1. Note the function $f$ is the convolution between the box function $\Pi$ and the unit Gaussian function $\gamma$, that is:

$$f(x) = \int_{-\infty}^{\infty} \Pi(u)\gamma(x-u)du \quad , \quad \Pi(u) = \left\{ \begin{array}{ccc} 1 & , & |u| < \frac{1}{2} \\ 0 & , & |u| > \frac{1}{2} \end{array} \right. \quad , \quad \gamma(u) = e^{-\pi|u|^2}$$

Thus the Fourier transform is the product of the Fourier transforms of $\Pi$ and $\gamma$, respectively:

$$F(s) = sinc(s)e^{-\pi|s|^2} = sinc(s)\gamma(s)$$

2. Note

$$f_1 = -\frac{1}{2}\frac{df_0}{dx}$$

The Fourier transforms of $f_0$ and $f_1$ are:

$$F_0(s) = \sqrt{\pi}e^{-\pi^2 s^2}$$

respectively

$$F_1(s) = -\pi i s F_0(s) = -is\pi\sqrt{\pi}e^{-\pi^2 s^2}.$$

Let $g_0 = f_0 * f_0$ and $g_1 = f_0 * f_1$. Then their Fourier transforms are:

$$G_0(s) = F_0(s)^2 = \pi e^{-2\pi^2 s^2}$$

$$G_1(s) = F_0(s)F_1(s) = -is\pi^2 e^{-2\pi^2 s^2}$$

Next we need to compute the inverse Fourier transforms. Again apply the dilation and the derivative rules, and get:

$$g_0(x) = \sqrt{\frac{\pi}{2}}e^{-x^2/2}$$

$$g_1(x) = -\frac{1}{2}\frac{dg_0}{dx} = \frac{x}{2}\sqrt{\frac{\pi}{2}}e^{-x^2/2}$$

3.
(Preferred Solution)

Let $F$ denote the Fourier transform of $f$. Define $g : R \to C$, $g(u) = \overline{f(x-u)}$. The Fourier transform of $g$ is $G : R \to C$ given by

$$G(s) = e^{-2\pi isx}\overline{F(s)}$$

Then the Parseval identity (which requires $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$) implies:

$$\int_{-\infty}^{\infty} f(u)f(x-u)du = \int_{-\infty}^{\infty} f(u)\overline{g(u)}du = \int_{-\infty}^{\infty} F(s)\overline{G(s)}ds = \int_{-\infty}^{\infty} e^{2\pi isx}(F(s))^2 ds$$

Thus $f$ (or $F$) must satisfy the following equation:

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi isx}(F(s))^2 ds$$

This is the inversion formula for $f$, hence $(F(s))^2$ must be the Fourier transform of $f$ at $s$, that is:

$$F(s) = (F(s))^2$$

A non-zero solution for this equation is:

$$F(s) = \left\{ \begin{array}{ccc} 1 & , & |s| \leq \frac{1}{2} \\ 0 & , & |s| > \frac{1}{2} \end{array} \right.$$

Then the function $f$ is the *sinc* function:

$$f(x) = sinc(s)$$

Note $\int_{-\infty}^{\infty} |sinc(x)|^2 dx < \infty$ as required by Parseval.

(Second Solution)

Note the integral equation is equivalent to the convolution equation:

$$f = f * f$$

Apply Fourier transform on both sides. We get:

$$F(s) = F(s)^2$$

A non-zero solution to this equation is:

$$F(s) = \begin{cases} 1 & , \quad |s| \leq \frac{1}{2} \\ 0 & , \quad |s| > \frac{1}{2} \end{cases}$$

Then the function $f$ is the *sinc* function:

$$f(x) = sinc(s)$$

(Why the first solution is the preferred solution: The function $f$ is not absolutely integrable. It is not obvious that the Fourier transform of the convolution is the product of the corresponding Fourier transforms. Instead the Parseval/Plancherel identities are suited to square-integrable functions.)

4. Apply Fourier transform to both sides of this equation. We obtain:

$$-2\pi i s F(s) + F(s) = \frac{1}{1 + 2\pi i s}$$

Thus:

$$F(s) = \frac{1}{1 + 4\pi^2 s^2}$$

Recall the Fourier transform of $g : R \to R$, $g(x) = e^{-|x|}$, is given by $G(s) = \frac{2}{1+4\pi^2 s^2}$. Thus $f$ is given by

$$f(x) = \frac{1}{2} e^{-|x|}$$

which is continuous (but not differentiable!).

Note: if you don't recall the Fourier transform of $e^{-|x|}$ you can obtain the inverse Fourier transform as follows:

$$\frac{1}{1 + 4\pi^2 s^2} = \frac{1}{2} \frac{1}{1 - 2\pi i s} + \frac{1}{2} \frac{1}{1 + 2\pi i s} = \frac{1}{2} E(-s) + \frac{1}{2} E(s)$$

where $E(s) = \frac{1}{1+2\pi i s}$. The inverse Fourier transform of $E$ is the truncated decaying exponential (the right hand side of the initial equation). The inverse Fourier transform of $E(-s)$ is the reflected truncated decaying exponential. Putting these two functions together we get again $f(x) = \frac{1}{2} e^{-|x|}$.