Worksheet 1 **Name:** ____________________

MATH 141 (Fall 2023)

Aug 29th, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

All problems are from Review Exercises on page 363.

Problem 1. (Exercise 4)

$$\int (3\cos x - 2\sin x)\,dx$$

Problem 2. (Exercise 7)

$$\int \frac{1+\sqrt{x+1}}{\sqrt{x+1}}\,dx$$

Problem 3. (Exercise 12)*

$$\int \frac{\tan^2 x}{x-\tan x}\,dx$$

Problem 4. (Exercise 17)

$$\int_{-8}^{-2} \frac{-1}{5u}\,du$$

Problem 5. (Exercise 18)*

$$\int_{-3}^{2} (u+|u|)\,du$$

Problem 6. (Exercise 36) Find the derivative of

$$G(y) = \int_{2y}^{\sin y} t\sin^2 t\,dt$$

Hint.

- (Problem 3) Use the following inequality: $\tan^2 x + 1 = \sec^2 x$
- (Problem 5) Break down the integral into two parts so that you don't have to work with the absolute value function.

Worksheet 2 **Name:** ______________________________
MATH 141 (Fall 2023)
Sep 5th, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

Problem 1. (Basic definition) A 20-pound bag of groceries is to be carreid up a flight of stairs 4 feet tall. Find the work W done on the bag.

Problem 2. (Basic definition) A bottle of wine has a cork 7 centimeters long. A person uncorking bottle exerts a force to overcome the force of friction between the cork and the bottle. Suppose applied force in dynes is given by

$$F(x) = 3 \times 10^3(2 - x)$$

for $0 \leq x \leq 2$ where x represents the length in centimeters of the cork extending from the bottle. Determine the work W done in removing the cork.

Problem 3. (Hooke's Law) Suppose a certain spring is expanded 6 centimeters from its natural position and held fixed, the force necessary to hold it is 12×10^3 dynes. Find the work W required to stretch the spring an additional 6 centimers.

Problem 4. (Pumping water from a tank) A tank in the shape of an inverted cone 18 feet tall and 6 feet in radius is full of water. Calculate the work W required to pump all the water over the edge of the tank.

Problem 5. Suppose that a ball of weight 0.3 pound is thrown vertically upward with its height after t seconds $h(t) = 3 + 6t - 9t^2$. Find the work W done on the ball by gravity while the ball descends from its maximum height to the ground.

Worksheet 3 **Name:** ______________________________
MATH 141 (Fall 2023)
Sep 12th, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

For problems 1-4, find an equation relating x and y when possible. Then sketch the curve C whose parametric equations are given, and indicate the direction $P(t)$ moves as t increases.

Problem 1. $x = 2 - \cos t$ and $y = -1 - \sin t$ for $0 \le t \le 2\pi$,

Problem 2. $x = 3$ and $y = -1 - t$ for $0 \le t < 1$,

Problem 3. $x = t^3$ and $y = t^2$ for all t,

Problem 4. $x = e^{-t}$ and $y = e^{3t}$ for all t,

Problem 5. Let $a, b > 0$, and consider the ellipse parametrized by $x = a\cos t$ and $y = b\sin t$ for $0 \le t \le 2\pi$. Find a representation of the ellipse in rectangular coordinates.

Problem 6. Consider the Folium of Descartes, parametrized by

$$x = \frac{3t}{1+t^3} \text{ and } y = \frac{3t^2}{1+t^3} \text{ for all } t \neq -1$$

Find an equation in rectangular coordinates for the folium. (*Hint*: Compute $x^3 + y^3$.)

For problems 7-8, find the length L of the graph of the given function.

Problem 7. $f(x) = x^2 - \frac{1}{8}\ln x$ for $2 \le x \le 3$,

Problem 8. $k(x) = x^4 + \frac{1}{32x^2}$ for $1 \le x \le 2$.

Problem 9. Let f be a function defined on $[2, 3]$ and $f'(x) = \sqrt{x^2 - 1}$. Find the length L of the graph of f.

Problem 10. Let $r > 0$. The graph of the equation $x^{2/3} + y^{2/3} = r^{2/3}$ is called an astroid.

(a) For the portion of the astroid in the first quadrant, express y as a function of x.

(b) Let $\varepsilon > 0$. Find the length L_ε of the portion of the astroid in the first quadrant for $\varepsilon \le x \le r$. Then find the limit $L = \lim_{\varepsilon \to 0^+} L_\varepsilon$, and thereby determine the length of one-fourth of the astroid and hence the length of the whole astroid.

For problems 11-12, find the length L of the curve described parametrically.

Problem 11. $x = 1 - t^2$ and $y = 1 + t^3$ for $0 \le t \le 1$,

Problem 12. $x = \sin t - t\cos t$ and $y = t\sin t + \cos t$ for $0 \le t \le \pi/2$.

Problem 13. Let $a, b > 0$, and consider the elllipse parametrized by $x = a\cos t$ and $y = b\sin t$ for $0 \le t \le 2\pi$.

(a) Find an integral that represents the circumference C_{ab} of the ellipse.

(b) Use the formula obtained in (a) to find C_{ab} when $a = b > 0$.

(c) Use the result of (a) to determine $\lim_{b \to 0^+} C_{ab}$.

Problem 14. Let $r > 0$. The equations $x = r\cos^3 t$ and $y = r\sin^3 t$ for $0 \le t \le 2\pi$ parametrize an astroid.

(a) Find the length L of the astroid.

(b) Show that the astroid is alternatively the graph of $x^{2/3} + y^{2/3} = r^{2/3}$.

Pratice Test 1 **Name:** ______________________________

MATH 141 (Fall 2023)

Sep 14th, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

Problem 1. (Spring 2019, Problem 1(a)) Let $f(x) = \frac{1}{\sqrt{4-x}}$ for $0 \le x \le 1$, and let R be the bounded region between the graph of f and the x-axis. Find the volume V of the solid obtained by revolving R about the x axis.

Problem 2. (Spring 2016, Problem 1(b)) Let $g(x) = x^3 + \frac{1}{12x}$, for $1 \le x \le 3$. Find the length L of the graph of g.

Problem 3. (Fall 2011, Problem 2(b)) Suppose that extending a spring 2 meters beyond its natural length requires 14 joules of work. Find the work W required to extend the spring from 2 meters beyond its natural length to 4 meters beyond its natural length.

Problem 4. (Spring 2018, Problem 2(a)) A tank has the shape of the surface obtained by rotating the curve $y = x^3$ for $0 \le x \le 2$ about the y-axis. Assume that the tank has water weighing 62.5 pounds per cubic foot that is 5-feet deep. Write the integral of the work W required to pump water to a level 1 foot above the top of the tank, until there is a depth of 1 foot of water left in the tank. Draw a picture of the situation, but DO NOT evaluate the integral.

Problem 5. (Fall 2015 Problem 2) Let R be the region between the graph of function $f(x) = \sin(x)$ and the x-axis on the interval $[0, \pi]$. Find the y-coordinate $\overline{y}$ of the center of gravity of the region R.

Problem 6. (Textbook, Exercise 6.7, Problem 4) Let C be a curve given parametrically by $x = -1 + \frac{3}{2}\sin t$ and $y = \frac{1}{2} - \frac{3}{2}\cos t$ for $0 \le t \le 2\pi$. Find an equation relating x and y. Then sketch the curve C, and indicate the direction $P(t)$ moves as t increases.

Problem 7. (Fall 2011 Problem 1(b)) Find the length of the curve described parametrically by: $x = t$ and $y = \ln\cos t$ for $0 \le t \le \pi/3$. (Hint: $\int \sec t \, dt = \ln|\sec t + \tan t| + C$).

Problem 8. (Fall 2005 Problem 2(a)) Let C be the arch of the cycloid given parametrically as $x = r(t - \sin t)$ and $y = r(1 - \cos t)$ for $0 \le t \le 2\pi$. Compute the length of C. (Hint: use the half-angle $\sin(\frac{t}{2}) = \frac{1-\cos(t)}{2}$)

Worksheet 4 **Name:** ____________________
MATH 141 (Fall 2023)
Sep 19th, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

Problem 1. Let $f(x) = 2x^7 - 5$.

(a) Explain why f has an inverse function.

(b) Find the inverse function f^{-1} of f.

(c) Find $(f^{-1})'(-3)$.

Problem 2. Let $f(x) = \frac{1}{x^2+2x+5}$.

(a) What is the domain of f?

(b) Compute $f'(x)$.

(c) Determine when $f'(x)$ is negative and is positive.

(d) Find the inverse f^{-1} if f has an inverse function. Otherwise, explain why f does not have an inverse function and find the subintervals such that f has an inverse.

(e) **(Challenge)** Repeat $(a) - (d)$ for $g(x) = \frac{1}{1-x^2}$.

Problem 3.

(a) Let $f(x) = 2x + \sqrt{x}$. Calculate $(f^{-1})'(3)$.

(b) Find the inverse function of $g(t) = \frac{t-3}{2t+7}$. What is the domain of g?

(c) Let $h(x) = \int_0^{x^3} \cos^2(t^2)\, dt$ for all x. Show that h has an inverse, and find $(h^{-1})'(c)$ where $c = h(\sqrt[6]{\pi/4})$.

Problem 4. (**Challenge**) Let $f(x) = \cos^3(x) + 2$. Find an interval I where f has an inverse function and find the inverse f^{-1}. Then determine the domain and the range of f^{-1}.

Worksheet 4 Solution
MATH 141 (Fall 2023)
Sep 19th, 2023

Problem 1.

(a) We have $f'(x) = 14x^6 \geq 0$ with only one zero. This means that f is an increasing function on all real number. Since f is one-to-one, f has an inverse function.

(b) Let $y = f(x)$ and solve for x. We have $y = 2x^7 - 5 \Rightarrow 2x^7 = y + 5 \Rightarrow x^7 = \frac{y+5}{2} \Rightarrow x = \sqrt[7]{\frac{y+5}{2}}$. Therefore, the inverse function is $f^{-1}(x) = \sqrt[7]{\frac{x+5}{2}}$.

(c) As $f(1) = -3$,

$$(f^{-1})'(-3) = \frac{1}{f'(1)} = \frac{1}{14}.$$

Problem 2.

(a) The domain of f is all real number except for x such that the denominator is zero. In other words, the zeros of $x^2 + 2x + 5$. The quadratic formula says that the roots are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

but this does not give (real) solutions because there is negative inside the square root. To conclude, the domain of f is all real numbers.

One other way to see is $x^2 + 2x + 5 = (x^2 + 2x + 1) + 4 = (x + 1)^2 + 4 \geq 4$, so it is never zero.

(b)

$$f'(x) = \frac{(x^2 + 2x + 5) \cdot 0 - 1 \cdot (2x + 2)}{(x^2 + 2x + 5)^2} = \frac{-(2x + 2)}{(x^2 + 2x + 5)^2}$$

(c) The sign of f' is determined by the numerator $-(2x + 2)$ as its denominator is a square. The zero of the numerator is $x = -1$, so we have a table

$(-\infty, -1)$	-1	$(-1, \infty)$
$+$	0	$-$

(d) The function f does not have an inverse because $f(0) = 5 = f(-2)$, so f is not one-to-one. However, f has an inverse function on the subintervals $(-\infty, 0]$ and $[0, \infty)$.

(e) The domain of g is all real number ± 1 which is the zeros of the denominator $1 - x^2$.

$$g'(x) = \frac{-(-2x)}{(1 - x^2)^2} = \frac{2x}{(1 - x^2)^2}$$

$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \infty)$
$-$	undefined	$-$	0	$+$	undefined	$+$

Then we claim that $(-\infty, -1)$, $(-1, 0]$, $[0, 1)$ and $(1, \infty)$ are subintervals where g has an inverse. Here notice that we excluded -1 because it is not in the domain of g.

The next explanation is only for the interested reader. One can even show that g has an inverse on $I_1 = (-\infty - 1) \cup (-1, 0]$ and $I_2 = [0, 1) \cup (1, \infty)$. This is because g is one-to-one on I_1 and I_2. To see this, suppose $g(x_1) = g(x_2)$, then $\frac{1}{1-x_1^2} = \frac{1}{1-x_2^2} \Rightarrow 1 - x_1^2 = 1 - x_2^2 \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$. This means that $g(x_1) = g(x_2)$ and $x_1 \neq x_2$, then $x_1 = -x_2$. This cannot happen on the subintervals I_1 and I_2 because they consist of numbers of the same sign.

Problem 3.

(a) By trial-and-error, $f(1) = 3$. Also $f'(x) = 2 + \frac{1}{2\sqrt{x}}$. Therefore,

$$(f^{-1})'(3) = \frac{1}{f(1)} = \frac{1}{2+\frac{1}{2}} = \frac{2}{5}$$

(b) The domain of g is all real numbers except $-\frac{7}{2}$ which is the zero of the denominator. Let $s = g(t)$. We solve for t. We have

$$s = \frac{t-3}{2t+7} \quad \Rightarrow \quad (2t+7)s = t-3 \quad \Rightarrow \quad 2st + 7s = t - 3 \Rightarrow (2s-1)t = -7s - 3$$
$$\Rightarrow \quad (1-2s)t = 7s + 3 \quad \Rightarrow \quad t = \frac{7s+3}{1-2s}$$

Therefore, $g^{-1}(t) = \frac{7t+3}{1-2t}$.

(c) By the fundamental theorem of calculus, one has

$$h'(x) = \cos^2(x^6) \cdot (3x^2)$$

Then $h'(x) \geq 0$ for all x, so h has an inverse function. Furthermore, we have

$$(h^{-1})'(c) = \frac{1}{h'(\sqrt[6]{\pi/4})} = \frac{1}{\cos^2(\pi/4)\cdot(3\sqrt[3]{\pi/4})} = \frac{1}{(\frac{1}{2})\cdot 3\sqrt[3]{\pi/4}} = \frac{2}{3\sqrt[3]{\pi/4}}$$

Problem 4. We have $f'(x) = -3\cos^2(x)\sin(x)$. Since $-3\cos^2(x) \leq 0$ for all x, the sign of $f'(x)$ is determined by $\sin(x)$. Consider the following table

	$-\pi$	$(-\pi, 0)$	0	$(0, \pi)$	π
$\sin(x)$	0	$-$	0	$+$	0
$f'(x)$	0	$+$	0	$-$	0

Hence we can choose $I = [0, \pi]$. To find the inverse function, consider $y = \cos^3(x) + 2 \Rightarrow y - 2 = \cos^3(x) \Rightarrow \cos(x) = \sqrt[3]{y-2} \Rightarrow x = \cos^{-1}(\sqrt[3]{y-2})$. Therefore $f^{-1}(x) = \cos^{-1}(\sqrt[3]{x-2})$. It suffices to find the domain and range of $f(x)$. The domain of f is $I = [0, \pi]$. We have $f(0) = 3$, $f(\pi) = 1$, and f is decreasing on $[0, \pi]$ (as $f'(x) < 0$ on $(0, \pi)$), the range of f is $[1, 3]$. Hence

- domain of f^{-1} = range of $f = [1, 3]$,
- range of f^{-1} = domain of $f = [0, \pi]$.

Worksheet 5 **Name:** ____________________

MATH 141 (Fall 2023)

Sep 21st, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

Problem 1. Find the integral

$$\int_1^4 \frac{e^{\frac{1}{x}}}{x^2}\, dx$$

Problem 2. Find a point at which the line $y = 3x + 4$ is tangent to the graph of $y = e^{x^3+1}$.

Problem 3.

(a) Let $f(x) = xe^x$. Find a formula for the nth derivative of f, where n is any positive integer.

(b) **(Challenge)** Let $f(x) = x^2e^x$. Find a formula for the nth derivative of f, where n is any positive integer. (Hint: First find the pattern and use mathematical induction.)

Problem 4. Find the area A of the region bounded by the graphs of $y = e^{3x}$ and $y = e^{-3x}$ and the line $x = 1$.

Worksheet 5 Solution
MATH 141 (Fall 2023)
Sep 21th, 2023

Problem 1. Let $u = \frac{1}{x} = x^{-1}$. Then $du = -x^{-2}\ dx = -\frac{1}{x^2}\ dx$. Therefore

$$\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\ dx = -\int_1^{\frac{1}{2}} e^u\ du = -(e^{1/2} - e) = e - e^{1/2}$$

Problem 2. If $y = 3x + 4$ is tangent to the graph of $y = e^{x^3+1}$, we have

$$\frac{dy}{dx} = e^{x^3+1} \cdot (3x^2) = 3$$

By trial-and-error, we have $x = -1$ gives us $\frac{dy}{dx}(-1) = e^0 \cdot 3 = 3$. Therefore the desired point is $(-1, 1)$.

Problem 3.

(a)

$$\begin{array}{rclcl} f'(x) & = & e^x + xe^x & = & (1+x)e^x \\ f''(x) & = & e^x + (1+x)e^x & = & (2+x)e^x \\ f'''(x) & = & e^x + (2+x)e^x & = & (3+x)e^x \end{array}$$

Then we see a pattern that $f^{(n)}(x) = (n+x)e^x$. To rigorously prove this, one needs to use mathematical induction, but we leave it only to interested students.

(b)

$$\begin{array}{rclcl} f'(x) & = & (2x)e^x + x^2e^x & = & (2x + x^2)e^x \\ f''(x) & = & (2+2x)e^x + (2x+x^2)e^x & = & (2+4x+x^2)e^x \\ f'''(x) & = & (4+2x)e^x + (2+4x+x^2)e^x & = & (6+6x+x^2)e^x \\ f^{(4)}(x) & = & (6+2x)e^x + (6+6x+x^2)e^x & = & (12+8x+x^2)e^x \\ f^{(5)}(x) & = & (8+2x)e^x + (12+8x+x^2)e^x & = & (20+10x+x^2)e^x \end{array}$$

You recognize the pattern the x-term is increasing by 2 and the constant term is a product of two conseuctive numbers, e.g. $0 \cdot 1 = 0, 1 \cdot 2 = 2, 2 \cdot 3 = 6, 3 \cdot 4 = 12, 4 \cdot 5 = 20$. Therefore, we might conjecture that $f^{(n)} = ((n-1)n + (2n)x + x^2)e^x$. Then one use induction to actually prove it.

Given that $f^{(n)}(x) = ((n-1)n + (2n)x + x^2)e^x$, we have

$$f^{(n+1)}(x) = (2n+2x)e^x + ((n-1)n + (2n)x + x^2)e^x = ((n-1)n + 2n + (2n+2)x + x^2)e^x$$

As $(n-1)n + 2n = (n-1+2)n = (n+1)n$ and $2n+2 = 2(n+1)$, we see that

$$f^{(n+1)}(x) = (n(n+1) + 2(n+1)x + x^2)e^x$$

which complete the induction step.

Problem 4.

$$A = \int_0^1 e^{3x} - e^{-3x}\ dx = \frac{1}{3}e^{3x} + \frac{1}{3}e^{-3x}\bigg|_0^1 = \frac{1}{3}e^3 + \frac{1}{3}e^{-3} - \left(\frac{1}{3} + \frac{1}{3}\right) = \frac{1}{3}(e^3 + e^{-3} - 2)$$

Worksheet 6 **Name:** ______________________________
MATH 141 (Fall 2023)
Sep 26st, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

Problem 1.

(a) $\tan^{-1}(1)$

(b) $\sec^{-1}(-2)$

(c) $\csc^{-1}(2\sqrt{3}/3)$

Problem 2.

(a) $\cot^{-1}(\tan \pi/3)$

(b) $\cos^{-1}(\sin \pi/4)$

(c) $\sin(\cos^{-1}(\sqrt{3}/2))$

(d) $\sec(\tan^{-1}(-\sqrt{3}))$

Problem 3.

(a) $\sin(\sec^{-1}(x/2))$

(b) $\csc(\tan^{-1}(x))$

(c) **(Challenge)** $\sin(2\sin^{-1} x)$.

Problem 4. Find the derivative of f.

(a) $f(x) = x\sin^{-1}(x^2)$

(b) $f(x) = \cot^{-1}\sqrt{1-x^2}$

Problem 5. Evaluate the indefinite integral.

(a)
$$\int \frac{1}{\sqrt{4-9x^2}}\,dx$$

(b)
$$\int \frac{1}{t^2-2t+5}\,dt$$

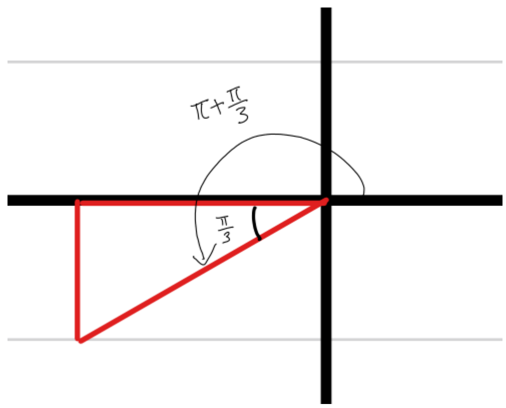
(c)
$$\int \frac{1}{x\ln x\sqrt{(\ln x)^2-1}}\,dx$$

Worksheet 6 Solution
MATH 141 (Fall 2023)
Sep 26th, 2023

Problem 1.

(a) $\tan^{-1}(1) = \frac{\pi}{4}$.

(b) $\sec^{-1}(-2) = \cos^{-1}(-\frac{1}{2}) = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$. To see this, you need to know $\cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$. Since the range of $\cos^{-1}(x)$ is $[0, \pi/2) \cup [\pi, 3\pi/2)$. Therefore we need to draw a triangle on the third quadrant of the form



Then we see that $\cos^{-1}(-\frac{1}{2}) = \pi + \frac{\pi}{3}$.

(c) $\csc^{-1}(2\sqrt{3}/3) = \sin^{-1}(3/2\sqrt{3}) = \sin^{-1}(\sqrt{3}/2) = \pi/3$.

Problem 2.

(a) $\tan\pi/3 = \sqrt{3}$. Then $\cot^{-1}(\sqrt{3} = \tan^{-1}(1/\sqrt{3}) = \frac{\pi}{6}$.

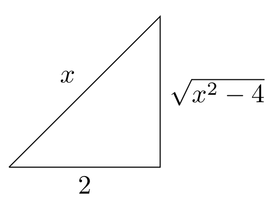
(b) $\sin\pi/4 = \sqrt{2}/2$, so $\cos^{-1}(\sqrt{2}/2) = \frac{\pi}{4}$.

(c) We have $\cos^{-1}(\sqrt{3}/2) = \frac{\pi}{6}$. Therefore, $\sin(\pi/6) = \frac{1}{2}$.

(d) One has $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$. Hence $\sec(-\pi/3) = \frac{1}{\cos(-\pi/3)} = \frac{1}{\cos(\pi/3)} = 2$.

Problem 3.

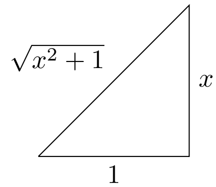
(a)



Therefore

$$\sin(\sec^{-1}(x/2)) = \frac{\sqrt{x^2-4}}{x}$$

(b)



Therefore

$$\csc(\tan^{-1}(x)) = \frac{\sqrt{x^2+1}}{x}$$

(c) Recall that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. Therefore,

$$\sin(2\sin^{-1}x) = 2\sin(\sin^{-1}x)\cos(\sin^{-1}x)$$

One have $\cos(\sin^{-1}x) = \sqrt{1-x^2}$, so we have $\sin(2\sin^{-1}x) = 2x\sqrt{1-x^2}$.

Problem 4.

(a) $f'(x) = \sin^{-1}(x^2) + x\dfrac{2x}{1-x^4} = \sin^{-1}(x^2) + \dfrac{2x^2}{\sqrt{1-x^4}}$.

(b) $f'(x) = \dfrac{-1}{(\sqrt{1-x^2})^2+1} \cdot \dfrac{-2x}{2\sqrt{1-x^2}} = \dfrac{x}{(2-x^2)\sqrt{1-x^2}}$.

Problem 5.

(a)

$$\int \frac{1}{\sqrt{4-9x^2}}\,dx = \int \frac{1}{3\sqrt{\frac{4}{9}-x^2}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 - x^2}}\,dx = \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C$$

(b) We want to make the denominator into the form x^2+a^2. Therefore $t^2-2t+5 = t^2-2t+1+4 = (t-1)^2+2^2$. Therefore

$$\int \frac{1}{t^2-2t+5}\,dt = \int \frac{1}{(t-1)^2+2^2}\,dt = \frac{1}{2}\tan^{-1}\left(\frac{t-1}{2}\right) + C$$

Here we used u-substitution $u = t-1$.

(c) Let $u = \ln x$, then $du = \frac{1}{x}dx$. Then

$$\int \frac{1}{x\ln x\sqrt{(\ln x)^2-1}}\,dx = \int \frac{1}{u\sqrt{u^2-1}}\,du = \frac{1}{1}\sec^{-1}(u/1) + C = \sec^{-1}(u) + C$$

Worksheet 7 **Name:** ______________________________

MATH 141 (Fall 2023)

Sep 28th, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

Use l'Hôpital's Rule to find the limit

Problem 1.

$$\lim_{x \to 0} \frac{\sin x}{x}$$

Problem 2.

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

Problem 3.

$$\lim_{x \to \pi/2^-} \frac{\ln(\cos x)}{\tan x}$$

Problem 4.

$$\lim_{x \to \infty} \frac{\tan 1/x}{1/x}$$

Problem 5.

$$\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x$$

Problem 6.

$$\lim_{x \to \pi/2^-} (\pi^2 - 4x^2) \tan x$$

Worksheet 7 Solution
MATH 141 (Fall 2023)
Sep 28th, 2023

Problem 1. Notice that $\lim_{x\to 0}\sin x = 0 = \lim_{x\to 0} x$. Hence we can use L'hôpital

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$$

Problem 2. Since we have $\lim_{x\to 1} = 0 = \lim_{x\to 1} x - 1$, we can use L'hôpital,

$$\lim_{x\to 0}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{1/x}{1} = 1$$

Problem 3. First, observe that $\lim -x \to \pi/2^- \cos x = 0$. As x appraches $\pi/2$ fro the let, $\cos x$ approaches to 0. If you don't see this, try to think about the graph of cosine. Therefore, $\lim_{x\to\pi/2^-}\ln(\cos x) = -\infty$.

Another way to write this would be to call $y = \cos x$, and see that $\lim_{x\to\pi/2^-}\ln(\cos x) = \lim_{y\to 0^+}\ln(y) = -\infty$. We also have $\lim_{x\to\pi/2^-}\tan(x) = \infty$, so we can use L'hôpital.

$$\lim_{x\to\pi/2^-}\frac{\ln(\cos x)}{\tan x} = \lim_{x\to\pi/2^-}\frac{-\sin x/\cos x}{\sec^2 x} = \lim_{x\to\pi/2^-}\frac{-\sin x}{\cos x}\cos^2 x = \lim_{x\to\pi/2^-} -\sin x\cos x = 0$$

Problem 4. As $x\to\infty$, $1/x\to 0$. Therefore $\lim_{x\to\infty}\tan(1/x) = 0 = \lim_{x\to\infty} 1/x$. Applying L'hôpital,

$$\lim_{x\to\infty}\frac{\tan 1/x}{1/x} = \lim_{x\to\infty}\frac{\sec^2(1/x)(-1/x^2)}{(-1/x^2)} = \lim_{x\to\infty}\sec^2(1/x) = 1$$

Problem 5. The limit has an indeterminant form 1^∞. Therefore we will use the identity $x = e^{\ln x}$.

$$\left(1-\frac{1}{x}\right)^x = e^{\ln(1-1/x)^x} = e^{x\ln(1-1/x)}$$

Therefore

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln(1-1/x)} = e^{\lim_{x\to\infty} x\ln(1-1/x)}$$

We compute

$$\lim_{x\to\infty} x\ln\left(1-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln(1-1/x)}{1/x} = \lim_{x\to\infty}\frac{1/x^2}{-1/x^2} = -1$$

Here we used the fact that $\lim_{x\to\infty}\ln(1-1/x) = 0 = \lim_{x\to\infty} 1/x$ to use L'hôpital. Finally, we can conclude that

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right) = e^{-1}$$

Problem 6. The limit has an interminant form $0\cdot\infty$. The idea is to make into $0/0$ form by taking the reciprocal of the ∞. To be precise,

$$\lim_{x\to\pi/2^-}(\pi^2-4x^2)\tan x = \lim_{x\to\pi/2^-}\frac{\pi^2-4x^2}{\cot x} = \lim_{x\to\pi/2^-}\frac{-8x}{-\csc^2 x} = \pi/4$$

Worksheet 8 **Name:** ______________________________

MATH 141 (Fall 2023)

Oct 3rd, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

For problems 1-4, find the indefinite integral.

Problem 1.

$$\int x \sec^2 x \, dx$$

Problem 2.

$$\int x^3 \ln x \, dx$$

Problem 3.

$$\int x^2 \sin x \, dx$$

Problem 4.

$$\int e^x \sin x \, dx$$

For problems 5-6, first make a substitution and then use integration by parts to evaluate the integral.

Problem 5.

$$\int 2e^{2x} \ln(2 + e^{2x}) \, dx$$

Problem 6.

$$\int \sin \sqrt{t} \, dt$$

Problem 7. (Challenge) Use integration by parts to establish the following formula.

$$\int \sin^n(x) \, dx = -\frac{1}{n} \sin^{n-1}(x) \cos(x) + \frac{n-1}{n} \int \sin^{n-2}(x) \, dx$$

Worksheet 8 Solution
MATH 141 (Fall 2023)
Oct 3rd, 2023

Problem 1. Let $u = x$ and $dv = \sec^2 x\ dx$. Here I chose dv because we know that $\sec^2 x$ is the derivative of $\tan x$. Then we have

$$\begin{array}{rclrcl} u & = & x & dv & = & \sec^2 x\, dx \\ du & = & dx & v & = & \tan x \end{array}$$

Then

$$\int x \sec^2\ dx = x \tan x - \int \tan x\ dx$$

For $\int \tan x\ dx = \int \sin x / \cos x\ dx$, use $u = \cos x$, then $du = -\sin x\ dx$. Therefore $\int \tan x\ dx = -\int 1/u\ du = -\ln|\cos x| + C$. We can conclude that

$$x \sec^2\ dx = x \tan x + \ln|\cos x| + C$$

Problem 2. Let $u = \ln x$ and $dv = x^3\ dx$. The main reason why we choose $u = \ln x$ is because integral of $\ln x$ is *more complicated* than $\ln x$.

$$\begin{array}{rclrcl} u & = & \ln x & dv & = & x^3\ dx \\ du & = & \frac{1}{x}\ dx & v & = & \frac{x^4}{4} \end{array}$$

Then

$$\int x^3 \ln x\ dx = \frac{1}{4} x^4 \ln x - \int \frac{x^3}{4}\ dx = \frac{1}{4} x^4 \ln x - \frac{x^4}{16} + C$$

Problem 3. We choose $u = x^2$ and $dv = \sin x\ dx$ so that we can *lower the degree* of x^2. We then have

$$\begin{array}{rclrcl} u & = & x^2 & dv & = & \sin x\ dx \\ du & = & 2x\ dx & v & = & -\cos x \end{array}$$

Then

$$\int x^2 \sin x\ dx = -x^2 \cos x + \int 2x \cos x\ dx$$

We use integration by parts one more time, namely,

$$\begin{array}{rclrcl} u & = & 2x & dv & = & \cos x\ dx \\ du & = & 2\ dx & v & = & \sin x \end{array}$$

$$\int 2x \cos x\ dx = 2x \sin x - \int 2 \sin x\ dx = 2x \sin x + 2 \cos x + C$$

In conclusion,

$$\int x^2 \sin x\ dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

Problem 4. Let $u = \sin x$ and $dv = e^x dx$. For this problem, switching u and dv would not make the problem more complicated. Then we have

$$\begin{array}{rclrcl} u & = & \sin x & dv & = & e^x\ dx \\ du & = & \cos x\ dx & v & = & e^x \end{array}$$

So

$$\int e^x \sin x\ dx = e^x \sin x - \int e^x \cos x\ dx$$

Let

$$\begin{array}{rclrcl} u & = & \cos x & dv & = & e^x\ dx \\ du & = & -\sin x\ dx & v & = & e^x \end{array}$$

Therefore

$$\int e^x \cos x\ dx = e^x \cos x + \int e^x \sin x\ dx$$

Here you should realize that we obtained the same integral that started with! However, this is not a problem. Combining the two results, we get

$$\int e^x \sin x\ dx = e^x \sin x - e^x \cos x - \int e^x \sin x\ dx$$

Moving the integral on the right-hand side to the left-hand side

$$2\int e^x \sin x\ dx = e^x \sin x - e^x \cos x \Rightarrow \int e^x \sin x\ dx = \frac{1}{2}(e^x \sin x - e^x \cos x)$$

Problem 5. Let $u = 2 + e^{2x}$. Then $du = 2e^{2x}\ dx$. Then

$$\int 2e^{2x} \ln(2 + e^{2x})\ dx = \int \ln u\ du$$

Choose v and dw such that

$$\begin{array}{rclrcl} v & = & \ln u & dw & = & du \\ dv & = & 1/u\ du & w & = & u \end{array}$$

$$\int \ln u\ du = u \ln u - \int du = u \ln u - u + C = (2 + e^{2x}) \ln(2 + e^{2x}) - (2 + e^{2x}) + C$$

Problem 6. Let $u = \sqrt{t}$, then $du = \frac{1}{2\sqrt{t}}\ dt = \frac{1}{2u}\ dt \Rightarrow 2u\ du = dt$. Hence we obtain

$$\int \sin \sqrt{t}\ dt = \int 2u \sin u\ du$$

Let

$$\begin{array}{rclrcl} v & = & 2u & dw & = & \sin u\ du \\ dv & = & 2\ du & w & = & -\cos u\ du \end{array}$$

$$\int 2u \sin u\ du = -2u \cos u + \int 2 \cos u\ du = -2u \cos u + 2 \sin u + C = -2\sqrt{t} \cos \sqrt{t} + 2 \sin \sqrt{t} + C$$

Problem 7. Let

$$\begin{array}{rclrcl} u & = & \sin^{n-1}(x) & dv & = & \sin(x)\ dx \\ du & = & (n-1) \sin^{n-2}(x) \cos(x)\ dx & v & = & -\cos(x) \end{array}$$

Then

$$\begin{aligned} \int \sin^n(x)\ dx &= -\sin^{n-1}(x) \cos(x) + \int (n-1) \sin^{n-2}(x) \cos^2(x)\ dx \\ &= -\sin^{n-1}(x) \cos(x) + \int (n-1) \sin^{n-2}(x)(1 - \sin^2(x))\ dx \\ &= -\sin^{n-1}(x) \cos(x) + \int (n-1) \sin^{n-2}(x) dx - \int (n-1) \sin^n(x)\ dx \end{aligned}$$

Then by moving the most-right term to the left-hand side, we et

$$n \int \sin^n(x)\ dx = -\sin^{n-1}(x) \cos(x) + \int (n-1) \sin^{n-2}(x)\ dx$$

Therefore

$$\int \sin^n(x)\ dx = -\frac{1}{n} \sin^{n-1}(x) \cos(x) + \frac{n-1}{n} \int \sin^{n-2}(x)\ dx$$

Practice Test 2 **Name:** ____________________
MATH 141 (Fall 2023)
Oct 5th, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

Problem 1. (Fall 2012, Midterm 2, Problem 4) Compute

$$\lim_{x\to 1^-} x^{\frac{1}{x-1}}$$

Problem 2. (Fall 2012, Problem 1) Let $f(x) = \frac{1}{5}\ln(1+e^x)$. Give the domain and the range of f. Show that f has an inverse function, and fine the derivative $(f^{-1})'(\frac{1}{5}\ln(1+e))$.

Problem 3. (Fall 2012, Midterm 2, Problem 2 & Spring 2017, Problem 6(a)) Compute and simplify

(a)

$$\tan\left(\cos^{-1}\left(\frac{-1}{2}\right)\right)$$

(b)

$$\int_0^{\frac{1}{3}} \frac{dx}{\sqrt{4-9x^2}}$$

(c)

$$\int_{-2}^{-1} \frac{1}{y^2+4y+7}\,dy$$

Problem 4. (Spring 2009, Problem 5(a)) Evaluate

$$\int t\sin(2t)\,dt$$

Problem 5. (Spring 2016, Problem 6(a) & Fall 2016, Problem 3(a))

(a) Compute with reasons

$$\lim_{x\to 0^+} \frac{\ln x}{\ln(\sin x)}$$

(b) Express $\cot(\sin^{-1}(x))$ as a function of x without trignometric functions.

Problem 6. (Fall 2016, Problem 6) Compute the integral

$$\int e^{2x}\cos(x)\,dx$$

Practice Test 2 Solution
MATH 141 (Fall 2023)
Oct 5th, 2023

Problem 1. Since $y = e^x$ is a continuous function

$$\lim_{x\to 1^-} x^{\frac{1}{x-1}} = \lim_{x\to 1^-} e^{\ln(x^{\frac{1}{x-1}})} = e^{\lim_{x\to 1^-} \ln(x^{\frac{1}{x-1}})}$$

It suffices to compute

$$\lim_{x\to 1^-} \ln\left(x^{\frac{1}{x-1}}\right) = \lim_{x\to 1^-} \frac{\ln x}{x-1} = \lim_{x\to 1^-} \frac{1}{x} = 1$$

Here we can use L'hôpital because $\lim_{x\to 1^-} \ln x = 0 = \lim_{x\to 1^-} x - 1$. In conclusion,

$$\lim_{x\to 1^-} x^{\frac{1}{x-1}} = e^1 = e$$

Problem 2. One has $e^x > 0 \Rightarrow 1 + e^x > 1$, so $\ln(1+e^x)$ is defined for all real number x. Since $\ln(x)$ is an increasing function $\ln(1+e^x) > 0$. Finally, this shows that $f(x) = \frac{1}{5}\ln(1+e^x) > 0$. Therefore

domain: $(-\infty, \infty)$ and range: $(0, \infty)$

Next, to show that f has an inverse function, consider the derivative

$$f'(x) = \frac{1}{5} \cdot \frac{e^x}{1+e^x}$$

It is clear that $f'(x) > 0$, so increasing. This shows that $f(x)$ has an inverse function.

Finally, note that $\frac{1}{5}\ln(1+e) = f(1)$. Hence

$$(f^{-1})'(\frac{1}{5}\ln(1+e)) = \frac{1}{f'(1)} = \frac{5+5e^x}{e^x}$$

Problem 3.

(a) $\cos^{-1}(-1/2) = \pi - \pi/3 = \frac{2\pi}{3}$. Therefore $\tan(2\pi/3) = -\sqrt{3}$.

(b)

$$\int_0^{\frac{1}{3}} \frac{dx}{\sqrt{4-9x^2}} = \int_0^{\frac{1}{3}} \frac{dx}{3\sqrt{\frac{4}{9}-x^2}} = \frac{1}{3}\int_0^{\frac{1}{3}} \frac{dx}{\sqrt{\left(\frac{2}{3}\right)^2 + x^2}} = \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right)\Bigg|_0^{\frac{1}{3}}$$
$$= \frac{1}{3}(\sin^{-1}(1/2) - \sin^{-1}(0)) = \frac{\pi}{18}$$

(c) We will complete the square in the denominator

$$\frac{1}{y^2+4y+7} = \frac{1}{y^2+4y+4+3} = \frac{1}{(y+2)^2+(\sqrt{3})^2}$$

Next, we will use u-substitution $u = y+2$ and $du = dy$, then

$$\int_{-2}^{-1} \frac{1}{y^2+4y+7} = \int_{-2}^{-1} \frac{1}{(y+2)^2+(\sqrt{3})^2}\,dy = \int_0^1 \frac{1}{u^2+(\sqrt{3})^2}\,du = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\Bigg|_0^1$$
$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6\sqrt{3}}$$

Problem 4. We will use integration by parts

$$\begin{array}{lllll} u & = & t & dv & = & \sin(2t)\,dt \\ du & = & dt & v & = & -\frac{1}{2}\cos(2t) \end{array}$$

Then

$$\int t\sin(2t)\,dt = -\frac{t}{2}\cos(2t) + \int \frac{1}{2}\cos(2t)\,dt = -\frac{t}{2}\cos(2t) + \frac{1}{4}\sin(2t) + C$$

Problem 5.

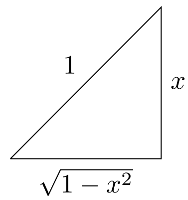
(a) $\lim_{x\to 0^+} \ln(x) = -\infty$. Also, $\lim_{x\to 0^+} \sin(x) = 0^+$ (meaning $\sin(x)$ approaches to 0 from the right, so we see that $\lim_{x\to 0^+} \ln(\sin(x)) = -\infty$. Hence $\lim_{x\to 0^+} \ln(x)/\ln(\sin x)$ is in indetrminant form $-\infty/-\infty$. By L'hôpital, we have

$$\lim_{x\to 0^+} \frac{\ln(x)}{\ln(\sin x)} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{1}{\sin x}} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

If you don't remember the last equality, one can use L'hôpital again as $\lim_{x\to 0^+} \sin x = 0 = \lim_{x\to 0^+} x$.

$$\lim_{x\to 0^+} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\cos x}{1} = 1$$

(b) To $\sin^{-1}(x)$, we can associate a triangle with hypothenus 1 and opposite x



The adjacent can be computed using the Pyhtagorean theorem. Then

$$\cot(\sin^{-1}(x)) = \frac{\sqrt{1-x^2}}{x}$$

Problem 6. We use integration by parts twice.

$$\begin{array}{rclcrcl} u & = & \cos(x) & & dv & = & e^{2x}\,dx \\ du & = & -\sin(x)dx & & v & = & \frac{1}{2}e^{2x} \end{array}$$

$$\int e^{2x}\cos(x)\,dx = \frac{1}{2}e^{2x}\cos(x) + \int \frac{1}{2}e^{2x}\sin(x)\,dx$$

$$\begin{array}{rclcrcl} u & = & \sin(x) & & dv & = & \frac{1}{2}e^{2x}\,dx \\ du & = & \cos(x)dx & & v & = & \frac{1}{4}e^{2x} \end{array}$$

$$\int e^{2x}\cos(x)\,dx = \frac{1}{2}e^{2x}\cos(x) + \int \frac{1}{2}e^{2x}\sin(x)\,dx = \frac{1}{2}e^{2x}\cos(x) + \frac{1}{4}e^{2x}\sin(x) - \frac{1}{4}\int e^{2x}\cos(x)\,dx$$

Therefore, we get

$$\frac{5}{4}\int e^{2x}\cos(x) = \frac{1}{2}e^{2x}\cos(x) + \frac{1}{4}e^{2x}\sin(x) + C$$

which then becomes

$$\int e^{2x}\cos(x) = \frac{4}{5}\left(\frac{1}{2}e^{2x}\cos(x) + \frac{1}{4}e^{2x}\sin(x)\right) + C$$

Worksheet 9 **Name:** ____________________

MATH 141 (Fall 2023)

Oct 10th, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

We begin with algebra problems. You should skip to problem 2 if you are comfortable with the algebra.

Problem 1. Simplify into expressions with no parenthesis or fractions.

(a) $-(1-u^2)u^3$

(b) $(1-u^2)^2u^4$

(c) $(1-u^2)^2/u^3$

(d) **(Challenge)** $(1-u)(1-2u)^2(1+2u)^2$

The next problem consists of Type A integrals $\int \sin^m(x)\cos^n(x)\ dx$.

Problem 2. Evaluate the integrals

(a) (A-1)
$$\int \sin^7(x)\cos^5(x)\ dx$$

(b) (A-2)
$$\int \sin^3(2x)\cos^6(2x)\ dx$$

(c) (A-3)
$$\int x\sin^2(x^2)\cos^2(x^2)\ dx$$

(d) (A-?)
$$\int \sin^3(x)\cos^3(x)\ dx$$

Next we consider Type B and D integrals $\int \tan^m(x)\sec^n(x)\ dx$

(a) (B-1)
$$\int \tan^2(x)\sec^2(x)\ dx$$

(b) (B-2)
$$\int \tan^3(x)\sec^5(x)\ dx$$

(c) (B-3)
$$\int \tan^2(x)\sec(x)\ dx$$

(d) (D)
$$\int \sin(2x)\cos(x)\ dx$$

Worksheet 9 Solution
MATH 141 (Fall 2023)
Oct 10th, 2023

Problem 1.

(a) $-(1-u^2)u^3 = -(u^3-u^5) = \boxed{-u^3+u^5}$

(b) $(1-u^2)^2u^4 = (1-2u^2+u^4)u^4 = \boxed{u^4-2u^6+u^8}$

(c) $(1-u^2)^2/u^3 = (1-2u^2+u^4)u^{-3} = \boxed{u^{-3}-2u^{-1}+u}$

(d) $(1-u)(1-2u)^2(1+2u)^2 = (1-u)((1-2u)(1+2u))^2 = (1-u)(1-4u^2)^2 = (1-u)(1-8u^2+16u^4) = 1-8u^2+16u^4-u+8u^3-16u^5 = \boxed{1-u-8u^2+8u^3+16u^4-16u^5}$.

Problem 2.

(a) We first deal with the integrand

$$\begin{aligned}\sin^7(x)\cos^5(x)\ dx = \sin^7(x)\cos^4(x)\cos(x)\ dx &= \sin^7(x)(\cos^2(x))^2\cos(x)\\ &= \sin^7(x)(1-\sin^2(x))^2\cos(x)\ dx = (*)\end{aligned}$$

Then using $u=\sin(x)$ and $du=\cos(x)\ dx$, we have

$$(*) = u^7(1-u^2)^2\ du = u^7(1-2u^2+u^4)\ du = u^7-2u^9+u^{11}\ du$$

Therefore

$$\begin{aligned}\int \sin^4(x)\cos^7(x)\ dx = \int u^7-2u^9+u^{11}\ du &= \frac{u^8}{8}-\frac{2u^{10}}{10}+\frac{u^{12}}{12}+C\\ &= \frac{\sin^8(x)}{8}-\frac{\sin^{10}(x)}{5}+\frac{\sin^{12}(x)}{12}+C\end{aligned}$$

(b) Let $y=2x$ with $dy=2\ dx$. This means that $dx=\frac{1}{2}dy$

$$\int \sin^3(2x)\cos^6(2x)\ dx = \frac{1}{2}\int \sin^3(y)\cos^6(y)\ dy$$

Now let's deal with the integrand

$$\sin^3(y)\cos^6(y)\ dy = \sin^2(y)\cos^6(y)\sin(y)\ dy = (1-\cos^2(y))\cos^6(y)\sin(y)\ dy = (*)$$

Let $u=\cos(y)$, then $du=-\sin(y)\ dy$, so we have

$$(*) = -(1-u^2)u^6\ du = -u^6+u^8\ du$$

Hence

$$\begin{aligned}\int \sin^3(2x)\cos^6(2x) = \frac{1}{2}\int -u^6+u^8du &= \frac{1}{2}\left(-\frac{u^7}{7}+\frac{u^9}{9}\right)+C\\ &= \frac{1}{2}\left(-\frac{\cos^7(2x)}{7}+\frac{\cos^9(2x)}{9}\right)+C\end{aligned}$$

(c) Let $y=x^2$ so that $dy=2x\ dx$. Then

$$\int x\sin^2(x^2)\cos^2(x^2)\ dx = \frac{1}{2}\int \sin^2(y)\cos^2(y)\ dy$$

We use identities $\sin(y)\cos(y)=\frac{1}{2}\sin(2y)$ and $\sin^2(y)=\frac{1}{2}(1-\cos(2y))$.

$$\sin^2(y)\cos^2(y) = (\sin(y)\cos(y))^2 = \frac{1}{4}\sin^2(2y) = \frac{1}{8}(1-\cos(4y))$$

Therefore

$$\begin{aligned}\int x\sin^2(x^2)\cos^2(x^2)\ dx = \frac{1}{16}\int 1-\cos(4y)\ dy &= \frac{1}{16}\left(y-\frac{1}{4}\sin(4y)\right)+C\\ &= \frac{1}{16}\left(x^2-\frac{1}{4}\sin(4x^2)\right)+C\end{aligned}$$

(d) Since both exponents are odd, you can use solve as both (A-1) and (A-2). We choose (A-1). Then

$$\sin^3(x)\cos^3(x)\ dx = \sin^3(x)\cos^2(x)\cos(x)\ dx = \sin^3(x)(1-\sin^2(x))\cos(x)\ dx = (*)$$

Let $u = \sin(x)$ to get

$$(*) = u^3(1-u^2)\ du = u^3 - u^5\ du$$

Hence

$$\int \sin^3(x)\cos^3(x)\ dx = \int u^3 - u^5\ du = \frac{\sin^4(x)}{4} - \frac{\sin^6(x)}{6} + C$$

Problem 3.

(a) If we let $u = \tan(x)$, then $du = \sec^2(x)\ dx$, so

$$\int \tan^3(x)\sec^2(x)\ dx = \int u^3 du = \frac{u^4}{4} + C = \frac{\tan^4(x)}{4} + C$$

You might be wondering why we didn't follow the *procedure* but we actually did!

(b) Now that n is odd, we factor out $\sec(x)\tan(x)$ to get

$$\tan^3(x)\sec^5(x) = \tan^2(x)\sec^4(x)\sec(x)\tan(x)\ dx = (\sec^2(x)-1)\sec^4(x)\sec(x)\tan(x)\ dx = (*)$$

Therefore, $u = \sec(x)$ and $du = \sec(x)\tan(x)\ dx$, we simplify to

$$(*) = (u^2-1)u^4\ du = u^6 - u^4\ du$$

Hence

$$\int \tan^3(x)\sec^5(x)\ dx = \int u^6 - u^4\ du = \frac{\sec^7(x)}{7} - \frac{\sec^5(x)}{5} + C$$

(c)

$$\int \tan^2(x)\sec(x)\ dx = \int (\sec^2(x)-1)\sec(x)\ dx = \int \sec^3(x) - \sec(x)\ dx$$

As a conclusion, we have

$$\begin{aligned}\int \sec^3(x) - \sec(x)\ dx &= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| - \ln|\sec(x)+\tan(x)| + C \\ &= \frac{1}{2}\sec(x)\tan(x) - \frac{1}{2}\ln|\sec(x)+\tan(x)| + C\end{aligned}$$

(d) We use $\sin(\alpha)\cos(\beta) = \frac{1}{2}\sin(\alpha-\beta) + \frac{1}{2}\sin(\alpha+\beta)$ where $\alpha = 2x$ and $\beta = x$. Then

$$\sin(2x)\cos(x) = \frac{1}{2}\sin(x) + \frac{1}{2}\sin(3x)$$

Hence

$$\int \sin(2x)\cos(x)\ dx = \frac{1}{2}\int \sin(x) + \sin(3x)\ dx = \frac{1}{2}(-\cos(x) - \frac{1}{3}\cos(3x)) + C = -\frac{1}{2}\left(\cos(x) + \frac{\cos(3x)}{3}\right) + C$$

Worksheet 10 **Name:** ____________________

MATH 141 (Fall 2023)

Oct 17th, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

§1 Warm Up

Problem 1. Evaluate the Integral

(a)
$$\int \frac{1}{2x-3}\,dx$$

(b)
$$\int \frac{1}{(x-3)^2}\,dx$$

(c)
$$\int \frac{1}{(-x+2)^3}\,dx$$

Problem 2. Factorize the following into linear and quadratic factors, i.e. expressions of the form $(ax+b)^r$ and $(ax^2+bx+c)^s$.

(a) $x^2-8x+15$ (b) x^3-2x^2+x (c) $x^3+6x^2+11x+6$

§2 Partial Fractions

Problem 3. Divide the numerator by its denominator and split the rational function so that the degree of the numerator is less than the degree of the denominator, e.g.

$$\frac{x-1}{x+1} = 1 - \frac{2}{x+1}$$

(a)
$$\frac{x^2}{x^2+1}$$

(b)
$$\frac{2x^3}{x^2+3}$$

(c)
$$\frac{2x^3+x^2+12}{x^2-4}$$

Problem 4. First factor the numerator and the denominator, and use partial fractions to find the integral

(a)
$$\int \frac{x^2}{x^2-4}\,dx$$

(b)
$$\int \frac{5x}{x^2+x-6}\,dx$$

(c)
$$\int \frac{-4}{x^2(x-2)}\,dx$$

§3 Trapezoidal Rule and Simpson's Rule

Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

with error

$$E_n^T \leq \frac{K_T}{12n^2}(b-a)^3 \text{ where } K_T = \text{ the maximum of } |f''(x)| \text{ for } a \leq x \leq b$$

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}[f(x_0) + 4(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

with error

$$E_n^S \leq \frac{K_S}{180n^4}(b-a)^5 \text{ where } K_T = \text{ the maximum of } |f^{(4)}(x)| \text{ for } a \leq x \leq b$$

Problem 5. Approximate the given integral by Trapezoidal and Simpson's Rule when $n = 4$. Find a value of n that guarantees an error of no more than 10^{-4}.

$$\int_1^5 \frac{1}{x}\,dx$$

Worksheet 10 Solution
MATH 141 (Fall 2023)
Oct 17th, 2023

Problem 1.

(a)
$$\int \frac{1}{2x-3}\,dx = \frac{1}{2}\ln|2x-3| + C$$
Use $u = 2x-3$.

(b)
$$\int \frac{1}{(x-3)^2}\,dx = \int (x-3)^{-2}\,dx = \frac{-1}{(x-3)} + C$$

(c)
$$\int \frac{1}{(-x+2)^3}\,dx = \int \frac{-1}{u^3}\,du = \frac{1}{2u^2} + C = \frac{1}{2(-x+2)^2} + C$$
Use $u = -x+2$.

Problem 2.

(a) $x^2 - 8x + 15 = (x-3)(x-5)$.

(b) $x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x-1)^2$.

(c) We find *a root* of the polynomial $x^3 + 6x^2 + 11x + 6$ by trying $0, \pm 1, \pm 2$, etc. We see that $x = -1$ is a root. This means that $x+1$ divides $x^3 + 6x^2 + 11x + 6$. The long division goes as follows:

$$
\begin{array}{r r}
 & x^2 \quad + 5x + 6 \\ \cline{2-2}
x+1 \big) & x^3 + 6x^2 + 11x + 6 \\
 & -x^3 \quad - x^2 \qquad\qquad\quad \\ \cline{2-2}
 & 5x^2 + 11x \qquad \\
 & -5x^2 \quad - 5x \qquad \\ \cline{2-2}
 & 6x + 6 \\
 & -6x - 6 \\ \cline{2-2}
 & 0
\end{array}
$$

Therefore, $x^3 + 6x^2 + 11x + 6 = (x+1)(x^2 + 5x + 6) = (x+1)(x+2)(x+3)$.

Problem 3.

(a) We have

$$
\begin{array}{r r}
 & 1 \\ \cline{2-2}
x^2+1 \big) & x^2 \qquad \\
 & -x^2 - 1 \\ \cline{2-2}
 & -1
\end{array}
$$

Therefore
$$1 - \frac{1}{x^2+1}$$

(b) We have

$$
\begin{array}{r r}
 & 2x \\ \cline{2-2}
x^2+3 \big) & 2x^3 \qquad \\
 & -2x^3 - 6x \\ \cline{2-2}
 & -6x
\end{array}
$$

Therefore

$$2x - \frac{6x}{x^2+3}$$

(c) We have

$$\begin{array}{r|l}
 & \quad\quad\quad\quad 2x \quad +1 \\ \hline
x^2-4\,) & 2x^3 + x^2 \quad\quad\quad + 12 \\
 & -2x^3 \quad\quad + 8x \\ \cline{2-2}
 & \quad\quad x^2 + 8x + 12 \\
 & \quad -x^2 \quad\quad\quad + 4 \\ \cline{2-2}
 & \quad\quad\quad\quad 8x + 16
\end{array}$$

Therefore

$$(2x+1) + \frac{8x+16}{x^2-4} = (2x+1) + \frac{8}{x-2}$$

Problem 4.

(a) Performing something similar to Problem 3(a),

$$\int \frac{x^2}{x^2-4}\,dx = \int 1 + \frac{4}{x^2-4}\,dx = x + \int \frac{4}{x^2-4}\,dx$$

Now let us focus on $\int \frac{4}{x^2-4}\,dx$. Since $x^2-4 = (x+2)(x-2)$, we try to solve

$$\frac{4}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

Multiplying $(x+2)(x-2)$ on both sides, we have

$$4 = A(x-2) + B(x+2) = (A+B)x + (2B-2A)$$

Therefore

$$\begin{cases} A+B=0 \\ 2B-2A=4 \end{cases} \Rightarrow \begin{cases} A+B=0 \\ B-A=2 \end{cases} \Rightarrow 2B=2 \Rightarrow B=1$$

Then by $A+B=0$, we get $A=-B=-1$.

$$\int \frac{4}{x^2-4}\,dx = \int \frac{1}{x+2} - \frac{1}{x-2}\,dx = \ln|x+2| - \ln|x-2| + C$$

Therefore,

$$\int \frac{x^2}{x^2-4}\,dx = x + \ln|x+2| - \ln|x-2| + C$$

(b)

$$\int \frac{5x}{x^2+x-6}\,dx = \frac{5x}{(x+3)(x-2)}\,dx$$

So we try to solve

$$\frac{5x}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

Multiplying $(x+3)(x-2)$ on both sides, we have

$$5x = A(x-2) + B(x+3) = (A+B)x + (3B-2A)$$

Now

$$\begin{cases} A+B=5 \\ 3B-2A=0 \end{cases} = \begin{cases} 2A+2B=10 \\ 3B-2A=0 \end{cases} \Rightarrow 5B=10 \Rightarrow B=2$$

Therefore $A=3$.

$$\int \frac{5x}{x^2+x-6}\,dx = \int \frac{3}{x+3} + \frac{2}{x-2}\,dx = 3\ln|x+3| + 2\ln|x-2| + C$$

(c)

$$\frac{-4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

We multiply $x^2(x-2)$ both sides to get

$$\begin{aligned} -4 &= Ax(x-2) + B(x-2) + Cx^2 \\ &= Ax^2 - 2Ax + Bx - 2B + Cx^2 \\ &= (A+C)x^2 + (B-2A)x - 2B \end{aligned}$$

Therefore $B = 2$. Then $2 - 2A = 0 \Rightarrow A = 1$ by comparing x-terms. Finally, $1 + C = 0 \Rightarrow C = -1$. Hence

$$\int \frac{-4}{x^2(x-2)}\, dx = \int \frac{1}{x} + \frac{2}{x^2} - \frac{1}{x-2}\, dx = \ln|x| - \frac{2}{x} - \ln|x-2| + C$$

Problem 5.

(a) Trapezoid rule for $n = 4$: we have $x_0 = 1, x_1 = 2, \ldots, x_4 = 5$. Using the formula above

$$\begin{aligned} \int_1^5 \frac{1}{x}\, dx &\approx \frac{5-1}{8}(f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)) \\ &= \frac{1}{2}(1 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{5}) \\ &= \frac{1}{2}\left(\frac{30+30+20+15+6}{30}\right) = \frac{101}{60}. \end{aligned}$$

(b) Simpson's rule for $n = 4$: we have $x_0 = 1, x_1 = 2, \ldots, x_4 = 5$. Using the formula above

$$\begin{aligned} \int_1^5 \frac{1}{x}\, dx &\approx \frac{5-1}{12}(f(1) + 4f(2) + 2f(3) + 4f(4) + f(5)) \\ &= \frac{1}{3}(1 + 2 + \frac{2}{3} + 1 + \frac{1}{5}) \\ &= \frac{1}{3}\left(\frac{15+30+10+15+3}{15}\right) = \frac{73}{45}. \end{aligned}$$

(c) Error term for Trapezoidal Rule: Since $f''(x) = \frac{2}{x^3}$, so we have $|f''(x)| = \left|\frac{2}{x^3}\right| \leq 2$ for $1 \leq x \leq 5$. As $2/x^3$ is a decreasing function on $[1, 5]$, we see that K_T = maximum of $|f''(x)| = 2$. Then we have

$$E_n^T \leq \frac{2}{12n^2} \cdot 4^3 = \frac{32}{3n^2}$$

So we want to find n such that $\frac{32}{3n^2} < 10^{-4}$. Hence we want to find n such that $\frac{32}{3} \cdot 10^4 < n^2 \Rightarrow \sqrt{32/3} \cdot 10^2 < n$. Therefore Since $\sqrt{32/3} < 4$, $n = 400$ should work.

(d) Error term for Simpson's Rule: Since $f^{(4)}(x) = \frac{24}{x^5}$, by the same reasoning as (c) above, $K_S = 24$. Therefore, we need to find n such that

$$\frac{24}{180n^4} \cdot 4^5 = \frac{2048}{15n^2} < 10^{-4}$$

We have $2048/15 = 136.5\overline{3} < 144 = 12^2$. Therefore, we need n such that $12^2 \cdot 10^4 \leq n^2 \Rightarrow n = 12 \cdot 10^2 = 1200$ should work.

Worksheet 11 **Name:** ____________________

MATH 141 (Fall 2023)

Oct 24th, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

Problem 1.

(a)
$$\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$$

(b)
$$\int_{2}^{\infty} \frac{1}{\sqrt{x^2-1}}\,dx$$

(Hint: Use trig substitution with $x = \sec(u)$.)

Problem 2. Use the comparison property to determine whether the integral converges.

(a)
$$\int_{0}^{\infty} \frac{1}{1+x^3}\,dx$$

(b)
$$\int_{1}^{\infty} \frac{\sin^2 x}{x^6}\,dx$$

(c)
$$\int_{0}^{\infty} \frac{1}{\sqrt{2+2\cos x}}\,dx$$

Problem 3. Find a formula for an Taylor polynomial for given n about $x = 0$.

(a) $f(x) = x^3 - 2x + 5$ for $n = 4$

(b) $f(x) = \frac{1}{1+3x}$ for $n = 3$

(c) $f(x) = e^{-2x}$ for $n = 5$

Problem 4. Calculate the third Taylor polynomial for $f(x) = 4\tan^{-1}(x)$ about $x = 0$. Use the polynomial to approximate π. (Hint: When is $4\tan^{-1}(x) = \pi$?)

Worksheet 11 Solution
MATH 141 (Fall 2023)
Oct 24th, 2023

Problem 1.

(a) Let $u = -x^2$ so that $du = -2x\ dx$. Therefore

$$\int 2xe^{-x^2}\ dx = -\int e^u\ du = -e^u + C = -e^{-x^2} + C$$

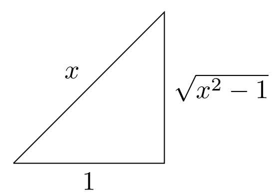
We have

$$\int_{-\infty}^{\infty} 2xe^{-x^2}\ dx = \lim_{c\to\infty} -e^{-x^2}\Big|_0^c + \lim_{d\to-\infty} -e^{-x^2}\Big|_d^0 = 0 + \frac{1}{2} - \frac{1}{2} + 0 = 0$$

(b) We first consider the indefinite integral. Let $x = \sec(u)$ so that $dx = \sec(u)\tan(u)\ du$.

$$\int \frac{1}{\sqrt{x^2-1}}\ dx = \int \frac{\sec(u)\tan(u)}{\sqrt{\sec^2(u)-1}}\ du = \int \sec(u)\ du = \ln|\sec(u)+\tan(u)| + C$$

Here we used $\sqrt{\sec^2(u)-1} = \sqrt{\tan^2(u)} = \tan(u)$. Now $u = \sec^{-1}(x)$.



Therefore

$$\ln|\sec(u)+\tan(u)| = \ln|x + \sqrt{x^2-1}|$$

We have

$$\lim_{c\to\infty} \frac{1}{\sqrt{x^2-1}}\ dx = \lim_{c\to\infty} \ln|x+\sqrt{x^2-1}|\Big|_2^c = \infty$$

Hence diverges.

(c)

Problem 2.

(a) For $1 \le x$, one has $x^2 \le x^3$. Therefore, $\frac{1}{1+x^3} \le \frac{1}{1+x^2}$. As

$$\int_0^{\infty} \frac{1}{1+x^2}\ dx = \lim_{c\to\infty} \int_0^c \frac{1}{1+x^2}\ dx = \lim_{c\to\infty} \tan^{-1}(x)\Big|_0^b = \lim_{b\to\infty} \tan^{-1} b = \frac{\pi}{2}.$$

By comparison theorem, $\int_0^{\infty} \frac{1}{1+x^3}\ dx$ converges.

(b) Since $\sin^2(x) \le 1$, we have $\frac{\sin^2(x)}{x^6} \le \frac{1}{x^6} = x^{-6}$.

$$\int_1^{\infty} x^{-6}\ dx = \lim_{c\to\infty} -\frac{1}{5x^5}\Big|_1^c = \frac{1}{5}$$

Hence the integral converges.

(c) We have $2 + 2\cos x \le 2 + 2 \Rightarrow \sqrt{2+2\cos(x)} \le 2$. Therefore $\frac{1}{2} \le \frac{1}{\sqrt{2+2\cos(x)}}$. Clearly,

$$\int_0^{\infty} \frac{1}{2}\ dx = \infty$$

hence by comparison theorem, $\int_1^{\infty} \frac{1}{\sqrt{2+2\cos(x)}}\ dx$ diverges.

Problem 3.

(a) We have

$$\begin{aligned} f(x) &= x^3 - 2x + 5 \\ f'(x) &= 3x^2 - 2 \\ f''(x) &= 6x \\ f'''(x) &= 6 \\ f^{(4)}(x) &= 0 \end{aligned}$$

Therefore

$$p_4 = 5 - 2x + 0x^2 + \frac{6}{3!}x^3 = 5 - 2x + x^3$$

(b) We have

$$\begin{aligned} f(x) &= \frac{1}{1+3x} = (1+3x)^{-1} \\ f'(x) &= -3(1+3x)^{-2} \\ f''(x) &= 18(1+3x)^{-3} \\ f'''(x) &= -162(1+3x)^{-4} \end{aligned}$$

Therefore

$$p_3 = 1 - 3x + \frac{18}{2}x^2 - \frac{162}{6}x^3 = 1 - 3x + 9x^2 - 27x^3$$

(c)

$$\begin{aligned} f(x) &= e^{-2x} \\ f'(x) &= -2e^{-2x} \\ f''(x) &= 4e^{-2x} \\ f'''(x) &= -8e^{-2x} \\ f^{(4)}(x) &= 16e^{-2x} \\ f^{(5)}(x) &= -32e^{-2x} \end{aligned}$$

Therefore

$$\begin{aligned} p_5(x) &= 1 - 2x + \frac{4}{2}x^2 - \frac{8}{6}x^3 + \frac{16}{24}x^4 - \frac{32}{120}x^5 \\ &= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5 \end{aligned}$$

Problem 4. We have

$$\begin{aligned} f(x) &= 4\tan^{-1}(x) \\ f'(x) &= \frac{4}{1+x^2} \\ f''(x) &= \frac{-8x}{(1+x^2)^2} \\ f'''(x) &= \frac{(1+x^2)^2(-8) - 2(-8x)(1+x^2)(2x)}{(1+x^2)^4} \end{aligned}$$

Therefore, $p_3(x) = 4x - \frac{8}{6}x^3 = 4x - \frac{4}{3}x^3$. Since $4\tan^{-1}(1) = \pi$, the approximation of π is given by

$$\pi \approx p_3(1) = 4 - \frac{4}{3} = \frac{8}{3}$$

Practice Test 3 **Name:** ____________________

MATH 141 (Fall 2023)

Oct 26th, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

Problem 1. Evaluate

(a) (Spring 2018, Question 5(b)) Evaluate

$$\int \sin^3(t)\cos^5(t)\ dt$$

(b) (Spring 2017, Question 4(b)) Evaluate

$$\int \tan^3(2t)\sec^3(2t)\ dt$$

Problem 2. (Fall 2016, Question 4) Evaluate

$$\int_1^\infty \frac{x-2}{x(x^2+1)}\ dx$$

Problem 3. Find the following integrals. Simplify your answers in both cases. Unless it is **not possible**, answeres should not involve trigonometric functions.

(a) (Variant of 8.3 Question 13)

$$\int \frac{1}{(4+t^2)^2}\ dt$$

(b) (Fall 2017, Question 3(b))

$$\int \frac{1}{\sqrt{(1-4x^2)^7}}\ dx$$

Problem 4. (Fall 2017, Question 4) Determine whether the following improper integrals converge or diverge. If any of them converges, find the value of the integral and simplify your answer.

(a)

$$\int_0^8 \frac{x-19}{(x+5)(x-3)}\ dx$$

(b)

$$\int_{-\infty}^{-1} \frac{1}{x^2-2x+5}\ dx$$

Practice Test 3 Solution
MATH 141 (Fall 2023)
Oct 26th, 2023

Problem 1.

(a)

$$\begin{aligned}\int \sin^3(t)\cos^5(t)\,dt &= \int \sin(t)(1-\cos^2(t))\cos^5(t)\,dt = -\int (1-u^2)u^5\,du \\ &= \int u^7 - u^5\,du = \frac{\cos^8(t)}{8} - \frac{\cos^6(t)}{6} + C\end{aligned}$$

where $u = \cos(t)$ and $du = -\sin(t)\,dt$.

(b)

$$\begin{aligned}\int \tan^3(2t)\sec^3(2t)\,dt &= \int \tan^2(2t)\sec^2(2t)(\tan(2t)\sec(2t))\,dt \\ &= \int (\sec^2(2t)-1)\sec^2(2t)(\tan(2t)\sec(2t))\,dt \\ &= \frac{1}{2}\int (u^2-1)u^2\,du = \frac{1}{2}\int u^4 - u^2\,du \\ &= \frac{1}{2}\left(\frac{\sec^5(2t)}{5} - \frac{\sec^3(2t)}{3}\right) + C\end{aligned}$$

where $u = \sec(2t)$ and $du = 2\sec(2t)\tan(2t)\,dt$.

Problem 2.

$$\frac{x-2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Then

$$\begin{aligned} & x - 2 = A(x^2+1) + (Bx+C)x \\ \Rightarrow\ & x - 2 = Ax^2 + A + Bx^2 + Cx \\ \Rightarrow\ & x - 2 = (A+B)x^2 + Cx + A\end{aligned}$$

Therefore $A = -2$, $C = 1$ and $B = 2$. Therefore

$$\begin{aligned}\int \frac{x-2}{x(x^2+1)}\,dx &= \int -\frac{2}{x} + \frac{2x+1}{x^2+1}\,dx = \int -\frac{2}{x} + \frac{2x}{x^2+1} + \frac{1}{x^2+1}\,dx \\ &= -2\ln|x| + \ln|x^2+1| + \tan^{-1}(x) + C\end{aligned}$$

As of now, we don't know whether or not the integral converges because

$$\lim_{c\to\infty} -2\ln|c| = -\infty \text{ and } \lim_{c\to\infty} \ln|x^2+1| = \infty$$

And $\infty - \infty$ is an indeterminant form. Instead, we write

$$-2\ln|x| + \ln|x^2+1| = \ln|x^2+1| - 2\ln|x| = \ln|x^2+1| - \ln|x^2| = \ln\left|\frac{x^2+1}{x^2}\right| = \ln\left|1 + \frac{1}{x^2}\right|$$

Then

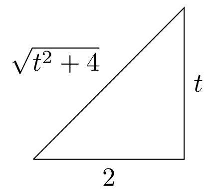
$$\begin{aligned}\int_1^\infty \frac{x-2}{x(x^2+1)}\,dx &= \lim_{c\to\infty} \ln\left|1+\frac{1}{x^2}\right| + \tan^{-1}(x)\Big|_1^c \\ &= \lim_{c\to\infty} \ln\left|1+\frac{1}{c^2}\right| + \tan^{-1}(c) - (\ln|2| + \frac{\pi}{4}) \\ &= \frac{\pi}{2} - \ln|2| - \frac{\pi}{4} = \frac{\pi}{4} - \ln|2|.\end{aligned}$$

Problem 3.

(a) Let $t = 2\tan(u)$ with $dt = 2\sec^2(u)\ du$. Then

$$\begin{aligned}
\int \frac{1}{(4+t^2)^2}\ dt &= \int \frac{2\sec^2(u)}{(4+4\tan^2(u))^2}\ du = \int \frac{2\sec^2(u)}{(4\sec^2(u))^2}\ du = \frac{1}{8}\int \cos^2(u)\ du \\
&= \frac{1}{16}\int 1+\cos(2u)\ du = \frac{1}{16}\left(u + \frac{1}{2}\sin(2u)\right) + C \\
&= \frac{1}{16}\left(u + \sin(u)\cos(u)\right) + C
\end{aligned}$$

Therefore with $u = \tan^{-1}(t/2)$



We have

$$\begin{aligned}
\int \frac{1}{(4+t^2)^2}\ dt &= \frac{1}{16}\left(\tan^{-1}(t/2) + \frac{t}{\sqrt{t^2+4}}\cdot\frac{2}{\sqrt{t^2+4}}\right) + C \\
&= \frac{1}{16}\left(\tan^{-1}(t/2) + \frac{2t}{t^2+4}\right) + C
\end{aligned}$$

Here we cannot further simplify $\tan^{-1}(t/2)$.

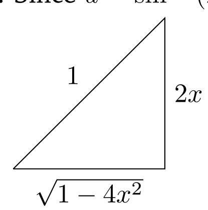
(b) Let $x = \frac{1}{2}\sin(u)$. Then $dx = \frac{1}{2}\cos(u)\ du$. Then substituting gives us

$$\int \frac{1}{\sqrt{(1-4x^2)^7}}\ dx = \frac{1}{2}\int \frac{\cos(u)}{\sqrt{(1-\sin^2(u))^7}}\ du = \frac{1}{2}\int \frac{\cos(u)}{\cos^7(u)}\ du = \frac{1}{2}\int \sec^6(u)\ du$$

Then $\sec^2(u) = \tan^2(u)+1 \Rightarrow \sec^4(u) = \tan^4(u) + 2\tan^2(u) + 1$ gives us

$$\begin{aligned}
\frac{1}{2}\int \sec^6(u)\ du &= \frac{1}{2}\int (\tan^4(u) + 2\tan^2(u) + 1)\sec^2(u)\ du = \frac{1}{2}\int v^4 + 2v^2 + 1\ dv \\
&= \frac{1}{2}\left(\frac{\tan^5(u)}{5} + \frac{2\tan^3(u)}{3} + \tan(u)\right) + C
\end{aligned}$$

with $v = \tan(u)$ and $dv = \sec^2(u)\ du$. Since $u = \sin^{-1}(2x)$.



Therefore, $\tan(u) = \frac{2x}{\sqrt{1-4x^2}}$. As a result,

$$\int \frac{1}{\sqrt{(1-4x^2)^7}}\ dx = \frac{1}{2}\left(\frac{1}{5}\left(\frac{2x}{\sqrt{1-4x^2}}\right)^5 + \frac{2}{3}\left(\frac{2x}{\sqrt{1-4x^2}}\right)^3 + \left(\frac{2x}{\sqrt{1-4x^2}}\right)\right) + C$$

Problem 4.

(a)

$$\frac{x-19}{(x+5)(x-3)} = \frac{A}{x+5} + \frac{B}{x-3}$$

Therefore $x - 19 = A(x-3) + B(x+5)$. Let $x = 3$, then $8B = -16 \Rightarrow B = -2$. If $x = -5$, then $-8A = -24 \Rightarrow A = 3$. Therefore

$$\int_0^8 \frac{3}{x+5} - \frac{2}{x-3}\,dx = 3\ln|x+5|\Big|_0^8 - \int_0^8 \frac{2}{x-3}\,dx = 3\ln(13/5) - \int_0^8 \frac{2}{x-3}\,dx$$

However, for the second integral, we need improper integral because the integrand is not defined for $x = 3$. Therefore

$$\int_0^8 \frac{2}{x-3}\,dx = \lim_{c\to 3^-}\int_0^c \frac{2}{x-3}\,dx + \lim_{d\to 3^+}\int_d^8 \frac{2}{x-3}\,dx$$

> The improper integral we have for this problem is mentioned at the bottom of page 554. We say that $\int_0^8 \frac{2}{x-3}\,dx$ **converges** if both the integrals $\int_0^3 \frac{2}{x-3}\,dx$ and $\int_3^8 \frac{2}{x-3}\,dx$ converge. Therefore, if **we need to only show that one of them diverges**, then the entire integral $\int_0^8 \frac{2}{x-3}\,dx$ diverges.

We have

$$\lim_{c\to 3^-}\int_0^c \frac{2}{x-3}\,dx = \lim_{c\to 3^-} 2\ln|x-3|\Big|_0^c = \lim_{c\to 3^-} 2\ln|c-3| - 2\ln(3).$$

We know that $\ln|c-3| \to -\infty$ as $c \to 3^-$, so the integral diverges. Therefore, the entire integral diverges.

(b)

$$\begin{aligned}\int_{-\infty}^{-1} \frac{1}{x^2-2x+5}\,dx &= \int_{-\infty}^{-1} \frac{1}{(x-1)^2+2^2}\,dx = \lim_{c\to-\infty} \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)\Bigg|_c^{-1} \\ &= \frac{1}{2}\left(\tan^{-1}(-1) - \left(-\frac{\pi}{2}\right)\right) = \frac{1}{2}\left(-\frac{\pi}{4} + \frac{\pi}{2}\right) = \frac{\pi}{8}\end{aligned}$$

Worksheet 12 **Name:** ____________________
MATH 141 (Fall 2023)
Oct 31st, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

Let $\{a_n\}_{n=m}^{\infty}$ be a sequence. To check that $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} a_n = \infty$, or $\lim_{n\to\infty} a_n = -\infty$, it suffices to

(a) find a function f on $[m, \infty)$ such that $f(n) = a_n$ for all n,

(b) $\lim_{x\to\infty} f(x) = L$, $\lim_{x\to\infty} f(x) = \infty$, or $\lim_{x\to\infty} f(x) = -\infty$.

Problem 1.

(a)

$$\lim_{n\to\infty} \int_{-n}^{n} \frac{1}{1+x^2}\, dx$$

(b)

$$\lim_{n\to\infty} \int_{-1/2n}^{1/2n} e^{3x}\, dx$$

Problem 2.

(a)

$$\lim_{n\to\infty} \frac{2n-11}{4n^2+5}$$

(b)

$$\lim_{n\to\infty} \frac{-3n^3-2}{4n^2+3n-7}$$

(c)

$$\lim_{n\to\infty} \frac{n^2+5n-8}{1-5n^2}$$

Problem 3.

(a)

$$\lim_{n\to\infty} \left(1-\frac{7}{n}\right)^n$$

(b)

$$\lim_{n\to\infty} \frac{e^{3n}}{2+e^{2n}}$$

(c)

$$\lim_{n\to\infty} \frac{\ln(n+5)}{\ln(3+4n)}$$

Problem 4. Using the following definition

> We say that $\lim_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$, there exists an integer N such that
> if $n \geq N$ then $|a_n - L| < \varepsilon$.

to prove that $\lim_{n\to\infty} \frac{1-5n}{n} = -5$.

Problem 5. (9.3) We have the following squeeze theorem

> For sequences $\{a_n\}, \{b_n\}$, and $\{c_n\}$, if $a_n \leq b_n \leq c_n$ for all n, then
> $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n$$

Evaluate

$$\lim_{n\to\infty} \frac{\cos(n)}{n}$$

Worksheet 12 Solution
MATH 141 (Fall 2023)
Oct 31st, 2023

Problem 1.

(a) Let $f(t) = \int_{-t}^{t} \frac{1}{1+x^2}\, dx$. Then

$$f(t) = \tan^{-1}(x)\Big|_{-t}^{t} = \tan^{-1}(t) - \tan^{-1}(-t)$$

Then

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \tan^{-1}(t) - \tan^{-1}(-t) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

(b) The function we want to use is $f(t) = \int_{-1/(2t)}^{1/(2t)} e^{3x}\, dx$. Then we have

$$f(t) = \frac{1}{3}e^{3x}\Big|_{-1/(2t)}^{1/(2t)} = \frac{1}{3}\left(e^{3/(2t)} - e^{3/(2t)}\right)$$

Therefore

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{1}{3}\left(e^{3/(2t)} - e^{3/(2t)}\right) = \frac{1}{3}(e^0 - e^0) = 0$$

Problem 2. To figure out whether or not the sequence converges, one needs to check the degrees of the numerator and the denominator.

(i) degree of the numerator $>$ degree of the denominator: limit diverges to ∞ or $-\infty$,

(ii) degree of the numerator $=$ degree of the denominator: fraction of the leading coefficients, e.g.

$$\lim_{n\to\infty} \frac{4x^3 - 2x + 1}{3x^3 + 1} = \frac{4}{3}$$

(iii) degree of the numerator $<$ degree of the denominator: limit converges to 0.

But, be careful, this is not showing work! The idea is to divide both the numerator and the denominator by the highest degree term of the denominator!

(a) Consider the function $f(x) = \frac{2x-11}{4x^2+5}$. Then it suffices to compute

$$\lim_{x\to\infty} \frac{2x - 11}{4x^2 + 5} = \lim_{x\to\infty} \frac{\frac{2}{x} - \frac{11}{x^2}}{4 + \frac{5}{x^2}} = \frac{0 - 0}{4 + 0} = 0.$$

Here we divided x^2 on both the numerator and the denominator.

(b) Similarly to (a), we have

$$\lim_{n\to\infty} \frac{-3x^3 - 2}{4x^2 + 3x - 7} = \lim_{x\to\infty} \frac{-3x - \frac{2}{x^2}}{4 + \frac{3}{x} - \frac{7}{x^2}} = \lim_{x\to\infty} \frac{-3x - 0}{4 + 0 - 0} = -\infty$$

Here we divided x^2 on both the numerator and the denominator.

(c) We have

$$\lim_{x\to\infty} \frac{x^3 + 5x - 8}{1 - 5x^3} = \lim_{x\to\infty} \frac{1 + \frac{5}{x^2} - \frac{8}{x^3}}{\frac{1}{x^3} - 5} = \frac{1 + 0 - 0}{0 - 5} = -\frac{1}{5}$$

Here we divided x^3 on both the numerator and the denominator.

Problem 3.

(a) As

$$\lim_{x\to\infty} \left(1 - \frac{7}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1-\frac{7}{x}\right)^x} = e^{\lim_{x\to\infty} \ln\left(1-\frac{7}{x}\right)^x}$$

We compute

$$\lim_{n\to\infty} \ln\left(1-\frac{7}{x}\right)^x = \lim_{n\to\infty} n\ln\left(1-\frac{7}{x}\right) = \lim_{x\to\infty} \frac{\ln(1-\frac{7}{x})}{1/x}$$

As the limit gives us the indeterminate form $\frac{0}{0}$, by L'hoptial, one has

$$\lim_{x\to\infty} \frac{\ln(1-\frac{7}{x})}{1/x} = \lim_{x\to\infty} \frac{\frac{1}{1-\frac{7}{x}}\cdot\frac{7}{x^2}}{-1/x^2} = \lim_{x\to\infty} \frac{-7}{1-\frac{7}{x}} = -7$$

We may conclude that

$$\lim_{x\to\infty} \left(1-\frac{7}{x}\right)^x = e^{-7}$$

(b) One gets the indeterminant form $\frac{\infty}{\infty}$, hence L'hoptial gives us

$$\lim_{x\to\infty} \frac{e^{3x}}{2+e^{2x}} = \lim_{x\to\infty} \frac{3e^{3x}}{2e^{2x}} = \lim_{x\to\infty} \frac{3}{2}e^x = \infty$$

(c) The limit yields an indeterminate form $\frac{\infty}{\infty}$, so by L'hopital, we get

$$\lim_{x\to\infty} \frac{\ln(x+5)}{\ln(3+4x)} = \lim_{x\to\infty} \frac{\frac{1}{x+5}}{\frac{1}{3+4x}} = \lim_{x\to\infty} \frac{3+4x}{x+5} = \lim_{x\to\infty} \frac{\frac{3}{x}+4}{1+\frac{5}{x}} = 4$$

Problem 4.

Let $\varepsilon > 0$. We want to find N such that whenever $n \geq N$, we have

$$\left|\frac{1-5n}{n} - (-5)\right| < \varepsilon$$

Since $|\frac{1-5n}{n} - (-5)| = |\frac{1}{n} - 5 + 5| = |\frac{1}{n}| = \frac{1}{n}$, choose N such that $N > \frac{1}{\varepsilon}$. Then one has

$$n \geq N \Rightarrow \left|\frac{1-5n}{n} - (-5)\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

Problem 5.

We use the inequality $-1 \leq \cos(n) \leq 1$. Therefore,

$$-\frac{1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n}$$

Then the squeeze theorem says that

$$0 = \lim_{n\to\infty} -\frac{1}{n} \leq \lim_{n\to\infty} \frac{\cos(n)}{n} \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

implies that

$$\lim_{n\to\infty} \frac{\cos(n)}{n} = 0.$$

Worksheet 13 **Name:** ______________________
MATH 141 (Fall 2023)
Nov 7th, 2023 **Section:** 0311 (12PM-12:50PM) / 0321 (1PM-1:50PM)

Review: Corollary 9.9 (Divergence Test), Theorem 9.10 (Geometric Series Thoerem), Theorem 9.12 (Integral Test), Theorem 9.13 (Comparison Test), and Theorem 9.14 (Limit Comparison Test)

SECTION 9.4

For problems 1-3, determine whether or not the series converges or diverges. If it convereges, find its sum.

Problem 1.

$$\sum_{n=0}^{\infty} 2\left(\frac{7}{11}\right)^n$$

Problem 2.

$$\sum_{n=1}^{\infty} \frac{2n^2-1}{n^2}$$

Problem 3.

$$\sum_{n=0}^{\infty} \frac{2^{n-5}}{7^{n+2}}$$

Problem 4. Express the repeating decimal $0.676767\ldots$ as fraction.

SECTION 9.5

For problems 5-8, use the Comparison Test, the Limit Comparison Test, or the Integral Test to determine whether the series converges or diverges.

Problem 5.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+4}}$$

Problem 6.

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^4-3}}$$

Problem 7.

$$\sum_{n=6}^{\infty} \frac{1}{(n-2)(n-5)}$$

Problem 8.

$$\sum_{n=2}^{\infty} \frac{n^3-1}{n^5-2n^2+1}$$

Problem 9.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$$

Problem 10. Use the Comparison Test and the Integral Test to prove the p-series Test. To be precise, prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Worksheet 13 Solution
MATH 141 (Fall 2023)
Nov 7th, 2023

Problem 1. Using the sum formula for geometric series (Theorem 9.10), we immediately have

$$\sum_{n=0}^{\infty} 2\left(\frac{7}{11}\right)^n = \frac{2}{1-\frac{7}{11}} = \frac{11}{2}$$

Problem 2. We have $\frac{2n^2-1}{n^2} = 2 - \frac{1}{n^2}$ and

$$\lim_{n\to\infty} 1 - \frac{1}{n^2} = 2 \neq 0$$

Then by the Divergence Test (Corollary 9.9), the series $\sum_{n=1}^{\infty} \frac{2n^2-1}{n^2}$ diverges.

Problem 3.

$$\sum_{n=0}^{\infty} \frac{2^{n-5}}{7^{n+2}} = \sum_{n=0}^{\infty} \frac{2^{-5}}{7^2}\left(\frac{2}{7}\right)^n = \frac{1}{2^5\cdot 7^2}\cdot\frac{1}{1-\frac{2}{7}} = \frac{5}{2^5\cdot 7^3}$$

Problem 4. We have

$$0.676767\ldots = 67\left(\frac{1}{100}\right) + 67\left(\frac{1}{100}\right)^2 + 67\left(\frac{1}{100}\right)^3 + \cdots = \sum_{n=1}^{\infty} 67\left(\frac{1}{100}\right)^n$$

Then

$$\sum_{n=1}^{\infty} 67\left(\frac{1}{100}\right)^n = \frac{\frac{67}{100}}{1-\frac{1}{100}} = \frac{\frac{67}{100}}{\frac{99}{100}} = \frac{67}{99}$$

Problem 5. First, one needs to **guess** whether or not the series will converge or diverge. Ignoring the lower term $\frac{1}{\sqrt{n^2+4}}$ is similar to $\sqrt{1}\sqrt{n^2} = \frac{1}{n}$, so the p-series test suggest that it must diverge. Be careful, this is just a **guess**.

In order to attempt to show divergence, one must find a sequence smaller than $\frac{1}{\sqrt{n^2+4}}$ that its associated series diverge. However, one has $\frac{1}{\sqrt{n^2+4}} \leq \frac{1}{\sqrt{n^2}} = \frac{1}{n}$, so we cannot use this. The key to finding such a sequence is to complete the square, i.e.

$$n^2+4 \rightsquigarrow n^2+4n+4$$

Then $n^2+4 \leq n^2+4n+4$ which implies $\sqrt{n^2+4} \leq \sqrt{n^2+4n+4} = n+2$ and implies $\frac{1}{n+2} \leq \frac{1}{\sqrt{n^2+4}}$. We then have

$$\sum_{n=1}^{\infty} \frac{1}{n+2} = \sum_{n=3}^{\infty} \frac{1}{n}$$

diverges, and by comparsion test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+4}}$ diverges.

Problem 6. For $n \geq 2$, one has $-2n^2 \leq -8$, so $-2n^2+4 \leq -4$, and finally,

$$n^4 - 2n^2 + 4 \leq n^4 - 4 \leq n^4 - 3$$

Therefore

$$\frac{1}{\sqrt{n^4-3}} \leq \frac{1}{\sqrt{n^4-2n^2+4}} = \frac{1}{n^2-2}$$

Therefore, it suffices to prove the convergence of

$$\sum_{n=2}^{\infty} \frac{1}{n^2-2} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2-2} = \sum_{n=1}^{\infty} \frac{1}{n^2+2n-1}$$

The above series is less or equal to $\sum_{n=1}^{\infty} \frac{1}{n^2}$ as $2n-1 \geq 0$ for $n \geq 1$. By using comparsion theorem two times, one sees that $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^4-3}}$ converges.

Problem 7. Since $\frac{1}{(n-2)(n-5)} \le \frac{1}{(n-5)^2}$ and

$$\sum_{n=6} \frac{1}{(n-5)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, by the comparision theorem, $\sum_{n=6}^{\infty} \frac{1}{(n-2)(n-5)}$ converges.

Problem 8. We will use limit comparison theorem (Theorem 9.14). Disregard all terms other than the highest term of the numerator and the denominator to obtain $n^3/n^5 = \frac{1}{n^2}$. We compute the limit of the ratio

$$\lim_{n\to\infty} \frac{\frac{n^3-1}{n^5-2n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^5-n^2}{n^5-2n^2+1} = \lim_{n\to\infty} \frac{1-\frac{1}{n^3}}{1-\frac{2}{n^3}+\frac{1}{n^5}} = 1$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, the series $\sum_{n=2}^{\infty} \frac{n^3-1}{n^5-2n^2+1}$ converges.

Problem 9. We will use the integral test (Theorem 9.12). The function $f(x) = \frac{1}{x(\ln(x))^3}$ defined on $[2,\infty)$ is decreasing and continuous because $x(\ln(x))^3$ is increasing and continuous and never equal to 0 on $[2,\infty)$. We compute the improper integral

$$\begin{aligned}\int_2^{\infty} f(x)\, dx = \int_2^{\infty} \frac{1}{x\ln(x)^3}\, dx = \int_{\ln(2)}^{\infty} \frac{1}{u^3}\, du &= \lim_{c\to\infty} -\frac{1}{2u^2}\Big|_{\ln(2)}^{c} \\ &= \lim_{c\to\infty} -\frac{1}{2c^2} + \frac{1}{2\ln(2)^2} \\ &= \frac{1}{2\ln(2)^2}\end{aligned}$$

which can be shown to converge. Therefore, $\sum_{n=2}^{\infty} \frac{1}{x(\ln(x))^3}$ converges.

Problem 10. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by example 4 of pg 594 in the textbook.

Now first observe that, $n^x = e^{\ln(n)x}$ is a strictly increasing function for $n \ge 1$ because the derivative is $\ln(n)e^{ln(n)x} > 0$. Therefore for $p < 1$, we have $\frac{1}{n} < \frac{1}{n^p}$. As $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges by the comparison test. Now for $p > 1$, consider consider the function $f(x) = \frac{1}{x^p}$. This is a continuous decreasing function on $[1,\infty)$. Therefore, we compute the improper integral

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{c\to\infty} \frac{1}{(1-p)x^{p-1}}\Big|_1^c = \lim_{c\to\infty} \frac{1}{(1-p)c^{p-1}} - \frac{1}{1-p}$$

For $p > 1$, we have $p-1 > 0$, so $\lim_{c\to\infty} \frac{1}{c^{p-1}} = 0$. This shows that

$$\lim_{c\to\infty} \frac{1}{(1-p)c^{p-1}} - \frac{1}{1-p} = -\frac{1}{1-p}$$

which converges. Therefore $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$.