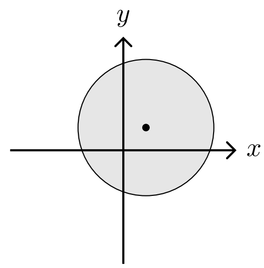
Groupwork 1 Solution
MATH 241 (Spring 2023)
01/31/2023

Problem 1. The inequality $(x-1)^2 + (y-1)^2 \leq 9$ in $\mathbb{R}^2$ would represent the closed disk of radius 3 with center $(1, 1)$ which is given in the figure below.



Since z can be any value, the region of $\mathbb{R}^3$ represented by the inequality would be an **infinite solid cylinder** such that its intersection with the xy-plane is the closed disk above.

Problem 2. The inequality for open region containing all points inside the sphere of radius R centered at the origin is given by

$$x^2 + y^2 + z^2 < R^2$$

The inequality for the open region containing all points outside the sphere of raidus r centered at the origin is given by

$$r^2 < x^2 + y^2 + z^2$$

In conclusion, the desired inequality is

$$r^2 < x^2 + y^2 + z^2 < R^2$$

Problem 3.

(a) $(\boldsymbol{a} \cdot \boldsymbol{b})$ is a scalar, so dot product with the vector $\boldsymbol{c}$ **does not make sense**.

(b) Both $||\boldsymbol{a}||$ and $\boldsymbol{b} \cdot \boldsymbol{c}$ are scalars, so the formula **makes sense**. We can evaluate to

$$||\boldsymbol{a}||(\boldsymbol{b} \cdot \boldsymbol{c}) = ||\hat{\boldsymbol{\imath}}||(2\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{k}}) = 1 \cdot 0 = 0$$

(c) As $(\boldsymbol{a} \cdot \boldsymbol{b})$ is a scalar and $\boldsymbol{c}$ is a vector, we cannot add them together, so it **does not make sense**.

(d) This is just a scalar multiplication, so it **makes sense**.

$$(\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} = (\hat{\boldsymbol{\imath}} \cdot 2\hat{\boldsymbol{\imath}})\hat{\boldsymbol{k}} = 2\hat{\boldsymbol{k}}$$

(e) Since $||\boldsymbol{a}||$ is a scalar, the dot product **does not make sense**.

Problem 4.

$$\begin{aligned} v_x &= ||\boldsymbol{v}|| \cos\frac{\pi}{6} = 50 \cdot \frac{\sqrt{3}}{2} = 25\sqrt{3} \\ v_y &= ||\boldsymbol{v}|| \sin\frac{\pi}{6} = 50 \cdot \frac{1}{2} = 25 \end{aligned}$$

Hence

$$\boldsymbol{v} = 25\sqrt{3}\hat{\boldsymbol{\imath}} + 25\hat{\boldsymbol{\jmath}}$$

Problem 5. We have

$$\begin{aligned} \overrightarrow{PQ} &= (2, 0, -4) - (1, -3, -2) &= (1, 3, -2) \\ \overrightarrow{PR} &= (6, -2, -5) - (1, -3, -2) &= (5, 1, -3) \\ \overrightarrow{QR} &= (6, -2, -5) - (2, 0, -4) &= (4, -2, -1) \end{aligned}$$

Then

$$||\overrightarrow{PQ}|| = \sqrt{1+9+4} = \sqrt{14} \quad ||\overrightarrow{PR}|| = \sqrt{25+1+9} = \sqrt{35} \quad ||\overrightarrow{QR}|| = \sqrt{16+4+1} = \sqrt{21}$$

The triangle formed by $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{QR}$ satisfies

$$||\overrightarrow{PR}||^2 = 35 = 14 + 21 = ||\overrightarrow{PQ}||^2 + ||\overrightarrow{QR}||^2$$

so the triangle ΔPQR **is a right triangle**.

Problem 6.

(a) $\boldsymbol{a}$ and $\boldsymbol{b}$ are not perpendicular because the dot product $\boldsymbol{a} \cdot \boldsymbol{b} = -4$ is nonzero. We have $||\boldsymbol{a}|| = \sqrt{2}$ and $||\boldsymbol{b}|| = 4$, so from the formula

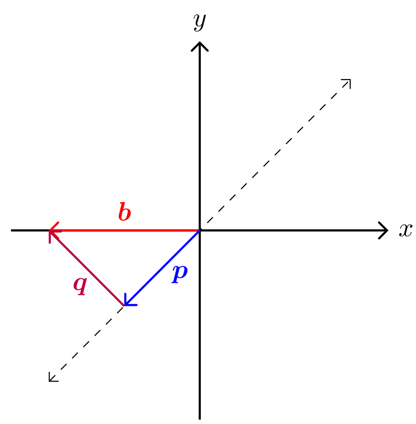
$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{||\boldsymbol{a}||||\boldsymbol{b}||} = \frac{-4}{\sqrt{2} \cdot 4}$$

we get $\cos\theta = -\frac{1}{\sqrt{2}}$. Then $\theta = \frac{3\pi}{4}$.

(b) The projection $\boldsymbol{p}$ is given by the formula

$$\boldsymbol{p} = \mathrm{pr}_{\boldsymbol{a}}\boldsymbol{b} = \left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{||\boldsymbol{a}||^2}\right)\boldsymbol{a} = -\frac{4}{2}\boldsymbol{a} = -2\boldsymbol{a} = (-2, -2)$$

(c)



Then geometrically, $\boldsymbol{b} = \boldsymbol{p} + \boldsymbol{q}$. Hence $\boldsymbol{q} = \boldsymbol{b} - \boldsymbol{p}$.

(d) From (c), $\boldsymbol{q} = (-4, 0) - (-2, -2) = (-2, 2)$. The vector $\boldsymbol{q}$ is perpendicular to $\boldsymbol{a}$ because $\boldsymbol{q} \cdot \boldsymbol{a} = (-2, 2) \cdot (1, 1) = -2 + 2 = 0$.