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Mid-Term Exam Solutions

MATH 464

1. We need to find the inverse Fourier transform of the function $F : R \to R$ defined by $F(s) = \frac{1}{2}(sinc(s/2))^2$. Using linearity and the property that the (inverse) Fourier transform of a product of functions is the convolution of the (inverse) Fourier transforms of the individual functions, we obtain:

$$\mathcal{F}^{-1}\{F\}(x) = \frac{1}{2}\mathcal{F}^{-1}\{sinc(\frac{s}{2})\} * \mathcal{F}^{-1}\{sinc(\frac{s}{2})\}$$

Now, using the dilation rule:

$$\mathcal{F}^{-1}\{sinc(\frac{s}{2})\}(x) = 2\mathcal{F}^{-1}\{sinc\}(2x) = 2\Pi(2x)$$

We obtain further:

$$f(x) = \mathcal{F}^{-1}\{F\}(x) = 2\int_{-\infty}^{\infty} \Pi(2(x-u))\Pi(2u)du = \int_{-\infty}^{\infty} \Pi(2x-t)\Pi(t)dt = \Pi * \Pi(2x)$$

The convolution of two box functions gives a "tent" function:

$$\Lambda(x) = \Pi * \Pi(x) = \begin{cases} x+1 & for -1 < x <= 0 \\ 1-x & 0 < x < 1 \\ 0 & otherwise \end{cases}$$

Hence the result is:

$$f(x) = \Lambda(2x) = \begin{cases} 2x+1 & for -\frac{1}{2} < x \leq 0 \\ 1-2x & for 0 < x < \frac{1}{2} \\ 0 & otherwise \end{cases}$$

2. Note:

$$\sum_k \frac{c_{k-5} - c_{k+5}}{2i} e^{2\pi ikx} = \frac{1}{2i}\sum_k c_{k-5}e^{2\pi i(k-5+5)x} - \frac{1}{2i}\sum_k c_{k+5}e^{2\pi i(k+5-5)x} =$$

$$= \frac{1}{2i}(e^{10\pi ix} - e^{-10\pi ix})\sum_n c_n e^{2\pi inx} = sin(10\pi x)f(x)$$

The plot of f is the indicator function of the interval $(0, 0.1)$. The plot of g is the first positive oscillation of the sinusoid squeezed in the interval $(0, 0.1)$ and then constant 0 on $(0.9, 1)$.

3.

(a) The signal has a bandwidth of 30KHz. Hence the largest sampling period that allows perfect recovery is $T_{max} = 1/(30,000)s = 33.333\mu s$.

(b) Use the formula we derived in class with $\Omega_1 = -10,000$ and $\Omega_2 = 20,000$ and we get

$$f(x) = \sum_{n=-\infty}^{\infty} f(nT_1)e^{10000\pi i(x-nT_1)} sinc\left(\frac{x-nT_1}{T_1}\right)$$

4. The Fourier transform of $e^{-|x|}$ is $\frac{2}{1+4\pi^2 s^2}$. Apply the polynomial rule to obtain

$$\mathcal{F}(xe^{-|x|})(s) = \frac{i}{2\pi}\frac{d}{ds}\left[\frac{2}{1+4\pi^2 s^2}\right] = -\frac{8\pi is}{(1+4\pi^2 s^2)^2}$$

5. Apply the Fourier transform to this equation and obtain:

$$(2\pi is + a)F(s) = G(s)$$

where $G(s)$ is the Fourier transform of the right hand side. This is given by

$$G(s) = \int_{-\infty}^{0} 2ae^{ax}e^{-2\pi isx}dx = 2a\frac{1}{a-2\pi is}(e^{ax-2\pi isx})|_{-\infty}^{0} = \frac{2a}{a-2\pi is}$$

Thus

$$F(s) = \frac{2a}{(a+2\pi is)(a-2\pi is)} = \frac{2a}{a^2+4\pi^2 s^2}$$

Its inverse Fourier transform is:

$$f(x) = e^{-a|x|}$$

which is the solution to this problem.