Solutions to Homework 1

1. (a) For $0 < t < 1$ use integration by parts in:

$$\int_t^1 log(x)dx = x\,log(x)|_t^1 - \int_t^1 x(log(x))'dx = -t\,log(t) - \int_t^1 1dx = -t\,log(t) - 1 + t$$

Then

$$\int_0^1 log(x)dx = \lim_{t \searrow 0} \int_t^1 log(x)dx = \lim_{t \searrow 0}(-t\,log(t) + t) - 1 = -1$$

Note:

$$\lim_{t \searrow 0} t\,log(t) = \lim_{t \searrow 0} \frac{log(t)}{\frac{1}{t}} \overset{L'Hospital}{=} \lim_{t \searrow 0} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t \searrow 0}(-t) = 0$$

(b) Fix an $a > 1$, and $M > 1$

$$\int_1^M \frac{1}{x^a}dx = \frac{x^{-a+1}}{-a+1}|_1^M = \frac{M^{1-a}}{1-a} - \frac{1}{1-a}$$

Since $1 - a < 0$ we get

$$\int_1^\infty \frac{1}{x^a}dx = \lim_{M \to \infty} \int_1^M \frac{1}{x^a}dx = -\frac{1}{1-a} = \frac{1}{a-1}$$

For $a = 1$ we obtain

$$\int_1^M \frac{1}{x}dx = log(x)|_1^M = log(M)$$

Thus

$$\int_1^\infty \frac{1}{x}dx = \lim_{M \to \infty} log(M) = \infty$$

2.

$$f : \mathbf{R} \to \mathbf{R} \quad , \quad f(x) = \begin{cases} x log(x^2) & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

At every point $x > 0$ the function is continuous, differentiable, and infinitely many times differentiable, since

$$f'(x) = 2log(x) + 2 \quad , \quad f''(x) = \frac{2}{x} \quad , \quad f^{(k)}(x) = (-1)^k \frac{2(k-2)!}{x^{k-1}} \ , \ k > 2$$

where $n! = n(n-1)\cdots 2\cdot 1$. Similarily for $x < 0$ since $f(-x) = -f(x)$.

At $x = 0$ the following happens:

$$lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{log(x^2)}{\frac{1}{x}} \stackrel{L'Hospital}{=} \lim_{x\to 0} \frac{\frac{2}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0}(-2x) = 0 = f(0)$$

Hence

(a): f is continuous at 0, and, based on the previous discussion, f is continuous on $\mathbf{R}$.

$$R(y) = \frac{f(y) - f(0)}{y - 0} = \frac{ylog(y^2)}{y} = log(y^2)$$

Since

$$\lim_{y\to 0} R(y) = \lim_{y\to 0} log(y^2) = -\infty$$

It follows that f has an infinite derivative at $x = 0$, but it is not differentiable (since the derivative is not finite).

Hence:

b: f is not differentiable on $\mathbf{R}$.

Because of this:

c. f is not of class C^1 on $\mathbf{R}$.

d. The largest k so that f is of class C^k is $k = 0$, that is, f is continuous on $\mathbf{R}$.

3. The function f is defined, explicitely, as follows:

$$f(x) = \begin{cases} \lim_{a\searrow 0}\int_a^x sin(\frac{1}{t})dt & if \quad x > 0 \\ 0 & if \quad x = 0 \\ \lim_{a\nearrow 0}\int_a^x sin(\frac{1}{t})dt & if \quad x < 0 \end{cases} = \begin{cases} \lim_{a\searrow 0}\int_a^x sin(\frac{1}{t})dt & if \quad x > 0 \\ 0 & if \quad x = 0 \\ \text{-}\lim_{a\nearrow 0}\int_x^a sin(\frac{1}{t})dt & if \quad x < 0 \end{cases}$$

Thus $f(-x) = f(x)$ and it is sufficient to analyze this function for $x \geq 0$. At $x > 0$

$$f'(x) = g(x) = sin(\frac{1}{x})$$

Hence f is continuous, differentiable, and, in fact, infinitely many times differentiable. At $x = 0$, we do the analysis as follows.

First, for $x > 0$:

$$|f(x)| = |\lim_{a\searrow 0}\int_a^x sin(\frac{1}{t})dt| = \lim_{a\searrow 0}|\int_a^x sin(\frac{1}{t})dt| \leq \lim_{a\searrow 0}\int_a^x |sin(\frac{1}{t})|dt \leq \lim_{a\searrow 0}\int_a^x 1dt = \lim_{a\searrow 0}(x-a) = x$$

Hence:

$$|f(x)| \leq |x|$$

and

$$lim_{x\to 0} f(x) = 0 = f(0).$$

Thus f is continuous at $x = 0$. Next we need to analyze the quotient

$$R(x) = \frac{f(x) - f(0)}{x - 0} = \frac{1}{x} \lim_{a \searrow 0} \int_a^x sin(\frac{1}{t})dt$$

and its behavior when $x \searrow 0$. Let

$$I(a,b) = b \int_a^{1/b} sin(\frac{1}{t})dt$$

Note

$$\lim_{x \searrow 0} R(x) = \lim_{b\to\infty} \lim_{a \searrow 0} I(a,b)$$

We change the variable $t \to u = \frac{1}{t}$ in the integral $I(a,b)$,

$$I(a,b) = b \int_b^{1/a} \frac{sin(u)}{u^2} du$$

Consider the case $b = 2\pi n$ and $a = \frac{1}{2\pi m}$ with $m > n$. Thus

$$I(\frac{1}{2\pi m}, 2\pi n) = 2\pi n \int_{2\pi n}^{2\pi m} \frac{sin(u)}{u^2} du = 2\pi n \sum_{k=n}^{m-1} \int_0^{2\pi} \frac{sin(u)}{(u + 2\pi k)^2} du$$

This last integral is next split as follows:

$$\int_0^{2\pi} \frac{sin(u)}{(u+2\pi k)^2} du = \int_0^{\pi} \frac{sin(u)}{(u+2\pi k)^2} du + \int_{\pi}^{2\pi} \frac{sin(u)}{(u+2\pi k)^2} du = \int_0^{\pi} (\frac{sin(u)}{(u+2\pi k)^2} + \frac{sin(u+\pi)}{(u+2\pi k+\pi)^2}) du$$

Since $sin(u+\pi) = -sin(u)$ we simplify further

$$\frac{sin(u)}{(u+2\pi k)^2} + \frac{sin(u+\pi)}{(u+2\pi k+\pi)^2} = \frac{sin(u)((u+2\pi k+\pi)^2 - (u+2\pi k)^2)}{(u+2\pi k)^2(u+2\pi k+\pi)^2} du = \frac{sin(u)(\pi^2 + 2\pi(u+2\pi k))}{(u+2\pi k)^2(u+2\pi k+\pi)^2}$$

Next, use $|\sin(u)| \leq 1$, $u + 2\pi k + \pi > u + 2\pi k$, and $\pi^2 \leq \pi(u + 2\pi k)$ for $k \geq 1$,

$$|\frac{sin(u)}{(u+2\pi k)^2} + \frac{sin(u+\pi)}{(u+2\pi k+\pi)^2}| \leq \frac{3}{(u+2\pi k)^3} \leq \frac{1}{k^3}$$

for $0 \leq u \leq \pi$. Thus

$$|I(\frac{1}{2\pi m}, 2\pi n)| \leq 2\pi n \sum_{k=n}^{m-1} \int_0^{\pi} \frac{1}{k^3} du \leq 2\pi^2 \sum_{k=n}^{m-1} \frac{1}{k^2}$$

Returning to $R(x)$:

$$\lim_{x\searrow 0} R(x) = \lim_{n\to\infty}\lim_{m\to\infty} I(\frac{1}{2\pi m}, 2\pi n)$$

Thus:

$$|\lim_{x\searrow 0} R(x)| \leq 2\pi^2 \lim_{n\to\infty}\lim_{m\to\infty}\sum_{k=n}^{m-1}\frac{1}{k^2} = 2\pi^2\lim_{n\to\infty}\sum_{k=n}^{\infty}\frac{1}{k^2}$$

Since $\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty$, it follows that last limit is zero (it is the "tail-sum" of a convergent series). Thus

$$\lim_{x\searrow 0} R(x) = 0 = g(0)$$

and thus $f'(x) = g(x)$ everywhere on $\mathbf{R}$.

This shows that f is differentiable on $\mathbf{R}$. However f is not C^1 since g is not continuous at 0.

4.

(a) $1 + e^{i\theta} = 0$ implies $1 + cos(\theta) = 0$ and $sin(\theta) = 0$. Hence $cos(\theta) = -1$ and $sin(\theta) = 0$ admit solutions $\theta = \pi + 2k\pi$ where k is an arbitrary integer.

(b) $1 + e^{i\theta} + e^{2i\theta} = 0$. Let z denote $z = e^{i\theta}$. We have $1 + z + z^2 = 0$ which has solutions $z = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Thus we need to solve $e^{i\theta} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ or $e^{i\theta} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

The first equation turns into

$$cos(\theta) = -\frac{1}{2}\ ,\ sin(\theta) = -\frac{\sqrt{3}}{2}$$

which admits as solution $\theta = \frac{4\pi}{3} + 2k\pi$, where k is an arbitrary integer.

The second equation becomes

$$cos(\theta) = -\frac{1}{2}\ ,\ sin(\theta) = +\frac{\sqrt{3}}{2}$$

which admits as solution $\theta = \frac{2\pi}{3} + 2k\pi$, where k is an arbitrary integer.

Thus the general solution of the equation $1 + e^{i\theta} + e^{2i\theta} = 0$ is any θ in the set

$$\{\frac{2\pi}{3} + 2k\pi\ ,\ k\ integer\} \cup \{\frac{4\pi}{3} + 2k\pi\ ,\ k\ integer\}.$$

(c) (d) For c and d we adopt a different strategy. First replace $z = e^{i\theta}$. The two equations become

$$1 + z + z^2 + z^3 = 0 \tag{1}$$

and

$$1 + z + z^2 + \cdots + z^n = 0 \tag{2}$$

respectively. Consider the general equation first. Use summation formula

$$0 = 1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

Thus we need to solve $z^{n+1} - 1 = 0$. However note that $z = 1$ is NOT a solution of (1) or(2). The $n + 1$ roots of $z^{n+1} = 1$ $z_1, \ldots, z_{n+1}$ are

$$z_r = e^{i \frac{2\pi r}{n+1}} \quad , \quad 1 \leq r \leq n + 1$$

We need to remove $z_{n+1} = 1$. Hence only $z_1, \ldots, z_n$ are roots of $1 + z + \cdots + z^n = 0$. Next we need to solve $e^{i\theta} = z_r$ for each $1 \leq r \leq n$. We obtain $e^{i\theta} = e^{i \frac{2\pi r}{n+1}}$. Hence the set of solutions is given by

$$\{\frac{2\pi r}{n+1} + 2k\pi \quad , \quad 1 \leq r \leq n \ , \ k \ integer\}$$

(c) For n=3 we obtain

$$\{\frac{2\pi r}{4} + 2k\pi \quad , \quad 1 \leq r \leq 3 \ , \ k \ integer\}$$

or

$$\{\frac{\pi}{2} + 2k\pi, \pi + 2k\pi, \frac{3\pi}{2} + 2k\pi \quad , \quad k \ integer\}$$