Groupwork 5 Solution
MATH 241 (Spring 2023)
03/07/2023

Problem 1. The equation of the tangent plane at $(x_0, y_0, f(x_0, y_0))$ is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

We have

$$f_x(x, y) = e^{xy} + xye^{xy} \quad \text{and} \quad f_y(x, y) = x^2 e^{xy}$$

Therefore, the equation of the tangent plane is

$$z = 1 + (x - 1) + (y - 0) = x + y$$

Then $f(1.1, -0.1) \approx 1.1 + (-0.1) = 1$ which is the z-value of the equation of the tangent plane when $x = 1.1$ and $y = -0.1$. We can also use

$$f(x_0 + h, y_0 + k) \approx f(x_0, y_0) + f_x(x_0, y_0)h + f_y(x_0, y_0)k$$

where $x_0 = 1$, $y_0 = 0$, $h = 0.1$ and $k = -0.1$. The two methods are actually the same.

Problem 2. Two planes are parallel if their normal vectors are parallel. The normal vector of the tangent plane at (x, y, z) is given by $(f_x, f_y, -1)$. Since $f_x(x, y) = -8x$ and $f_y(x, y) = -2y$, we have that a normal vector of the tangent plane is $(-8x, -2y, -1)$.

The equation $z = 4y$ is $0 = 4y - z$, so a noraml vector of this plane is $(0, 4, -1)$. The two vectors are parallel if they are multiples of each other, i.e. there is a number c such that

$$(-8x, -2y, -1) = c(0, 4, -1)$$

By comparing the $\hat{\boldsymbol{k}}$-component, we see that $c = 1$. Then $x = 0$ and $y = -2$. By plugging it back to the equation of the paraboloid, the desired point is $(0, -2, 5)$.

Problem 3. Given a unit vector $\vec{u}$ and a point (x_0, y_0), the directional vector can be given by $D_{\vec{u}}\, f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$. Hence it is a scalar.

Problem 4. (See Theorem 13.16 from the textbook) $D_{\vec{u}} f$ attains its maximum (i.e. increase most rapidly) when $\vec{u}$ points in the same direction as $\text{grad } f(x_0, y_0, z_0)$. We have

$$\text{grad } f(x_0, y_0, z_0) = (yze^{xy}, xze^{xy}, e^{xy})$$

$f(x, y, z)$ increase most rapidly at the point $(0, 1, 2)$ in the same direction as $\text{grad } f(0, 1, 2) = (2, 0, 1)$. The maxium value is $||\text{grad}(0, 1, 2)|| = \sqrt{5}$.

Problem 5. We have

$$f_x(x, y) = 4x^3 - 4y \qquad \text{and} \qquad f_y(x, y) = 4y^3 - 4x$$

Hence we need to solve $\begin{cases} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{cases}$ or equivalently, $\begin{cases} x^3 - y = 0 \\ y^3 - x = 0 \end{cases} \Rightarrow \begin{cases} y = x^3 \\ x = y^3 \end{cases}$. By substituting, we get $x = y^3 = x^9$. Hence $0 = x^9 - x = x(x^8 - 1)$. Then $x = \pm 1$ or 0. By the same reasoning, $y = \pm 1$ or 0. If $x = 0$, then by $y = x^3$, $y = 0$. Repeating this, we see that the critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

We have

$$\begin{aligned} f_{xx} &= 12x^2 \\ f_{xy} &= -4 \\ f_{yx} &= -4 \\ f_{yy} &= 12y^2 \end{aligned}$$

	$D(x_0, y_0)$	$f_{xx}(x_0, y_0)$	
$(0, 0)$	<0	=0	saddle point
$(1, 1)$	>0	>0	relative minimum
$(-1, -1)$	>0	>0	relative minimum

Problem 6. We take derivative term by term

$$\frac{d}{dt}\left(\frac{1}{2}m||\boldsymbol{v}(t)||^2\right) = \frac{1}{2}m \cdot \frac{d}{dt}(\boldsymbol{v}(t)\cdot\boldsymbol{v}(t)) = \frac{1}{2}m \cdot (2\boldsymbol{a}(t)\cdot\boldsymbol{v}(t)) = m\boldsymbol{a}(t)\cdot\boldsymbol{v}(t) = \boldsymbol{F}(t)\cdot\boldsymbol{v}(t)$$

On the other hands, if we write $\boldsymbol{r}(t) = (x(t), y(t), z(t))$, we obtain

$$\frac{d}{dt}V(\boldsymbol{r}(t)) = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt} = \nabla V(x,y,z)\cdot\boldsymbol{v}(t) = -\boldsymbol{F}(t)\cdot\boldsymbol{v}(t)$$

If we combine these two, we obtain that

$$\frac{d}{dt}E(t) = \boldsymbol{F}(t)\cdot\boldsymbol{v}(t) - \boldsymbol{F}(t)\cdot\boldsymbol{v}(t) = 0$$

We can now conclude that $E(t)$ is constant.