I.

1. First note that

$$(x^4-1)\cdot\delta(x) = -\delta(x)$$

Hence $-\delta(x)$ is one particular solution. Thus the general solution we seek must satisfy:

$$(x^4-1)\cdot(f(x)+\delta(x)) = 0$$

Since $x^4-1 = (x^2+1)(x+1)(x-1)$ we obtain, first that

$$(x-1)(x+1)(f(x)+\delta(x)) = 0$$

And then

$$f(x)+\delta(x) = c_1\delta(x-1)+c_2\delta(x+1)$$

Hence the general solution is:

$$f(x) = -\delta(x)+c_1\delta(x-1)+c_2\delta(x+1)$$

2. One particular solution is δ since $(x^4+1)\cdot\delta(x) = \delta(x)$. Thus f must satisfy:

$$(x^4+1)\cdot(f(x)-\delta(x)) = 0$$

Since x^4+1 does not have real roots, the only solution (multiply both sides by $\frac{1}{1+x^4}$) is given by

$$f(x)-\delta(x) = 0$$

Hence:

$$f(x) = \delta(x)$$

is the unique solution.

II.

3. First note: $f(x) = e^{2\pi ix}\cdot\delta(x) = \delta(x)$. Thus

$$\hat{f}(s) = \hat{\delta}(s) = 1$$

4. Note $f(x) = sin^2(\pi x) = \frac{1-cos(2\pi x)}{2} = \frac{1}{2}-\frac{1}{4}e^{2\pi ix}-\frac{1}{4}e^{-2\pi ix}$. Thus

$$\hat{f}(s) = \frac{1}{2}\delta(s)-\frac{1}{4}\delta(s-1)-\frac{1}{4}\delta(s+1)$$

5. First note $f\{\Phi\} = \delta'\{x^2\Phi\} = -\delta\{2x\Phi+x^2\Phi'\} = 0$. Thus $f = 0$. Hence

$$\hat{f}(s) = 0$$

6. Recall $\widehat{p_{-1}}(s) = -i\pi sign(s)$ and $\widehat{sign}(s) = \frac{1}{i\pi}p_{-1}(s)$. Thus

$$\hat{f}(s) = \widehat{sign}(s-2) = -\frac{i}{\pi}p_{-1}(s-2)$$

7. Note $f(x) = \frac{1}{4}\frac{1}{x-2}-\frac{1}{4}\frac{1}{x+2} = \frac{1}{4}p_{-1}(x-2)-\frac{1}{4}p_{-1}(x+2)$. Thus

$$\hat{f}(s) = \frac{1}{4}e^{-4\pi is}\widehat{p_{-1}}(s)-\frac{1}{4}e^{4\pi is}\widehat{p_{-1}}(s) = -\frac{i\pi}{4}e^{-4\pi is}sign(s)+\frac{i\pi}{4}e^{4\pi is}sign(s) =$$

$$= -\frac{\pi}{2}sin(4\pi s)sign(s) = -\frac{\pi}{2}sin(4\pi|s|)$$

III

8. Fourier transform the equation and obtain

$$((2\pi is)^4-1)\cdot F(s) = 1$$

Thus

$$F_0(s) = \frac{1}{(2\pi)^4 s^4 - 1}$$

is a particular solution of this algebraic equation. Since $(2\pi)^4 s^4 - 1 = (4\pi^2 s^2 + 1)(2\pi s - 1)(2\pi s + 1)$ the general solution is given by

$$F(s) = \frac{1}{16\pi^4 s^4 - 1} + c_1 \delta(s - \frac{1}{2\pi}) + c_2 \delta(s + \frac{1}{2\pi})$$

Next we need to Fourier invert this distribution. First:

$$\frac{1}{16\pi^4 s^4 - 1} = \frac{1}{2}\frac{1}{4\pi^2 s^2 - 1} - \frac{1}{2}\frac{1}{4\pi^2 s^2 + 1} = \frac{1}{4}\frac{1}{2\pi s - 1} - \frac{1}{4}\frac{1}{2\pi s + 1} - \frac{1}{2}\frac{1}{4\pi^2 s^2 + 1} =$$

$$= \frac{1}{8\pi} p_{-1}(s - \frac{1}{2\pi}) - \frac{1}{8\pi} p_{-1}(s + \frac{1}{2\pi}) - \frac{1}{4}\frac{2}{1 + 4\pi^2 s^2}$$

Thus its Fourier transform inverse is

$$f_0(x) = \frac{1}{8} e^{ix} \pi i sign(x) - \frac{1}{8} e^{-ix} \pi i sign(x) - \frac{1}{4} e^{-|x|} = -\frac{\pi}{4} sin(|x|) - \frac{1}{4} e^{-|x|}$$

Then the general solution becomes

$$f(x) = -\frac{\pi}{4} sin(|x|) - \frac{1}{4} e^{-|x|} + c_1 e^{ix} + c_2 e^{-ix}$$

9. Fourier transform the equation and obtain

$$((2\pi i s)^4 + 1) \cdot F(s) = 1$$

or

$$((16\pi^4 s^4 + 1) \cdot F(s) = 1$$

The unique solution is

$$F(s) = \frac{1}{16\pi^4 s^4 + 1} = \sum_{k=0}^{3} \frac{\alpha_k}{2\pi s - z_k}$$

where $z_k = e^{i(2k+1)\pi/4}$ are the 4 roots of -1, and α_k are obtained from:

$$\alpha_k = \lim_{2\pi s \to z_k} \frac{2\pi s - z_k}{16\pi^4 s^4 + 1} = \frac{1}{4} e^{-i3\pi(2k+1)/4} = \frac{z_k^3}{4} = -\frac{\bar{z_k}}{4}$$

Explicitely:

$$F(s) = -\frac{1}{4}\frac{\bar{z_0}}{2\pi s - z_0} - -\frac{1}{4}\frac{\bar{z_1}}{2\pi s - z_1} - \frac{1}{4}\frac{\bar{z_2}}{2\pi s - z_2} - \frac{1}{4}\frac{\bar{z_3}}{2\pi s - z_3}$$

Finally we need to Fourier invert this function. Note for a real and $b > 0$,

$$F_+(s) = \frac{1}{2\pi s - (a - ib)} \longrightarrow f_+(x) = -ie^{iax} e^{bx} h(-x) = -ie^{i(a-ib)x} h(-x)$$

$$F_- = \frac{1}{2\pi s - (a + ib)} \longrightarrow f_-(x) = ie^{iax} e^{-bx} h(x) = ie^{i(a+ib)x} h(x)$$

where $h(x)$ is the Heaviside step function (see equations (22) in Chapter 3). Note $imag(z_0) = imag(z_1) > 0$ and $imag(z_2) = imag(z_3) < 0$. Thus

$$f(x) = -\frac{\bar{z_0}}{4} e^{iz_0 x} i h(x) - \frac{\bar{z_1}}{4} e^{iz_1 x} i h(x) - \frac{\bar{z_2}}{4} e^{iz_2 x}(-i) h(-x) - \frac{\bar{z_3}}{4} e^{iz_3 x}(-i) h(-x)$$

To simplify this expression, note $z_2 = -z_0$ and $z_3 = -z_1$, hence

$$f(x) = i\frac{z_2}{4} e^{-1 z_2 |x|} + i\frac{z_3}{4} e^{-iz_3|x|} = i\frac{-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}{4} e^{-i(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})|x|} + i\frac{\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}{4} e^{-i(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})|x|} =$$

$$= e^{-\frac{\sqrt{2}}{2}|x|}\left[\frac{\sqrt{2}}{8}(1-i)e^{i\frac{\sqrt{2}}{2}|x|}+\frac{\sqrt{2}}{8}(1+i)e^{-i\frac{\sqrt{2}}{2}|x|}\right] = \frac{\sqrt{2}}{4}e^{-\frac{\sqrt{2}}{2}|x|}\left[cos(\frac{\sqrt{2}}{2}|x|)+sin(\frac{\sqrt{2}}{2}|x|)\right]$$

Note that $f(x)$ is a real valued continuous function. To test:

$$Df(x) = -\frac{1}{2}e^{-\frac{\sqrt{2}}{2}|x|}sin(\frac{\sqrt{2}}{2}|x|)sign(x)$$

$$D^2f(x) = \frac{\sqrt{2}}{4}e^{-\frac{\sqrt{2}}{2}|x|}\left(sin(\frac{\sqrt{2}}{2}|x|)-cos(\frac{\sqrt{2}}{2}|x|)\right)$$

$$D^3f(x) = \frac{1}{2}e^{-\frac{\sqrt{2}}{2}|x|}cos(\frac{\sqrt{2}}{2}|x|)sign(x)$$

$$D^4f(x) = -\frac{\sqrt{2}}{4}e^{-\frac{\sqrt{2}}{2}|x|}\left[cos(\frac{\sqrt{2}}{2}|x|)+sin(\frac{\sqrt{2}}{2}|x|)\right]+\delta(x)$$

IV.
10.
First we solve the equation

$$(D^2-1)g(x) = \delta(x)$$

Fourier transform to obtain

$$((2\pi is)^2-1)G(s) = 1$$

or

$$(4\pi^2s^2+1)G(s) = -1$$

The unique solution is

$$G(s) = -\frac{1}{4\pi s^2+1}$$

whose inverse Fourier transform is

$$g(x) = -\frac{1}{2}e^{-|x|}$$

Then the solution of $(D^2-1)f = sin(x)$ is $f = g * sin$:

$$f(x) = -\frac{1}{2}\int_{-\infty}^{\infty}e^{-|y|}sin(x-y)dy = -\frac{1}{2}sin(x)\int_{-\infty}^{\infty}e^{-|y|}cos(y)dy+\frac{1}{2}cos(x)\int_{-\infty}^{\infty}e^{-|y|}sin(y)dy$$

The two integrals are:

$$\int_{-\infty}^{\infty}e^{-|y|}sin(y)dy = 0$$

because the integrand is odd, and

$$\int_{-\infty}^{\infty}e^{-|y|}cos(y)dy = 2\int_{0}^{\infty}e^{-y}cos(y)dy = 2I$$

where, integrating by parts,

$$I = \int_0^{\infty}e^{-y}cos(y)dy = 1-\int_0^{\infty}e^{-y}sin(y)dy = 1-(0+\int_0^{\infty}e^{-y}cos(y)dy) = 1-I$$

Hence $I = 1/2$ and

$$\int_{-\infty}^{\infty}e^{-|y|}cos(y)dy = 1$$

Assembling the result we get:

$$f(x) = -\frac{1}{2}sin(x)$$