AREA OF THE ELLIPSE IN POLAR COORDINATES

This article will guide you to compute the area of an ellipse using polar coordinates. You can check the next page for missing details if you get stuck. Let R be the region bounded by an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{*}$$

Step 1: Change the Cartesian coordinate to polar coordinate to show that

$$R = \left\{ (r, \theta) \;\middle|\; 0 \le \theta \le 2\pi \text{ and } 0 \le r \le \frac{ab}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} \right\}$$

Step 2: Show that

$$A = \iint_R dA = \frac{a^2 b^2}{2} \int_0^{2\pi} \frac{d\theta}{b^2 \cos^2 \theta + a^2 \sin^2 \theta}$$

where A is the area of the ellipse given by $(*)$.

Step 3: Use the change of varaibles $a \tan \theta = b \tan \phi$ to rewrite the integral as

$$\frac{a^2 b^2}{2} \int_0^{2\pi} \frac{d\theta}{b^2 \cos^2 \theta + a^2 \sin^2 \theta} = \frac{a^2 b^2}{2} \int_0^{2\pi} \frac{(b/a) d\phi}{b^2}$$

(**Hint:** $1 + \tan^2 \theta = \sec^2 \theta$)

Step 4: Show that the area of the ellipse given by $(*)$ is $ab\pi$.

Step 1: If we let $x = r\cos\theta$ and $y = r\sin\theta$, then we obtain

$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$

By multiplying a^2b^2 on both sides, one gets

$$r^2b^2\cos^2\theta + r^2a^2\sin^2\theta = a^2b^2 \quad \Rightarrow \quad r^2 = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$\Rightarrow \quad r = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

Hence the region R is given by

$$R = \left\{ (r,\theta) \;\middle|\; 0 \le \theta \le 2\pi \text{ and } 0 \le r \le \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \right\}$$

Step 2: We have

$$\iint_R dA = \int_0^{2\pi}\int_0^{\dagger} r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi} r^2\Big|_0^{\dagger}\,d\theta = \frac{1}{2}\int_0^{2\pi}\frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}\,d\theta = \frac{a^2b^2}{2}\int_0^{2\pi}\frac{d\theta}{b^2\cos^2\theta + a^2\sin^2\theta}$$

where $\dagger = (ab)/\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}$.

Step 3: Since $a\tan\theta = b\tan\phi$, we have $a\sec^2\theta\,d\theta = b\sec^2\phi\,d\phi$. Then

$$\frac{d\theta}{b^2\cos^2\theta + a^2\sin^2\theta} = \frac{\sec^2\theta\,d\theta}{b^2 + a^2\tan^2\theta} = \frac{\frac{b}{a}\sec^2\phi\,d\phi}{b^2\sec^2\phi} = \frac{\frac{b}{a}\,d\phi}{b^2}$$

Plugging back, we get

$$\frac{a^2b^2}{2}\int_0^{2\pi}\frac{d\theta}{b^2\cos^2\theta + a^2\sin^2\theta} = \frac{a^2b^2}{2}\int_0^{2\pi}\frac{(b/a)d\phi}{b^2}$$

Step 4:

$$\frac{a^2b^2}{2}\int_0^{2\pi}\frac{(b/a)d\phi}{b^2} = \frac{a^2b^2}{2}\cdot\frac{b}{ab^2}\int_0^{2\pi}d\phi = \frac{ab}{2}\cdot 2\pi = ab\pi$$