Rotation in 2D **Name:** ____________________
MATH 241 (Spring 2023) **Section:** 0112 (8AM-9AM) / 0122 (9:30AM-10:20AM)
02/07/2023 **TA:** Shin Eui Song

Complex numbers are used to describe rotations in 2D. Recall that **complex numbers** are numbers of the form

$$a + bi$$

where a and b are real numbers and i is an imaginary unit such that $i^2 = -1$. The addition operation is defined as

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and the multiplication operation is defined as

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

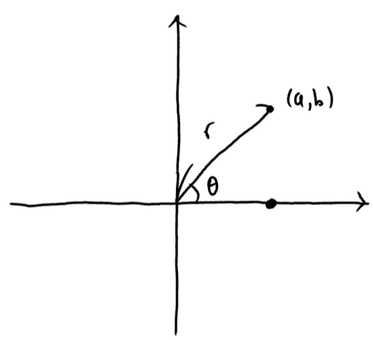
Then **Euler's identity** shows that

$$e^{i\theta} = \cos(\theta) + \sin(\theta)i$$

For this to make sense, we have explain what it means by raising to the ith power, but we ignore such technicalities in this handout. Now we are ready to see why complex numbers describe rotations in 2D. Let $a + bi$ be a complex number. Let $r = \sqrt{a^2 + b^2}$.

Problem 1. Show that the point $(\frac{a}{r}, \frac{b}{r})$ is on a unit circle. In particular, there exists a real number $0 \le \theta \le 2\pi$ such that $\frac{a}{r} = \cos\theta$ and $\frac{b}{r} = \sin\theta$.

By Problem 1, complex number $a + bi = r\cos\theta + (r\sin\theta)i$ for some $0 \le \theta \le 2\pi$ and $r = \sqrt{a^2 + b^2}$.



In fact, θ is the angle between vector (a, b) and the x-axis.

Problem 2. Show that rotating a complex number $a + bi$ by an angle θ' is same thing as multiplying $e^{i\theta'}$ on $a + bi$ using the Euler's identity.

One can also use the Euler's identity to prove the following trig identities.

Problem 3. Show that $\sin(\theta + \theta') = \sin(\theta)\cos(\theta') + \cos(\theta)\sin(\theta')$ and $\cos(\theta + \theta') = \cos(\theta + \theta') = \cos(\theta)\cos(\theta') - \sin(\theta)\sin(\theta')$ using the Euler's identity. (**Hint:** $e^{i(\theta+\theta')} = e^{i\theta} \cdot e^{i\theta'}$)

We will see later that there is a generalziation to this story. The generalization called the quaternions gave birth to dot products and corss products.