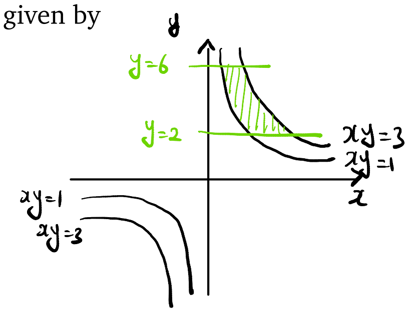
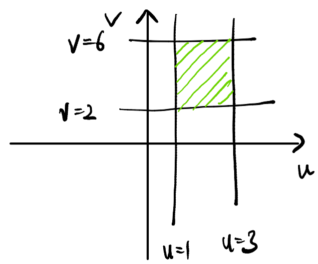
Groupwork 8 Solution
MATH 241 (Spring 2023)
04/11/2023

Problem 1. Note that the bound is given by



$$\{(x, y) \mid 1 \le xy \le 3 \text{ and } 2 \le y \le 6\}$$

If we set $u = xy$ and $v = y$, then the region R becomes



$$\{(u, v) \mid 1 \le u \le 3 \text{ and } 2 \le v \le 6\}$$

a rectangular region. We have $u = xy = xv$, so

$$x = \frac{u}{v} \text{ and } y = v$$

Hence the Jacobian of the transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

Therefore

$$\iint_R xy^3 \, dA = \int_2^6 \int_1^3 uv^2 \left|\frac{1}{v}\right| du\, dv = \int_2^6 \int_1^3 uv \, du\, dv = \int_2^6 \frac{u^2}{2} v \Big|_1^3 dv = 4 \int_2^6 v \, dv = 4 \left(\frac{v^2}{2}\right)\Big|_2^6 = 64$$

Problem 2.

(a) We keep the variables x, y. Then $6x + y + 3z = 9 \Rightarrow 3z = 9 - 6x - y \Rightarrow z = \frac{1}{3}(9 - 6x - y)$. Therefore, **a** vector parametrization is

$$\boldsymbol{r}(x, y) = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + \frac{1}{3}(9 - 6x - y)\hat{\boldsymbol{k}}$$

and we need to figure out the ranges of the parameters. Since being in first octant means that x, y, z are all ≥ 0, we first figure out the range of x. Since $y, z \ge 0$, we have

$$6x \le 6x + y + 3z = 9 \Rightarrow x \le \frac{3}{2}$$

On the other hand, the range of y depends on x.

$$0 \le z = \frac{1}{3}(9 - 6x - y) \Rightarrow 0 \le 9 - 6x - y \Rightarrow y \le 9 - 6x$$

Therefore, the ranges are given by

$$0 \le x \le \frac{3}{2} \text{ and } 0 \le y \le 9 - 6x$$

(b) For circular paraboloid, it is better to use the *cylindrical coordinate*. Hence we are going to use r, θ for the vector parametrization. Observe that the intersection of the circular paraboloid and the cylinder is $z = x^2+y^2 = 9$. Hence the part of the circular paraboloid inside the cylinder is given by $0 \le z \le 9$. As $z = x^2 + y^2 = r^2$, we obtain the inequality $0 \le r \le 3$. Therefore, a vector parametrization is

$$\boldsymbol{s}(r,\theta) = r\cos\theta\hat{\boldsymbol{\imath}} + r\sin\theta\hat{\boldsymbol{\jmath}} + r^2\hat{\boldsymbol{k}}$$

with $0 \le \theta \le 2\pi$ and $0 \le r \le 3$.

Problem 3.

(a) Here dummy variable change means

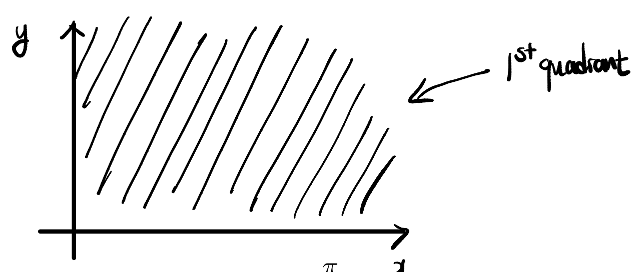
$$\int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-y^2}\,dy$$

By setting $y = x$. Then we have

$$\begin{aligned}\left(\int_0^\infty e^{-x^2}\,dx\right)^2 &= \int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-x^2}\,dx \\ &= \int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy \\ &= \int_0^\infty e^{-y^2}\left(\int_0^\infty e^{-x^2}\,dx\right)\,dy \\ &= \int_0^\infty \int_0^\infty e^{-x^2}\cdot e^{-y^2}\,dx\,dy \\ &= \int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dx\,dy\end{aligned}$$

Here we have repeatedly used the *constant rule* $C(y)\int f(x)\,dx = \int C(y)f(x)\,dx$ where $C(y)$ is a function of y because in terms of x, $C(y)$ is a constant.

(b) The bound $0 \le x$ and $0 \le y$ in polar coordinate is given by



$$0 \le r \text{ and } 0 \le \theta \le \frac{\pi}{2}$$

Then we have

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r\,dr\,d\theta$$

(c) Let $u = -r^2$, then $du = -2r\,dr$, so we have

$$\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left(-\frac{1}{2}\int_0^{-\infty} e^u\,du\right)\,d\theta = -\frac{1}{2}\int_0^{\frac{\pi}{2}} e^u\Big|_0^{-\infty}\,d\theta = -\frac{1}{2}\int_0^{\frac{\pi}{2}} -1\,d\theta = \frac{\pi}{4}$$

In particular,

$$\left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \frac{\pi}{4} \Rightarrow \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$