Homework #4
Due: Tuesday, October 16, 2007

1. Draw a phase portrait for each of the following autonomous systems of differential equations:
   a. (2pt)
   $$\dot{x} = xy^2$$
   $$\dot{y} = x^2 y$$
   b. (2pt)
   $$\dot{x} = -y(x^2 + y^2 - 1)$$
   $$\dot{y} = x(x^2 + y^2 - 1)$$
   c. (2pt)
   $$\dot{x} = -y((x+1)^2 + y^2)$$
   $$\dot{y} = x((x+1)^2 + y^2)$$

2. (4pt) Find an integral for the system
   $$\dot{x} = y$$
   $$\dot{y} = x + x^2$$
   and sketch its phase portrait.

3. (10pt: 2 for writing the system, 2 for each case)Consider the scalar second-order differential equation $\ddot{x} = F(x)$, where F:R$\to$R is locally Lipschitz. Suppose that F(x) is a restoring force, xF(x)<0 for x$\neq$0, and F(0)=0. Show that $V(x) = -\int_o^x F(w)dw$ is increasing for x>0 and decreasing for x<0. Let $\alpha = \lim_{x\to-\infty} V(x)$ and $\beta = \lim_{x\to\infty} V(x)$. Write $\ddot{x} = F(x)$ as a system of differential equations on R^2 and sketch the phase portrait in each of the following situations:
   a. $\alpha = \beta = \infty$,
   b. $\alpha < \beta = \infty$,
   c. $\alpha < \beta < \infty$,
   d. $\alpha = \beta < \infty$.

4. (10pt: 2 for each case) Each of the scalar autonomous differential equations shown below contains a real parameter μ. Sketch all possible phase portraits for them as μ varies through R.
   a. $\dot{x} = x^2 + \mu^2 + 1$,
   b. $\dot{x} = x^2 + \mu^2$,
   c. $\dot{x} = x^2 + 2\mu x + 1$,

d. $\dot{x} = x(x^2 - \mu)$,

e. $\dot{x} = (x^2 - \mu)(x^2 - \mu^3)$.

5. (10pt) Sketch the phase portrait for the system:

$$\dot{x} = -y + x(1 - x^2 - y^2 - z^2)$$
$$\dot{y} = x + y(1 - x^2 - y^2 - z^2)$$
$$\dot{z} = 0.$$

Total: 40 pts