Chapter 6: Applications of the Integral

Section 6.1: Volume

(6.1.1) Volume. Let D be a solid region and $A(x)$ be the cross-sectional area of D at x. The volume of D is defined to be the integral of $A(x)$ that is

$$V = \int_a^b A(x)\, dx$$

(6.1.2) Disc Method. Let f be a nonnegative function on an interval $[a, b]$.

- the solid region D generated by revolving the graph of f about the x-axis
- the cross section at x is a circle with radius $f(x)$, so the cross-sectional area at x is given by

$$A(x) = \pi[f(x)]^2$$

- the volume of D is

$$V(x) = \int_a^b \pi[f(x)]^2\, dx$$

(6.1.3) Washer Method. Let f and g be nonnegative functions on an interval $[a, b]$ such that $g(x) \leq f(x)$ for $a \leq x \leq b$.

- the solid region D generated by revolving a region between the graph of f and the graph of g.
- the cross section of D at x is an annulus with radii $f(x)$ and $g(x)$, so the cross-sectional area at x is given by

$$A(x) = \pi[f(x)^2 - g(x)^2]$$

- the volume of D is

$$V(x) = \int_a^b \pi[f(x)^2 - g(x)^2]\, dx$$

(6.1.4) Shell Method. Let f and g be continuous functions on $[a, b]$ with $a \geq 0$ and $g(x) \leq f(x)$,

- the solid region D generated by revolving about the y-axis the region R between the graphs of f and g on $[a, b]$.
- the cylindrical shell at x has base $2\pi x$ and the height is $f(x) - g(x)$, so the area of the cylindrical shell is

$$A(x) = 2\pi x(f(x) - g(x))$$

- the volume of D is

$$\int_a^b 2\pi x(f(x) - g(x))\, dx$$

Section 6.2: Length of a Curve

Let f have a continuous derivative on $[a, b]$. Then the length L of the graph of f on $[a, b]$ is defined by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Section 6.4: Work

(6.4.1) Work. Let F be a work function that is continuous on $[a,b]$. Then the work W done by the force F is defined by

$$W = \int_a^b F(x)\, dx$$

(6.4.2) Hooke's Law. We want to figure out the force $F(x)$ needed in order to hold a spring extended x units beyond its natural length. This is the opposite of the elastic force $G(x)$ exterted by a spring which has been extended x units beyond its natural length. We have that $F(x) = -G(x) = -(-kx) = kx$ where k is the constant factor characteristic of the spring.

(6.4.3) Pumping water from a tank. We want to compute the work required to pump water from a tank. Note that the weight density of water is approximately $62.5\text{lb}/\text{ft}^3$. Therefore the weight of water is $62.5V$ where V is the volume of the water in cubic feet. Let l be the level to which the water is to be pumped. The volume of the infinitesimal cylinder at x can be written as $A(x)\, dx$. In conclusion the work done is given by

$$W = \int_a^b 62.5(l - x)A(x)\, dx$$

(6.4.4) Recollection From Physics. $F = ma$. Pound is a unit of weight (a type of force). Density ρ is mass divided by volume, or $\rho = m/V$. However, we can also consider weight density, i.e. $D = \rho g$ where g is the gravity on the surface of the Earth (i.e. an acceleration). The weight w which is a force is given by $w = mg$ where m is the mass. There may be some confusions because lb is interchangeably used to be mass or weight. However, in this course, pound is a unit of weight, namely $1\text{lb} \approx 4.44\text{N}$. For mass, some people use lbm, so that $1\text{lbm} \approx 0.453\text{kg}$.

Section 6.5: Moments and Center of Gravity

(6.5.1) Moments. Let f and g be continuous functions on $[a,b]$ such that $g(x) \leq f(x)$ for $a \leq x \leq b$. Let R be the region between the graphs of f and g on $[a,b]$. Then the moment M_x of R about the x-axis is given by

$$M_x = \int_a^b \frac{1}{2}\left(f(x)^2 - g(x)^2\right)\, dx$$

and the moment M_y of R about the y-axis is given by

$$M_y = \int_a^b x(f(x) - g(x))\, dx$$

(6.5.2) Moments: Background. Let us first consider an idealized situation. An object of positive mass m is concentrated at a point (x,y) is called a point mass. Roughly, the moment M_y about the x-axis measures the tendency of the point mass to rotate about the y-axis. Thinking of a seesaw, the moment should be proportional to the distance of the point mass to the y-axis which is given by x and to the mass m. Therefore, we define the moment about the y-axis to be $M_y = mx$.

Now we generalize the situation to 2-dimensions. Let R be a region between the graph of f and g. We assume that the mass is distributed uniformly throughout a plane region R and the mass of the subregion of R is given by its area. Therefore, we have that the infinitesimal moment is

$$\text{(distance to the } y\text{-axis)} \times \text{(area of the infinitesimal area)} = x \cdot (f(x) - g(x))\, dx$$

Here x can be thought of the distance between the y-axis and the center of the subregion, $f(x) - g(x)$ is the height, and dx is the width of the subregion. This defines the M_y.

Similarly, the moment M_x of a point mass about the $x-$axis is again given by $M_x = my$. If we have infinitesimal subregion, then we have to pick a point. The y-coordinate of the center of the subregion is $\frac{f(x)-g(x)}{2}$, therefore, we have the infinitesimal moment is

$$\begin{aligned}\text{(distance between the } x\text{-axis and the center)} \times \text{(area of the infinitesimal area)} &= \frac{1}{2}(f(x) + g(x)) \cdot (f(x) - g(x))\, dx \\ &= \frac{1}{2}(f(x)^2 - g(x)^2)\, dx\end{aligned}$$

(6.5.3) Center of Gravity. If R has a positive area A, then the center of gravity (or center of mass, or centroid) of R is the point $(\overline{x}, \overline{y})$ defined by

$$\overline{x} = \frac{M_y}{A} \qquad \text{and} \qquad \overline{y} = \frac{M_x}{A}$$

Section 6.6: Parametrized Curves

(6.6.1) Graph of a function. Let f be a continuous function on $[a, b]$. Then the graph C of f is a parametrized curve. A parametrization of C is

$$x = t \qquad \text{and} \qquad y = f(t)$$

(6.6.2) Lines. Let $f(x) = mx + b$ be a linear function. Then the graph C of f is a line. By (6.6.1), a parametrization of C is

$$x = t \qquad \text{and} \qquad y = mt + b$$

(6.6.3) Circles and Ellipses. Let r be the radius of a circle C with center $(0, 0)$. Then a parametrization of C is

$$x = r\cos(t) \qquad \text{and} \qquad y = r\sin(t)$$

More generally, let C be an ellipse with length $2a$ of horizontal axis and length $2b$ the vertical axis, and center $(0, 0)$. Then a parametrization of C is

$$x = a\cos(t) \qquad \text{and} \qquad y = b\sin(t)$$

(6.6.4) Translations and Dilations.

Section 6.7: Length of a Curve given Parametrically

(6.7.1) Arc Length. Let C be a curve parametrized by

$$x = f(t) \qquad \text{and} \qquad y = g(t)$$

Then the length L of the curve C is given by

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt$$

Chapter 7: Inverse Functions, L'Hôpital's Rule, and Differential Equations

Section 7.1: Inverse Functions

(7.1.1) Keywords. Inverse functions, Criteria of the existence of inverse functions, Derivative of inverse functions

(7.1.2) Inverse Function.

Let f be a function. A function g is an inverse function of f if

- the domain of g is the range of f,
- $f(x) = y$ if and only if $g(y) = x$ for all x in the domain of f and all y in the range of f.

If f has an inverse function g, then we say that f is invertible. One can show that an inverse function of f is unique, so we will denote by f^{-1}.

(7.1.3) Properties of inverse functions.

(i) To find an inverse function g of f, one solves $y = f(x)$ for x in terms of y. Then switch x and y. Consider $y = x^5$ for example. Solve for x to get $x = y^{1/5}$. Then if we switch x and y, we get $y = x^{1/5}$. In other words, $g(x) = x^{1/5}$.

(ii) An inverse function doesn't always exist. The constant function $y = 5$ does not have an inverse.

(iii) An inverse function may exist if we *restrict* the domain of a function. The usual $y = \sin x$ does not have an inverse function on $(-\infty, \infty)$. However, if we restrict our domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then $y = \sin x$ has the inverse $y = \sin^{-1} x$ on $[-1, 1]$.

(iv) The graph of the inverse function f^{-1} of f is the reflection of the graph of f with respect to $y = x$.

Let I be an interval. We say that f has an inverse function on an interval I (or is invertible on I) if function f restricted to I has an inverse function. By (iii) above, we see that $f(x) = \sin x$ is not invertible, but invertible on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(v) The domain of f is the range of f^{-1} and the range of f is the domain of f.

(vi) The inverse of f^{-1} is f, i.e. $(f^{-1})^{-1} = f$.

(vii) $f^{-1}(f(x)) = x$ for all x in the domain of f and $f(f^{-1}(y)) = y$ for all y in the range of f.

(7.1.4) Criteria of the existence of inverse functions.

A function f has an inverse if and only if two numbers x_1 and x_2 in the domain of f, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. If a function f satisfies such condition, we say that f is one-to-one.

Then we have the following special case.

Every increasing function and every decreasing fuction has an inverse.

Let f be a function with domain I. Then we can divide I into subintervals $I_1, \ldots, I_n$ such that $f'(x) > 0$ or $f'(x) < 0$ for all $x \in I_i$. Then f restricted to I_i is invertible. Therefore, first find critical points of f.

(7.1.5) Derivative of Inverse Functions.

Assume $f'(a)$ exists, $f'(a) \neq 0$, and $f(a) = c$. Then $(f^{-1})'(c) = \dfrac{1}{f'(a)}$

Section 7.2: The Natural Exponential Function

The natural logarithm function is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

on $(0, \infty)$. The derivative of $\ln x$ is $\frac{1}{x}$ (by fundamental theorem of calculus), so it is always positive. This shows that $\ln x$ is an increasing function, and has an inverse function. We denote by $y = e^x$ the inverse function and call it the natural exponential function.

- $e^{a+b} = e^a \cdot e^b$
- $(e^x)' = e^x$
- $\displaystyle\int e^x\,dx = e^x + C$

Section 7.3: General Exponential and Logarithmic Functions

For $a > 0$, we define

$$a^r = e^{r \ln a}$$

for any real number r.

- $a^0 = 1$
- $a^1 = a$
- $\ln a^r = r \ln a$
- $a^{b+c} = a^b a^c$
- $a^{-b} = 1/a^b$
- $(a^b)^c = a^{bc}$

$$\frac{d}{dx}a^x = (\ln a)a^x \qquad \text{and} \qquad \int a^x\,dx = \frac{1}{\ln a}a^x + C$$

$$\log_a x = \frac{\ln x}{\ln a}$$

For a, b, and c positive real numbers,

$$\log_a bc = \log_a b + \log_a c$$

Section 7.5: The Inverse Trigonometric Functions

Function	Domain	Range
$\sin^{-1}(x)$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}(x)$	any x	$(-\pi/2, \pi/2)$
$\cot^{-1}(x)$	any x	$(0, \pi)$
$\sec^{-1}(x)$	$(-\infty, 1] \cup [1, \infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1}(x)$	$(-\infty, 1] \cup [1, \infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}+C$$

$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = -\cos^{-1}\frac{x}{a}+C$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1} \qquad \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C$$

$$\frac{d}{dx}\cot^{-1}x = \frac{-1}{x^2+1} \qquad \int \frac{1}{x^2+a^2}\,dx = -\frac{1}{a}\cot^{-1}\frac{x}{a}+C$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}} \qquad \int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a}+C$$

$$\frac{d}{dx}\csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}} \qquad \int \frac{1}{x\sqrt{x^2-a^2}}\,dx = -\frac{1}{a}\csc^{-1}\frac{x}{a}+C$$

SECTION 7.6: L'HÔPITAL'S RULE

Suppose that $\frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ and $\frac{\infty}{\infty}$, and assume that $g'(x) \neq 0$ for x near $*$. Then

$$\lim_{x\to *}\frac{f(x)}{g(x)} = \lim_{x\to *}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists (as a number, or as ∞ or $-\infty$).

There might be cases where we are in the following indeterminate form $0\cdot\infty, 0^0, 1^\infty, \infty^0$ and $\infty-\infty$. In this case, we try to change to the limit in a form where we can use l'Hôpital's Rule.

CHAPTER 8: TECHNIQUES OF INTEGRATION

SECTION 8.1:

SECTION 8.2: TRIGONOMETRIC INTEGRALS

Type A: $\int \sin^m(x)\cos^n(x)\,dx$.

(A-1) n is odd:

(Ex)
$$\int \sin^3(x)\cos^7(x)\,dx$$

(a) Factor out $\cos(x)$.

$$\int \sin^3(x)\cos^6(x)\cos(x)\,dx$$

(b) Write the rest of the integrand in terms of $\sin x$ using the identity $\cos^2(x) = 1-\sin^2(x)$.

$$\int \sin^3(x)(\cos^2(x))^3\cos(x)\,dx = \int \sin^3(x)(1-\sin^2(x))^3\cos(x)\,dx$$

(c) do u-substitution with $u = \sin(x)$ and $du = \cos(x)\,dx$.

$$\int u^3(1-u^2)^3du$$

(A-2) m is odd:

The procedure is analogous to Type A-1, but you will factor out $\sin(x)$ and express the rest of the integrand in terms of $\cos(x)$ using $\sin^2(x) = 1-\cos^2(x)$. You have to be careful with the sign because now we will have $u = \cos(x)$ and $du = -\sin(x)\,dx$.

(A-3) Both m and n are even:

Use the following identities to reduce the powers m and n,

$$\begin{aligned} \sin(x)\cos(x) &= \frac{1}{2}\sin(2x) \\ \sin^2(x) &= \frac{1-\cos(2x)}{2} \\ \cos^2(x) &= \frac{1+\cos(2x)}{2} \end{aligned}$$

Type B: $\int \tan^m(x)\sec^n(x)\,dx$.

(B-1) n is even:

(Ex) $$\int \tan^5(x)\sec^6(x)\,dx$$

(a) Factor out $\sec^2(x)$

$$\int \tan^5(x)\sec^4(x)\sec^2(x)\,dx$$

(b) Write the rest of the integrand in terms of $\tan(x)$ using the identity $\tan^2(x)+1=\sec^2(x)$.

$$\int \tan^5(x)(\sec^2(x))^2\sec^2(x)\,dx = \int \tan^5(x)(\tan^2(x)+1)^2\sec^2(x)\,dx$$

(c) do u-substitution with $u=\tan(x)$ and $du=\sec^2(x)\,dx$

$$\int u^5(u^2+1)\,du$$

(B-2) m is odd:

(Ex) $$\int \tan^5(x)\sec^6(x)\,dx$$

(a) Factor out $\sec(x)\tan(x)$

$$\int \tan^4(x)\sec^5(x)\sec(x)\tan(x)\,dx$$

(b) Write the rest of the integrand in terms of $\sec(x)$ using the identity $\tan^2(x)=\sec^2(x)-1$

$$\int (\tan^2(x))^2\sec^5(x)\sec(x)\tan(x)\,dx = \int (\sec^2(x)-1)^2\sec^5(x)\sec(x)\tan(x)\,dx$$

(c) do u-substitution with $u=\sec(x)$ and $du=\sec(x)\tan(x)\,dx$.

$$\int (u^2-1)^2u^5\,du$$

(B-3) m is even and n is odd:

Since m is even, we have $\tan^m(x) = (\tan^2(x))^{m/2} = (\sec^2(x)-1)^{m/2}$. So we can reduce the problem to finding integrals of the form $\int \sec^n(x)\,dx$ where n is odd. When $n=1$,

$$\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)|+C$$

and when $n=3$,

$$\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x)+\frac{1}{2}\ln|\sec(x)+\tan(x)|+C$$

But for $n\geq 5$, you have to use integration by parts.

Type C: $\int \cot^m(x)\csc^n(x)\ dx$.

This is similar to Type B except you need to be careful with the signs are

$$(\cot(x))' = -\csc^2(x) \quad \text{and} \quad (\csc(x))' = -\csc(x)\cot(x)$$

Type D: $\int \sin(ax)\cos(ax)\ dx$, $\int \sin(ax)\sin(bx)\ dx$ and $\int \cos(ax)\cos(bx)\ dx$. Use the following identities

$$\begin{aligned} \sin(\alpha)\cos(\beta) &= \frac{1}{2}\sin(\alpha-\beta)+\frac{1}{2}\sin(\alpha+\beta) \\ \sin(\alpha)\sin(\beta) &= \frac{1}{2}\cos(\alpha-\beta)-\frac{1}{2}\cos(\alpha+\beta) \\ \cos(\alpha)\cos(\beta) &= \frac{1}{2}\cos(\alpha-\beta)+\frac{1}{2}\cos(\alpha+\beta) \end{aligned}$$

Type E: Otherwise. We use trig identities to use u-substitution or to write the integrand in one of Types A-D.