Homework #2
Due: Tuesday, September 25, 2007

1. Consider the initial value problem: $\dot{x} = f(t), x(0) = 0$
   a. (3pt) Give an example when f in not continuous at 0, and yet the IVP has a differentiable solution.
   b. (4pt) Does the above IVP have a solution for

   $$f(t) = \begin{cases} 0 & t = 0 \\ \sin(1/t) & t \neq 0 \end{cases} \quad ?$$

   Prove the claim.

2. (3pt) Suppose f(t,x) is continuous and uniformly bounded on R^d. Show that for all x_0 the initial-value problem

   $$\begin{aligned} \dot{x} &= f(t,x) \\ x(t_0) &= x_0 \end{aligned} \qquad (*)$$

   has a solution defined on an arbitrary long interval.

3. (5pt) Suppose f(t,x) is continuous and satisfies $|f(t,x)| \leq \log(1+|t|+|x|)$ on R^d. Show that for all x_0 the initial-value problem (*) of problem (2) has a solution defined on an arbitrarily long interval.

4. Let $f{:}R^2 \to R$ be continuous and suppose that for every t the function f(t,x) is nonincreasing in x, that is, $f(t,x) \leq f(t,y)$, when x>y.
   a. (3pt) Let $\varphi_1(t)$ and $\varphi_2(t)$ be two solutions of $\dot{x} = f(t,x)$ defined on an open interval I. Show that $\varphi_1(t_0) = \varphi_2(t_0)$ for some t_0 in I, then $\varphi_1(t) = \varphi_2(t)$ for $t \geq t0$.
   b. (2pt) Show that

   $$f(t,x) = \begin{cases} |x|^{1/2} & \text{for } x < 0 \\ 0 & \text{otherwise} \end{cases}$$

   satisfies the above condition and the initial-value problem

   $$\begin{aligned} \dot{x} &= f(t,x) \\ x(0) &= 0 \end{aligned}$$

   does not have a unique solution.

Total: 20 pts