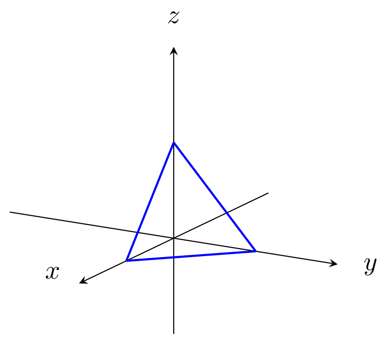
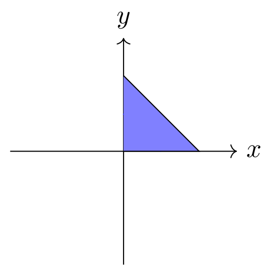
**Groupwork 7 Solution**
**MATH 241 (Spring 2023)**
**04/04/2023**

**Problem 1.** In order to sketch the tetrahedron $E$, consider the case when $x = 0$, $y = 0$, and $z = 0$. If $x = 0$, then you have the line $y + z = 1$ on the $yz$-plane. Similarly, If $y = 0$, then you have the line $x + z = 1$ on the $xz$-plane. Lastly, when $z = 0$, you have $x + y = 1$ in the $xy$-plane which results in the following graph.




We first set up the first two integral which will describe the region $R$ formed by projecting the tetrahedron $E$ down to the $xy$-plane. Then the region $R$



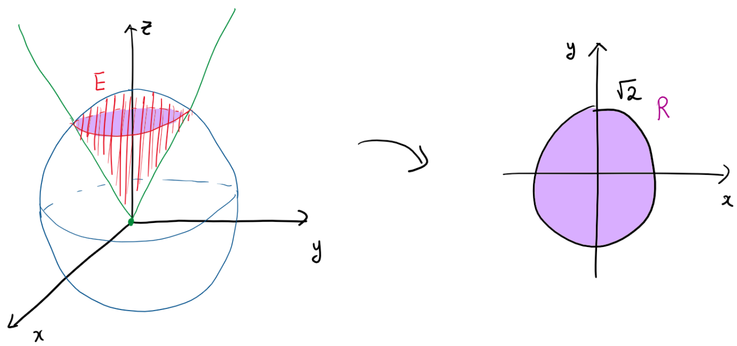

is bounded by $x = 0$, $y = 0$, and $x + y = 1$, so the first two integral will be

$$\iiint_E z\ dV = \iint_R \int_?^? z\ dz\ dA = \int_0^1 \int_0^{1-x} \int_?^? z\ dz\ dy\ dx$$

To figure out the third integral, note that the *top function* is given by $x + y + z = 1 \Rightarrow z = 1 - x - y$ and the *bottom function* is given by $z = 0$, hence the integral is

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\ dz\ dy\ dx$$

**Problem 2.** The solid region $E$ can be sketched as follows




Then the region $R$ is formed by projecting $E$ onto the $xy$-plane. This is exactly the intersection between the cone and the sphere. Namely, substituting $z^2 = x^2 + y^2$ into the equation of the sphere, we obtain $x^2 + y^2 + (x^2 + y^2) = 4 \Rightarrow x^2 + y^2 = 2$. Hence $R$ is enclosed by the circle of radius $\sqrt{2}$ centered at the origin. Using the sphere as the top function and cone as the bottom function, we get

$$\iiint_E f\, dV = \iint_R \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} z\, dz\, dA$$

(a) In cylindrical coordiante, we get

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} zr\, dz\, dr\, d\theta$$

using $r^2 = x^2 + y^2$.

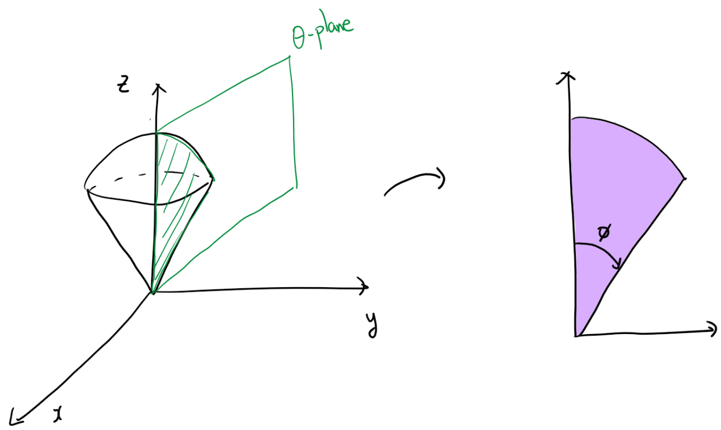
(b) To figure out the range for $\phi$, we use the equation of the cone. This says that

$$\rho^2 \cos^2 \phi = \rho^2 \sin^2 \phi \Rightarrow \cos^2 \phi = \sin^2 \phi$$

Since $z > 0$ in $E$ and $0 \leq \phi \leq \pi$, the only solution is $\phi = \frac{\pi}{4}$. Hence the integral in spherical coordinate is

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 (\rho \cos \phi)(\rho^2 \sin \phi)\, d\rho\, d\phi\, d\theta$$

In order to see this, the $\theta$-cross section is all of the same shape




As $\phi$ ranges from $0$ to $\frac{\pi}{4}$, $\rho$ is always from $0$ to $2$.

**Problem 3.** Since we use $r$ as a variable, we use $R$ for the radius of the cylinder. Then the volume of a cylinder is given by the following iterated integral

$$\int_0^{2\pi}\int_0^R\int_0^h r dz\; dr\; d\theta = \int_0^{2\pi}\int_0^R hr\; dr\; d\theta = h\int_0^{2\pi} \left.\frac{r^2}{2}\right|_0^R d\theta = h\int_0^{2\pi}\frac{R^2}{2}\; d\theta = \frac{R^2 h}{2}\cdot(2\pi) = \pi R^2 h$$

**Problem 4.**

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \; dz \; dy \; dx &= \int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{2} \; dy \; dx \\
&= \frac{1}{2} \int_0^1 \int_0^{1-x} (1-x)^2 - 2(1-x)y + y^2 \; dy \; dx \\
&= \frac{1}{2} \int_0^1 \left. (1-x)^2 y - (1-x)y^2 + \frac{y^3}{3} \right|_0^{1-x} \; dx \\
&= \frac{1}{2} \int_0^1 \frac{(1-x)^3}{3} \; dx \\
&= -\frac{1}{6} \int_1^0 u^3 \; du \\
&= -\frac{1}{6} \left. \frac{u^4}{4} \right|_1^0 = \frac{1}{24}
\end{aligned}$$

(a)

$$\begin{aligned}
\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} zr \; dz \; dr \; d\theta &= \int_0^{2\pi} \int_0^{\sqrt{2}} \left. r\frac{z^2}{2} \right|_r^{\sqrt{4-r^2}} \; dr \; d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} \frac{r}{2}((4-r^2) - r^2) \; dr \; d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} 2r - r^3 \; dr \; d\theta \\
&= \int_0^{2\pi} \left. r^2 - \frac{r^4}{4} \right|_0^{\sqrt{2}} \; d\theta \\
&= \int_0^{2\pi} d\theta = 2\pi
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^2 (\rho\cos\phi)(\rho^2\sin\phi) \; d\rho d\phi \; d\theta &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^2 \rho^3 \cos\phi \sin\phi \; d\rho d\phi \; d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \left. \frac{\rho^4}{4} \cos\phi \sin\phi \right|_0^2 \; d\phi \; d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} 4\cos\phi\sin\phi \; d\phi \; d\theta \\
&= \int_0^{2\pi} \left. -\cos(2\phi) \right|_0^{\frac{\pi}{4}} \; d\theta \\
&= \int_0^{2\pi} -(0-1) \; d\theta = 2\pi
\end{aligned}$$

Recall here that $\sin(2\phi) = 2\sin\phi\cos\phi$, so $\sin\phi\cos\phi = \frac{1}{2}\sin(2\phi)$. Then

$$\int \cos\phi\sin\phi \; d\phi = \int \frac{1}{2}\sin(2\phi) \; d\phi = -\frac{1}{4}\cos(2\phi) + C$$