

Homework #3
Due: Tuesday, October 9, 2007

1. (5pt) Let $f:(c-2a,c+2a)\rightarrow\mathbb{R}$ be a continuous function so that $f(x)=0$ has unique solution $x=c$. Show that the scalar initial-value problem:

$$\dot{x} = f(x)$$

$$x(0) = c$$

has a unique solution if and only if the improper integrals

$$\int_{c-a}^c \frac{dx}{f(x)} \quad \text{and} \quad \int_c^{c+a} \frac{dx}{f(x)}$$

are both infinite.

2. (5pt) Consider the scalar autonomous differential equations $\dot{x} = f(x)$, where $f(x)$ has a continuous first derivative on \mathbb{R} . Suppose $f(c)=0$ and $f(x)>0$, when $x>c$. For $\xi>c$, show that $t\rightarrow x(t,0,\xi)$ is defined on the interval

$$-\infty < t < \int_{\xi}^{\infty} \frac{dw}{f(w)}$$

3. The scalar differential equation $\dot{x} = f(t, x)$ on a domain $D\subset\mathbb{R}^2$ is said to be *exact* if there exists a function $E:D\rightarrow\mathbb{R}$ such that:

$$f(t, x) = -\frac{\frac{\partial E}{\partial t}}{\frac{\partial E}{\partial x}}.$$

- a. (1pt) Show that separable equations are exact.
- b. (1pt) Let $\varphi(t)$ be a solution of an exact differential equation. Show that $t\rightarrow E(t,\varphi(t))$ is a constant function.
- c. (3pt) Find the solutions of

$$\dot{x} = -\frac{bx + ct}{ax + bt}$$

on $D = \{(t, x) : ax + bt \neq 0\}$.

4. (5pt) Let $E(x)$ be an integral for $\dot{x} = f(x)$ on an open set D , and assume that f is locally Lipschitz on D . Show that if the set $R = \{x \in D : E(x) = C\}$ is compact and $E(\xi)=C$, then the solution $t\rightarrow x(t,0,\xi)$ is defined on \mathbb{R} .

Total: 20 pts.