1. First we need to estimate an upper bound on $|F[k]|$, say $|F[k]| \leq c_k$. The log-log plot suggests an upper bound (blue line) described by

$$log(c_k) = a_0 - \alpha log(k)$$

where a_0 and α are estimated from the graph. At $k = 1$ $log(k) = 0$ and we get $a_0 = log(c_1) = log(10)$. Hence $e^{a_0} = 10$. We estimate the slope of the blue line using for instance the points $(k = 1, c_1 = 10)$ and $(k = 10, c_{10} = 10^{-1} = 0.1)$. In the log-log scale we get:

$$log(10) = a_0 \quad , \quad log(10^{-1}) = a_0 - \alpha log(10)$$

Substituting a_0 in the second equation we get:

$$-log(10) = log(10) - \alpha log(10)$$

or

$$-1 = 1 - \alpha \rightarrow \alpha = 2$$

Thus we get

$$|F[k]| \leq c_k \leq exp(a_0 - \alpha log(k)) = \frac{e^{a_0}}{k^\alpha} = \frac{10}{k^2}$$

Next we apply the summation formula we derived in class:

$$|f(x) - \sum_{k=-N}^{N} F[k]e^{2\pi ikx}| \leq \sum_{|k|>N} |F[k]| \leq 2\sum_{k=N+1}^{\infty} \frac{10}{k^2} \leq \frac{20}{N}$$

Thus to obtain a level of accuracy of 1% we need to have

$$\frac{20}{N} \leq 0.01 \rightarrow N \geq 2000$$

That is, 2000 rank partial Fourier sum, or for any N larger than 2000.

2. We derive an error for the bound $|F[k]| \leq 9 \cdot 10^{-|k|}$.

$$|f(x)\sum_{k=-N}^{N} F[k]e^{2\pi inx}| \leq \sum_{|k|>N} |F[k]| \leq 2\sum_{k=N+1}^{\infty} \frac{9}{10^k} = 18\frac{10}{9}10^{-N-1} = 2 \cdot 10^{-N}$$

Thus, for an error of 0.002 we need $2 \cdot 10^{-N} \leq 0.002$ or $10^{-N} \leq 10^{-3}$. Hence $N = 3$ (or any integer larger than 3) would achieve the desired level of accuracy.

3.

$$f_1 = 1_{[1,2]} \quad , F_1(\omega) = e^{-3\pi i\omega}\frac{sin(\pi\omega)}{\pi\omega}$$

4.

$$f_2(x) = e^{-a|x|} \quad , \quad F_2(\omega) = \frac{2a}{4\pi^2\omega^2 + a^2}$$

5.

$$f_3(x) = \frac{1}{\alpha^2 + x^2} = \frac{\pi}{\alpha}F_2(x)|_{a=2\pi\alpha} \quad , \quad F_3(\omega) = \frac{\pi}{\alpha}e^{-2\pi\alpha|\omega|}$$

Then replace α by a:

$$f(x) = \frac{1}{x^2 + a^2} \quad \rightarrow \quad F_3(\omega) = \frac{\pi}{a}e^{-2\pi a|\omega|}$$

6.

$$f_4(x) = sin(2\pi x) f_1(x) = \frac{1}{2i}(e^{2\pi i x} f_1(x) - e^{-2\pi i x} f_1(x))$$

$$F_4(\omega) = \frac{1}{2i}[F_1(\omega - 1) - F_1(\omega + 1)] = e^{-3\pi i \omega} \frac{sin(\pi\omega)}{i\pi(\omega^2 - 1)}$$

7.

$$f_5(x) = e^{-|x|} cos(2\pi\alpha x) = \frac{1}{2}(e^{2\pi i \alpha x} + e^{-2\pi i \alpha x}) f_2(x)|_{a=1}$$

$$F_5(\omega) = \frac{1}{2}(F_2(\omega - \alpha) + F_2(\omega + \alpha)) = \frac{1}{1 + 4\pi^2(\omega - \alpha)^2} + \frac{1}{1 + 4\pi^2(\omega + \alpha)^2}$$

Then replace α by a:

$$f(x) = e^{-|x|} cos(2\pi a x) \quad \rightarrow \quad F_5(\omega) = \frac{1}{1 + 4\pi^2(\omega - a)^2} + \frac{1}{1 + 4\pi^2(\omega + a)^2}$$

8.

$$f_6(x) = \frac{cos(2\pi x)}{x^2 + a^2} = \frac{1}{2}(e^{2\pi i x} + e^{-2\pi i x}) f_3(x)$$

$$F_6(\omega) = \frac{1}{2}(F_3(\omega - 1) + F_3(\omega + 1)) = \frac{\pi}{2a}\left[e^{-2\pi a|\omega - 1|} + e^{-2\pi a|\omega + 1|}\right]$$