MATH 241 Calculus III Spring 2023
Groupwork 6: Sections 14.1-14.2, Answers

1. If the following equality is true, explain why. If false, fix the integral bounds on the right-side so the two iterated integrals are equal.

$$\int_0^4 \int_0^{\sqrt{x}} \sqrt{x+y^2}\, \mathrm{d}y\mathrm{d}x = \int_0^{\sqrt{x}} \int_0^4 \sqrt{x+y^2}\, \mathrm{d}x\mathrm{d}y.$$

The left side of the equality is a constant, and the right is a function of x, so the equality can't be true. The integration region R for the left-hand side is $R = \{0 \le x \le 4, 0 \le y \le \sqrt{x}\}$. Rewriting R as a "vertically simple region", the right-hand iterated integral should be $\int_0^2 \int_{y^2}^4 \sqrt{x+y^2}\, \mathrm{d}x\mathrm{d}y$.

2. Evaluate the iterated integral

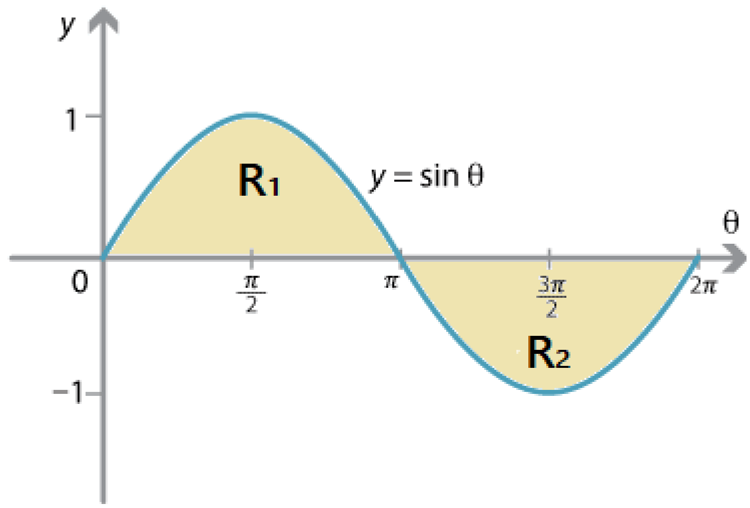
$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\mathrm{d}x\mathrm{d}y.$$

Switching the order of integration, this iterated integral is equivalent to $\int_0^{\sqrt{\pi}} \int_0^x \cos(x^2)\, \mathrm{d}y\mathrm{d}x = \int_0^{\sqrt{\pi}} \cos(x^2)x\, \mathrm{d}x = 0$ (after a a u-substitution).

3. (a) What is the geometric meaning of $\iint_R f(x,y)\, \mathrm{d}A$ if $f(x,y) \ge 0$? What if f takes both positive and negative values? If f has units of mass density, i.e. mass per unit area, what does this integral represent?

   If $f \ge 0$, the integral represents the volume of the solid bounded by f and the xy-plane over the region R. If f takes both positive and negative values, it represents a signed volume (same as single integrals represent signed areas). If f is a mass density, then the integral represents the total mass of the region R. (Similarly for one-dimensional integrals, if $\rho(x)$ describes mass per unit length, i.e. for an object having only mass variation in only one direction, then $\int_a^b \rho(x)\mathrm{d}x$ is the total mass on the interval $[a,b]$.)

   (b) Consider the region $R = R_1 \cup R_2$ pictured below:



If $f(\theta, y) = 1$ for all (θ, y) in R_1, and $f(\theta, y) = -1$ for all (θ, y) in R_2, evaluate the integral

$$\iint_R f \, \mathrm{d}A.$$

Hint: Consider your answer to part (a).

Since the areas of regions R_1 and R_2 are equal, we have

$$\iint_R f\,dA = \iint_{R_1} 1\,dA + \iint_{R_2} (-1)\,dA = \iint_{R_1} 1\,dA - \iint_{R_2} 1\,dA = \text{area}(R_1) - \text{area}(R_2) = 0$$

(c) Write down $\displaystyle\iint_{R_1} f(\theta, y)\,\mathrm{d}A$ as an iterated integral in two different ways (with R_1 as a "vertically simple region" and also as a "horizontally simple region"). Evaluate one of these.

As a vertically simple region, we have

$$\iint_{R_1} 1\,\mathrm{d}A = \int_0^{\pi} \int_0^{\sin\theta} 1\,\mathrm{d}y\mathrm{d}\theta = 2,$$

which is just the area of R_1.

Describing R_1 as a horizontal region is not so easy since sine is not invertible on $[0, \pi]$. As far as MATH241 is concerned, you can ignore this particular question as it was harder than intended. How it would need to be done is to first note that $\sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$ is the usual definition for sine inverse, which can be used for the left-bounding function. But for the right-bounding function, we need an inverse sine with (at least) the range $[\pi/2, \pi]$. Since changing the domain and/or range results in a different function, we can't call it $\sin^{-1}$, so let's call this function $\sin^{-1}_{[\pi/2,\pi]} : [0, 1] \to [\pi/2, \pi]$, where the interval subscript denotes that we are inverting the sine function on the interval $[\pi/2, \pi]$. (This is not standard notation, and I am not aware of a better notation.) So as a horizontal region, we can write

$$\iint_{R_1} 1\,\mathrm{d}A = \int_0^1 \int_{\sin^{-1}(y)}^{\sin^{-1}_{[\pi/2,\pi]}(y)} 1\mathrm{d}\theta\,\mathrm{d}y = \int_0^1 \sin^{-1}_{[\pi/2,\pi]}(y) - \sin^{-1}(y)\,\mathrm{d}y$$

This can be carefully evaluated (it ends up being $(\pi/2+1)-(\pi/2-1)=2$ but will not be done here.

4. Set up the iterated (double) integral that gives the volume of the solid that lies under the paraboloid $z=x^2+y^2$, above the xy-plane, and inside the cylinder $x^2+y^2=2x$.

   In polar coordinates, the cylinder has the equation $r=2\cos\theta$. So the integration region R is $\{-\pi/2 \le \theta \le \pi/2, 0 \le r \le 2\cos\theta\}$ and the double integral is

$$\iint_R (x^2+y^2)\,\mathrm{d}A = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} (r^2)\,r\mathrm{d}r\mathrm{d}\theta$$

5. Evaluate the double integral $\iint_R (2x-y)\,\mathrm{d}A$ where R is the region in the first quadrant enclosed by the circle $x^2+y^2=4$ and the lines $x=0$ and $y=x$.

   Converting $y=x$ to polar, we get $\theta=\pi/4$ in the first quadrant. Similarly the line $x=0$ in the first quadrant becomes $\theta=0$, and also the circle bound is $r=2$ in polar. Therefore we can evaluate the double integral as

$$\int_0^{\pi/4}\int_0^2 (2r\cos\theta - r\sin\theta)\,r\mathrm{d}r\mathrm{d}\theta = \frac{8}{3}\left[\frac{3}{2}\sqrt{2}-1\right]$$