we cannot find a function $h(y)$ that satisfies this equation.

While this equation is not exact, we might be able to fix this issue by multiplying the entire equation by some function. Such a function that makes the equation exact is called an integrating factor.

We are given $M(x,y) + N(x,y)\dfrac{dy}{dx} = 0$

which is not exact:

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

We seek a function $\mu(x,y)$ such that:

$$\mu \cdot M + \mu \cdot N \frac{dy}{dx} = 0$$

becomes an exact equation, that is:

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu \cdot N)$$

Expand out the derivatives:

$$\frac{\partial \mu}{\partial y} \cdot M + \mu \cdot \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}$$

and re-arrange the terms:

$$M \cdot \frac{\partial \mu}{\partial y} - N \frac{\partial \mu}{\partial x} + \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \cdot \mu = 0 \qquad (**)$$

In general this equation is as hard to solve as the initial equation ☹

However there are two cases when this equation can be easily solved.

Case 1. When:

$$\alpha = \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{M}$$

is a function on y only: $\alpha = \alpha(y)$.

Then we can choose $\mu = \mu(y)$ and $(**)$ becomes:

$$M \cdot \frac{d\mu}{dy} + \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu = 0$$

$$\frac{d\mu}{dy} + \alpha \cdot \mu = 0$$

This equation is linear and separable:

$$\frac{1}{\mu}\frac{d\mu}{dy} = -\alpha \quad \Rightarrow \quad \frac{d}{dy}\left[\ln|\mu|\right] = -\alpha.$$

$$\ln|\mu| = -\int \alpha(y)\,dy$$

$$\boxed{\mu(y) = \exp\left(-\int \alpha(y)\,dy\right).}$$

Case 2. When:

$$\beta = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

is a function on x only: $\beta = \beta(x)$.

Then we can choose $\mu = \mu(x)$ and $(**)$ becomes:

$$-N\frac{d\mu}{dx} + \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 0$$

or: $$\frac{d\mu}{dx} = \beta \cdot \mu$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = \beta(x) \quad \Rightarrow \quad \frac{d}{dx}\left[\ln|\mu|\right] = \beta$$

$$\boxed{\mu(x) = \exp\left(\int \beta(x)\,dx\right)}$$

Example Consider again:

$$\underbrace{(3xy+y^2)}_{M(x,y)} + \underbrace{(x^2+xy)}_{N(x,y)}\, y' = 0$$

Since $\frac{\partial M}{\partial y} = 3x + 2y \neq 2x + y = \frac{\partial N}{\partial x}$

this equation is not exact.

Let us compute an integrating factor.

1. Multiply by μ, an unknown function that is yet to be determined:

$$\mu \cdot (3xy + y^2) + \mu \cdot (x^2 + xy)\, y' = 0$$

2. We need μ to satisfy:

$$\frac{\partial}{\partial y}\left[\mu \cdot (3xy + y^2)\right] = \frac{\partial}{\partial x}\left[\mu \cdot (x^2 + xy)\right]$$

Expand out:

$$\frac{\partial \mu}{\partial y} \cdot (3xy + y^2) + \mu \cdot (3x + 2y) = \frac{\partial \mu}{\partial x} \cdot (x^2 + xy) + \mu \cdot (2x + y)$$

Simplify to:

$$\frac{\partial \mu}{\partial y} \cdot y(3x + y) - \frac{\partial \mu}{\partial x}\, x(x+y) + \mu \cdot (x + y) = 0.$$

Remark: $\frac{x+y}{x(x+y)} = \frac{1}{x}$ depends on x only!

Hence we choose a solution: $\mu = \mu(x)$.

$$\Rightarrow -\frac{d\mu}{dx}\, x(x+y) + \mu (x+y) = 0.$$

$$\frac{d\mu}{dx} = \frac{\mu}{x}$$

A solution: $\mu = x$

Thus:

$$x(3xy+y^2) + x(x^2+xy)\cdot y' = 0$$

is an exact equation. Indeed, let's check:

$$\frac{\partial}{\partial y}\left(x\cdot(3xy+y^2)\right) = 3x^2+2xy.$$

$$\frac{\partial}{\partial x}\left(x\cdot(x^2+xy)\right) = 3x^2+2xy$$

They are equal!

Now let's find ψ:

$$\begin{cases} \dfrac{\partial \psi}{\partial x} = x(3xy+y^2) = 3x^2y+xy^2 \\ \dfrac{\partial \psi}{\partial y} = x(x^2+xy) = x^3+x^2y. \end{cases}$$

Integrate the first equation:

$$\psi(x,y) = x^3y+\frac{1}{2}x^2y^2+h(y)$$

Plug into the second equation:

$$\underbrace{x^3+x^2y+h'(y)}_{\frac{\partial \psi}{\partial y}} = x^3+x^2y.$$

$$\Rightarrow h'(y)=0. \quad \text{Choose } h=0$$

Thus: $\psi(x,y) = x^3y+\frac{1}{2}x^2y^2$

and the solution of our differential equation: $3xy+y^2+(x^2+xy)\cdot y'=0$

is given implicitly by:

$$\boxed{x^3y+\frac{1}{2}x^2y^2 = C.}$$

Appendix: Differential Forms

The differential equation:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

admits a different, formal (for now), rewriting:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

The expression:

$$\omega = M(x,y)\,dx + N(x,y)\,dy$$

is called a differential form.

A differential form ω is said exact if there exists a function ψ such that $\omega = d\psi$

The differential $d\psi$ is defined by: $d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy$

Hence ω is exact if and only if $M(x,y) = \frac{\partial\psi}{\partial x}$, $N(x,y) = \frac{\partial\psi}{\partial y}$

The (exterior) differential of the form ω is defined by:

$$d\omega = dM \wedge dx + dN \wedge dy$$

where the exterior product $\wedge$ is a bilinear operation that satisfies:

$$dx \wedge dx = 0$$

$$dx \wedge dy = -dy \wedge dx$$

$$dy \wedge dy = 0.$$

Then, formally we obtained:

$$d\omega = dM \wedge dx + dN \wedge dy =$$

$$= \left(\frac{\partial M}{\partial x} dx + \frac{\partial M}{\partial y} dy\right) \wedge dx + \left(\frac{\partial N}{\partial x} dx + \frac{\partial N}{\partial y} dy\right) \wedge dy$$

$$= \frac{\partial M}{\partial x} \underbrace{dx \wedge dx}_{0} + \frac{\partial M}{\partial y} \underbrace{dy \wedge dx}_{-dx \wedge dy.} + \frac{\partial N}{\partial x} dx \wedge dy + \frac{\partial N}{\partial y} \underbrace{dy \wedge dy}_{0}$$

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \wedge dy$$

Note the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is equivalent to $d\omega = 0$.

<u>Definition</u>. A differential form ω is said <u>closed</u> if $d\omega = 0$.

With these two definitions (exact and closed) in place, Theorem 2.6.1. can be restated:

<u>Theorem</u>. Assume the differential form $\omega = M(x,y)dx + N(x,y)dy$ is defined in a rectangular region $R: \alpha < x < \beta, \gamma < y < \delta$.

Then ω is exact if and only if ω is closed. $\square$