1. Background

Let $f(x)$ be a real-valued function. We review what it means for a function to have a limit L as the x approaches to a. This is denoted by

$$\lim_{x\to a} f(x) = L.$$

Informally, L is the y-value that the function approaches as x approaches to a. How can we make this definition mathematically rigorous? Following the work of Bolzano (1817), Cauchy (1821), and Weierstrass (1861), mathematicians use the *epsilon-delta* definition to define limits.

2. Epsilon-Delta Definition

The variable epsilon ε will represent how close $f(x)$ is to our limit L, and the variable delta δ will represent how close x is to a. Mathematically, we express via

$$|f(x) - L| < \varepsilon \quad \text{and} \quad |x - a| < \delta$$

We want to say that $f(x)$ can be as close as we want to L if x is close enough to a. Therefore, **for all** small ε, we want to **conclude** that $|f(x) - L| < \varepsilon$. How do we say x is close enough to a? We say that there **exists** δ such that $|x - a| < \delta$ would conclude that $|f(x) - L| < \varepsilon$. To summarize, we obtain the following definition.

Definition 2.1. Let $f(x)$ be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then

$$\lim_{x\to a} f(x) = L$$

if for all $\varepsilon > 0$, there exists some number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \varepsilon \tag{$\dagger$}$$

Let us look at an example.

Example 2.2. Let $f(x) = x$. Then

$$\lim_{x\to 3} f(x) = 3$$

Proof. Let $\varepsilon > 0$ which is chosen arbitrarily. We need to find δ (which is usually in terms of ε) such that ($\dagger$) in the definition above would hold. In this case, we can simply choose $\delta = \varepsilon$ because if $0 < |x - 3| < \delta \Rightarrow |f(x) - 3| < \delta = \varepsilon$. □

Exercise 2.3. Let $f(x) = 2x + 3$. Show that

$$\lim_{x\to 2} f(x) = 7$$

Using this definition, one can prove some properties of limits.

Example 2.4. Let $f(x) = \sqrt{x}$ and let $g(x) = \sin(\pi x)$. Suppose that we know that $\lim_{x\to 4} f(x) = 2$ and $\lim_{x\to 4} g(x) = 0$. Then

$$\lim_{x\to 4} f(x) + g(x) = 2$$

Proof. Let $\varepsilon > 0$. Then there exists $\delta_1, \delta_2 > 0$ such that

$$\text{if } 0 < |x - 4| < \delta_1 \text{ then } |f(x) - 2| < \frac{\varepsilon}{2}$$

$$\text{if } 0 < |x - 4| < \delta_2 \text{ then } |g(x) - 0| < \frac{\varepsilon}{2}$$

Let $\delta = \min(\delta_1, \delta_2)$. If $0 < |x - 4| < \delta$,

$$|f(x) + g(x) - 2| = |(f(x) - 2) + (g(x) - 0)| \le |f(x) - 2| + |g(x) - 0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Exercise 2.5. Let $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$. Following the example 2.4, Prove

$$\lim_{x\to a} f(x) + g(x) = L_1 + L_2$$

3. Multivariable Epsilon-Delta

For a vector-valued function, the definition is the same except, we use vectors and magnitudes.

Definition 3.1. Let $\boldsymbol{F}$ be a vector-valued function defined at each point in some open interval containing a, except possibly at a itself. Then

$$\lim_{x\to a} \boldsymbol{F}(x) = \boldsymbol{L}$$

if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta \text{ then } ||\boldsymbol{F}(x) - \boldsymbol{L}|| < \varepsilon \tag{††}$$

A vector-valued function $\boldsymbol{F}(x) = f_1(x)\hat{\boldsymbol{\imath}} + f_2(x)\hat{\boldsymbol{\jmath}} + f_3(x)\hat{\boldsymbol{k}}$ can be thought as three real-valued functions f_1, f_2, f_3. We like to see the relationship between the limits of f_1, f_2, f_3, and $\boldsymbol{F}$.

Exercise 3.2. Suppose

$$\lim_{x\to a} f_1(x) = L_1, \quad \lim_{x\to a} f_2(x) = L_2, \quad \lim_{x\to a} f_3(x) = L_3$$

Then prove that $\lim_{x\to a} \boldsymbol{F}(x) = (L_1, L_2, L_3)$.

(**Hint:** Find δ_i such that $|f_i(x) - a| < \frac{\varepsilon}{\sqrt{3}}$ for $i = 1, 2, 3$.)