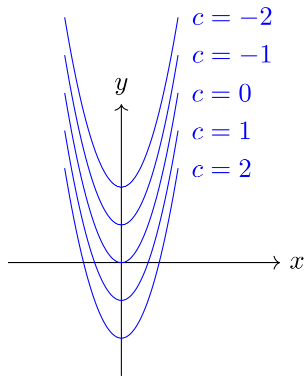
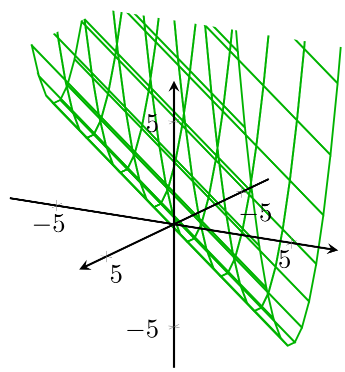
Groupwork 4 Solution
MATH 241 (Spring 2023)
02/28/2023

Problem 1. The level curve at c is given by $c = x^2 - y$. Rearranging gives $y = x^2 - c$. Hence we have that the level curves are just parabolas but shifted along the y-axis.



Each level curve at c will be on $z = c$ plane. Hence



It will be a slanted parabolic cylinder.

Problem 2.

$$f_x(x, y) = 3x^2 + 2xy^3 \text{ and } f_y(x, y) = 3x^2y^2 - 4y$$

Hence $f_x(2, 1) = 12 + 4 = 16$ and $f_y(2, 1) = 12 - 4 = 8$. The partial derivative $f_x(2, 1)$ is the instantaneous rate of change of $f(x, y)$ at $(2, 1)$ with respect to x when $y = 1$. Similarly, the partial derivative $f_y(2, 1)$ is the instantaneous rate of change of $f(x, y)$ at $(2, 1)$ with respect to y when $x = 2$.

$$\begin{aligned} f_{xx} &= 6x + 2y^3 \\ f_{xy} &= 6xy^2 \\ f_{yx} &= 6xy^2 \\ f_{yy} &= 6x^2y - 4 \end{aligned}$$

Problem 3.

$$\begin{aligned} \frac{\partial f}{\partial x} &= \cos(2x + 3y) \cdot 2 = 2\cos(2x + 3y) \\ \frac{\partial f}{\partial y} &= \cos(2x + 3y) \cdot 3 = 3\cos(2x + 3y) \end{aligned}$$

Problem 4. We do partial implicit differentiation. We take partial-x keeping in mind that $z = z(x, y)$ is a function of x and y. We take partial-x on the left-hand side and the right hand side.

$$\frac{\partial}{\partial x}(x^2 + y^2 + z^2) = 2x + 0 + 2z \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial}{\partial x}((3x)(yz)) = 3(yz) + (3x) \cdot \left(y \cdot \frac{\partial z}{\partial x}\right)$$

Therefore,

$$2x + 2z \cdot \frac{\partial z}{\partial x} = 3yz + 3xy \cdot \frac{\partial z}{\partial x}$$

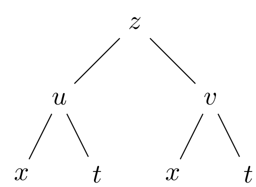
Reordering gives us

$$(2z - 3xy)\frac{\partial z}{\partial x} = 3yz - 2x$$

Therefore

$$\frac{\partial z}{\partial x} = \frac{3yz - 2x}{2z - 3xy}$$

Problem 5. Let $u = x + at$ and $v = x - at$. Then $z = f(u) + g(v)$, and we have the dependency diagram



In particular, we have

$$z_t = \frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t}$$

Since

$$\frac{\partial z}{\partial u} = f_u(u), \quad \frac{\partial u}{\partial t} = a, \quad \frac{\partial z}{\partial v} = g_v(v), \quad \frac{\partial v}{\partial t} = -a$$

we have

$$z_t = af_u(u) - af_v(v)$$

By the same procedure, one has

$$z_{tt} = a^2 f_{uu}(u) + a^2 f_{vv}(v)$$

On the other hand,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 1$$

so one has

$$z_x = f_u(u) + f_v(v) \text{ and } z_{xx} = f_{uu}(u) + f_{vv}(v)$$

Hence, we conclude that

$$z_{tt} = a^2 z_{xx}$$