Calc I & II Review **Name:** ______________________
MATH 241 (Spring 2023) **Section:** 0112 (8AM-9AM) / 0122 (9:30AM-10:20AM)
01/26/2023 **TA:** Shin Eui Song

Instructions:
This is a review worksheet for Calc I and Calc II that will not be graded.

Problem 1. Use product rule, quotient rule, power rule, and chain rule to **find the derivative** of each of the following functions. Don't simplify your answers. Can you identify which rules you used?

(a) $f(x) = \left(\dfrac{3x+5}{x^2+7}\right)^2$

(b) $g(t) = t^{1/3}(e^{\sec t}) + \ln(4\pi)$

Problem 2. Find the integrals of the following functions. If you use substitution in your solution, indicate the substitution.

(a) $\displaystyle\int_{-1}^{0} t^2\sqrt{1+t}\, dt$

(b) $\displaystyle\int 4x\cos(2-3x)\, dx$

(Hint: use integration by parts for (b).)

Problem 3. Find the Cartesian coordinates of the points having the given polar coordinates.

(a) $(-2, -\pi/6)$

(b) $(4, 3\pi/4)$

Problem 4. Find the polar coordinate (r, θ) with $r \geq 0$ and angles $0 \leq \theta < 2\pi$ of the points having the given Cartesian coordinates.

(a) $(4, -4)$

(b) $(3, 3\sqrt{3})$

Problem 5. Find the integrals of the following function.

$$\int \frac{x - 11}{x^2 + 5x - 14}\, dx$$

(Hint: use partial fractions.)

Problem 6.

(a) Find a parametrization for a circle of radius 4 with the center $(2, -1)$. Don't forget to specify the domain.

(b) Any set of parametric equations of the form

$$x = a + bt \text{ and } y = c + dt \text{ with } b \neq 0 \text{ or } d \neq 0$$

represents a straight line. **Find a parametrization of a line** that passes through the point $(3, 4)$ with slope -2. Can you give more than one?

Problem 7. Find $f(4)$ for the continuous function f satisfying

$$x \sin \pi x = \int_0^{x^2} f(t)\, dt$$

(Hint: use the fundamental theorem of calculus.)

Problem 1.

(a)

$$f'(x) = 2\left(\frac{3x+5}{x^2+7}\right)\frac{(x^2+7)\cdot 3 - (3x+5)\cdot(2x)}{(x^2+7)^2}$$

We used power rule, chain rule, and quotient rule.

(b)

$$g'(t) = \frac{1}{3}t^{-2/3}e^{\sec(t)} + t^{1/3}e^{\sec(t)}\sec(t)\tan(t)$$

We used product rule, power rule, and chain rule.

Problem 2.

(a) Let $u = 1+t$, then $du = dt$, and $t = u-1$. Then the bound $-1 \le t \le 0$ becomes $0 \le u \le 1$. After u-substitution, we have

$$\begin{aligned}\int_0^1 (u-1)^2\sqrt{u}\,du &= \int_0^1 (u^2-2u+1)\cdot u^{1/2}\,du &= \int_0^1 u^{5/2} - 2u^{3/2} + u^{1/2}\,du \\ &= \left.\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right|_0^1 &= \left(\frac{2}{7} - \frac{4}{5} + \frac{2}{3}\right) - 0 = \frac{16}{105}\end{aligned}$$

(b) We use integration by parts. Set

$$\begin{aligned} u &= 4x & dv &= \cos(2-3x) \\ du &= 4 & v &= -\tfrac{1}{3}\sin(2-3x)\end{aligned}$$

then

$$\begin{aligned}\int 4x\cos(2-3x)\,dx &= 4x\left(-\frac{1}{3}\sin(2-3x)\right) - \int -\frac{4}{3}\sin(2-3x)\,dx \\ &= -\frac{4}{3}x\sin(2-3x) + \frac{4}{9}\cos(2-3x) + C\end{aligned}$$

Problem 3. We use the fact that a polar coordinate (r,θ) corresponds to the Cartesian coordinate $(r\cos\theta, r\sin\theta)$.

(a)

$$(x,y) = (-2\cos(-\pi/6), -2\sin(-\pi/6)) = \left(-2\cdot\frac{\sqrt{3}}{2}, -2\cdot -\frac{1}{2}\right) = (-\sqrt{3}, 1)$$

(b)

$$(x,y) = (4\cos(3\pi/4), 4\sin(3\pi/4)) = \left(4\cdot -\frac{\sqrt{2}}{2}, 4\cdot\frac{\sqrt{2}}{2}\right) = (-2\sqrt{2}, 2\sqrt{2})$$

Problem 4. Given a Cartesian coordinate (x,y), the r-coordinate of the polar coordinate is given by

$$r = \pm\sqrt{x^2+y^2}$$

Since the problem said $r \ge 0$, we choose $r = \sqrt{x^2+y^2}$. To figure out the angle θ, one needs to solve the inverse tangent problem

$$\tan\theta = \frac{y}{x}$$

so that θ is in the interval $[0, 2\pi)$.

(a) First, $r = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$. We need to find $\theta \in [0, 2\pi)$ such that $\tan\theta = \frac{-4}{4} = -1$. Then $\theta = -\frac{\pi}{4} + \pi k$ for some integer k. The point $(4, -4)$ is in the 4^{th} quadrant, so $\theta = \frac{3\pi}{4}$ with $k = 2$. Hence the corresponding polar coordinate is $(4\sqrt{2}, \frac{3\pi}{4})$.

(b) First, $r = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{36} = 6$. We solve $\tan\theta = \frac{3\sqrt{3}}{3} = \sqrt{3}$. Then $\theta = \frac{\pi}{3} + \pi k$ for some integer k. The point $(3, 3\sqrt{3})$ is in the 1^{st} quadrant, so $\theta = \frac{\pi}{3}$ with $k = 0$. Hence the corresponding polar coordinate is $(6, \frac{\pi}{3})$.

Problem 5. We use partial fractions. First, we must factor the denominator $x^2 + 5x - 14$. From

$$\begin{array}{ccccc} x^2 & + & 5x & - & 14 \\ 1 & & & & 7 \\ 1 & & & & -2 \end{array}$$

we see that $x^2 + 5x - 14 = (x + 7)(x - 2)$. Next, we want to find A and B such that

$$\frac{x - 11}{x^2 + 5x - 14} = \frac{A}{x + 7} + \frac{B}{x - 2}$$

By multiplying both sides by $x^2 + 5x - 14$, we get an equation

$$x - 11 = A(x - 2) + B(x + 7)$$

If $x = 2$, we get $-9 = 9B$, so $B = -1$. Similarly, if $x = -7$, then $-18 = -9A$, so $A = 2$. With all these being done, we have

$$\int \frac{x - 11}{x^2 + 5x - 14}\, dx = \int \frac{2}{x + 7} - \frac{1}{x - 2}\, dx = 2\ln|x + 7| - \ln|x - 2| + C$$

Problem 6.

(a)

$$\gamma(\theta) = (4\cos(\theta) + 2, 4\sin(\theta) - 1)$$

for $0 \le \theta \le 2\pi$.

(b) We can set $a = 3$ and $c = 4$ because the line will pass $(3, 4)$ when $t = 0$. One can show that the slope of the parametrized line is $\frac{d}{b}$ assuming that the slope is not 0. Hence we can choose any b and d such that $\frac{d}{b} = -2$. For example,

$$x = 3 - t \text{ and } y = 4 + 2t$$

for all real numbers t is a desired parametrization. We can choose any b and d with $\frac{d}{b} = -2$, so you have infinitely many parametrization, e.g. $(b, d) = (-2, 4)$ or $(b, d) = (-100, 200)$.

Extra Explanation: To see that $\frac{d}{b}$ is the slope of the line, we write the parametrization as

$$x - a = bt \text{ and } y - c = dt \tag{†}$$

Then we get

$$\frac{x - a}{b} = t = \frac{y - c}{d}$$

assuming that $b \neq 0$ and $d \neq 0$. Both cannot be 0 because then the parametrization is for a point. If one of them is 0, then the line is either a horizontal line or a vertical line. This means that the slope is either undefined or is 0. Since our slope is -2, we know that b and d are both nonzero.

After multiplying bd on both sides of (†) and solving for y, we get

$$y = \frac{d}{b}x + \frac{bc - ad}{b}$$

Therefore, $\frac{d}{b}$ is the slope of the line.

Problem 7. We use the (application of) fundamental theorem of calculus which says that

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\ dt = f(g(x))g'(x) - f(h(x))h'(x)$$

We first compute the derivative of the right-hand side.

$$\frac{d}{dx}\int_{0}^{x^2} f(t)\ dt = f(x^2)\cdot 2x - f(0)\cdot 0 = 2xf(x^2)$$

Next, we compute the derivative of the left-hand side.

$$\frac{d}{dx}(x\sin\pi x) = \sin\pi x + \pi x\cos\pi x$$

Since we want to compute $f(4)$, let $x = 2$. Then we get

$$\begin{array}{rrcl} & \sin(2\pi) + 2\pi\cos(2\pi) & = & 4f(4) \\ \Rightarrow & 2\pi & = & 4f(4) \\ \Rightarrow & f(4) & = & \frac{\pi}{2} \end{array}$$

References

[1] Gulick D., Ellis R. (2006) *Calculus with Analytic Geometry*, Cengage Learning, 6th ed. (Exercise 10.1: Problems 1(b), 1(g), 2(b), 2(e))

[2] Math Testbank, *Final Examination - Math 140 - Fall 2014*. (Problems 3, 9(b))

[3] Paul's Online Notes, *Integration By Parts*, https://tutorial.math.lamar.edu/problems/calcii/integrationbyparts.aspx. (Problem 1)

[4] MITOpenCourseware, *18.01SC Single Variable Calculus*, https://ocw.mit.edu/courses/18-01sc-single-variable-calculus-fall-2010/resources/mit18_01scf10_exam3/. (Problem 3(b))